7 Scattering theory and relativistic theory of particles interactions

In this chapter we lay foundations for the first approach to formulating relativistic quantum field theories. This approach is close in spirit to the view expressed by Steven Weinberg - one of the founders of the Standard Theory of elementary particle interactions - that quantum field theory is merely a convenient machinery allowing to systematically construct amplitudes of particle reactions (comprising together the S-matrix) satisfying a certain set of physically motivated requirements such as Poincaré covariance, unitarity, cluster decomposition and analyticity (which were formulated in the historical development of high energy particle physics quite independently of the field theory principles). This view, while convenient as a starting point for our considerations, seems, however, too restricted. A more balanced one is probably that quantum field theory is just a quantum theory of some physical system. But what this system really is? In other words, what is the "ontology" underlying the quantum field theory? We will see that to some extent the ultimate formalism we will come to know dispenses us of such questions.[1] Nevertheless, in formulating quantum field theory one has to stick to some "ontology". The two obvious possible choices which lead to the quantum field theory as we know it are particles and fields (but one cannot exclude that the true underlying physical system may ultimately prove to be something else). Therefore, in this chapter and in the two following ones (Chapters 8 and 9) our underlying ontology will be particles. Quantum field theory as a quantum theory of a system of fields[2] will be developed in Chapter 11. We decided to present both formulations because this allows to better understand the foundations of quantum field theory and makes also clear similarities and differences between its versions used in particle physics and in condensed matter and solid state physics.

Adopting particles as the basic ontology is natural in condensed matter and solid state physics. Systems considered in these branches of physics can certainly be treated as composed of well known particles[3] (although in order to capture essential properties of some systems one nowadays frequently considers idealized systems consisting of spins at fixed positions or allows particles to only hop from one site to another of a prescribed lattice - these are effective, purely theoretical constructions treating systems of ordinary

[1] In this sense quantum field theory seems to favour the view, nowadays widespread, if not prevailing among theoreticians, that only the mathematical formalism matters and the ontology is largely irrelevant; this was most probably the attitude to physical theories of Dirac, but certainly not the one of Bohr!

[2] It is of course possible to formulate quantum field theory as a quantum theory of a mixed system consisting of fields and particles. (This was the approach adopted in Section 3.8 in which quantum theory of radiation was presented as a prototype quantum field theory). In fact, it seems that this may be the most natural point of view on the physical system underlying the quantum field theory: fermionic fields are Grassman algebra valued mappings (see Section 11.8) which hardly, if at all, can be ascribed any physical reality - because of this fermions most probably should be considered true particles. Bosons, in contrast, are most naturally interpreted as quantum excitations of continuous fields.

[3] That is, the question what these particles are made of and why they have properties they have - masses, spins - is entirely irrelevant for problems which are of interest in these areas of physics.

particles at a - so to say - higher emergent level) the properties of which - masses, spins, charges, (long distance) interactions, etc. - are well known. The Hilbert space $\mathcal{H}$ of a theory constructed adopting this ontology is naturally a multiparticle space of the same kind as the spaces built in Chapter 5, possessing the vector $|\text{void}\rangle$ from which other vectors can be obtained by the action of an arbitrary number (which can also be infinite) of creation operators corresponding to the kinds of particles which are "put in" into the system (as its "fundamental" constituents or "fundamental" buildingblocks); the theory is constructed by adding to the free Hamiltonian H_0 an interaction operator V_{int} acting in $\mathcal{H}$. The resulting quantum mechanics of a many particle system with the Hamiltonian similar to the ones considered in Chapter 5 *is* a model of (nonrelativistic) quantum field theory and in the thermodynamic limit, $V \to \infty$, $N \to \infty$, N/V fixed, properties of excitations of the resulting system, interpreted in terms of quasi-particles, are in general very different than properties of the "fundamental" particles "put" in the system - see the discussion in Section 5.7.

In the approach developed in this and in the two following chapters, relativistic field theories will be formulated in the similar spirit, as quantum theories of interacting relativistic particles. Therefore the starting point will be the relativistic theory of *free* particles of a finite number of definite kinds, a, b, ..., constructed on the basis of the second quantization formalism of Section 6.5. The "arena" of the latter theory is the big multiparticle Hilbert space $\mathcal{H}$ which as in Section 5.1 is a direct sum of multiple tensor products of single-particle Hilbert spaces $\mathcal{H}_a^{(1)}$, $\mathcal{H}_b^{(1)}$, ..., of several types of particles and of the one dimensional space $\mathcal{H}^{(0)}$. The big Hilbert space is therefore spanned by the vector $|\text{void}\rangle$ (which spans $\mathcal{H}^{(0)}$) and all possible multi-particle state-vectors

$$|(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2, \ldots, \mathbf{p}_N\sigma_N)_0\rangle\,, \tag{7.1}$$

constructed as (appropriately symmetrized/antisymmetrized) tensor products of one-particle state-vectors.[4] In the infinite space volume the vectors (7.1) are normalized so that (somewhat symbolically)

$$\begin{aligned}\langle(\mathbf{p}'_N\sigma'_N, \mathbf{p}'_{N-1}\sigma'_{N-1}, \ldots, \mathbf{p}'_1\sigma'_1)_0|(\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2, \ldots, \mathbf{p}_M\sigma_M)_0\rangle \\ = \delta_{NM}\sum_P(-1)^P\delta_\Gamma(\mathbf{p}'_1 - \mathbf{p}_{P(1)})\ldots\delta_\Gamma(\mathbf{p}'_N - \mathbf{p}_{P(N)})\,.\end{aligned} \tag{7.2}$$

The sum in (7.2) is over permutations within groups of labels corresponding to identical particles and $(-1)^P$ is the sign of the permutation of fermionic labels in a given

[4] If the vector (7.1) represents N_a particles of type a, N_b particles of type b, etc. ($N_a + N_b + \ldots = N$), different groups of labels, e.g. $(\mathbf{p}_i\sigma_i, \ldots, \mathbf{p}_{i+N_a}\sigma_{i+N_a})$ correspond to different types of particles but we do not introduce any additional index to distinguish which labels correspond to which type of particles. Only labels corresponding to identical particles are symmetrized or antisymmetrized as described in Chapter 5. Basis N-particle states constructed as appropriate linear combinations of the states (7.1) are also in use (see Section 6.4 for examples of such alternative bases of the $\mathcal{H}^{(2)}$ subspace). The subscript "0" is used to distinguish these vectors from the *in* and *out* vectors (to be introduced in Section 7.3) which will also be labeled by listing the momenta and spin variables of the particles they represent.

permutation P. The symmetrizations and antisymmetrizations are implicitly understood to be done only within groups of identical particles. The symbol $\delta_\Gamma(\mathbf{p}'-\mathbf{p})$ which here is assumed to include also the Kronecker delta of the spin variables σ, depends on the normalization of the one-particle states: if it is the one usually adopted in nonrelativistic applications, $\delta_\Gamma(\mathbf{p}'-\mathbf{p}) = (2\pi)^3\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})$; in relativistic theories more convenient is the normalization such that $\delta_\Gamma(\mathbf{p}'-\mathbf{p}) = (2\pi)^3 2E_{\mathbf{p}}\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})$, corresponding to $2E_{\mathbf{p}}$ particles in the unit volume (see Section 10.2). Because in general considerations we will be not interested in the detailed particle composition of the multi-particle state-vectors, it is practical to introduce a compact notation, in which $|\alpha_0\rangle$ stands for state-vectors of the form (7.1) and the scalar product (7.2) is concisely written as

$$\langle\beta_0|\alpha_0\rangle = \delta(\beta-\alpha) \equiv \delta_{\beta\alpha}\,. \tag{7.3}$$

The completeness relation

$$\hat{1} = |\text{void}\rangle\langle\text{void}| + \sum_{N=1}^{\infty}\left(\sum_{N_a}\sum_{N_b}\ldots\right)\delta_{N,(N_a+N_b+\ldots)}\,\frac{1}{N_a!N_b!\ldots} \tag{7.4}$$
$$\sum_{\sigma_1,\ldots,\sigma_N}\int\prod_{i=1}^{N}d\Gamma_{\mathbf{p}_i}|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_0\rangle\langle(\mathbf{p}_N\sigma_N,\ldots,\mathbf{p}_1\sigma_1)_0|\,,$$

in which the summation is over different numbers N_a, $N_b,\ldots$ of indistinguishable particles of distinct types a, $b,\ldots$, will be then compactly written as

$$\hat{1} = \int d\alpha\,|\alpha_0\rangle\langle\alpha_0|\,, \qquad \text{i.e.} \qquad |\Psi\rangle = \int d\alpha\,|\alpha_0\rangle\langle\alpha_0|\Psi\rangle\,. \tag{7.5}$$

where $|\Psi\rangle$ is any state-vector belonging to the Hilbert space $\mathcal{H}$.

The vectors (7.1) are the eigenvenctors of the free Hamiltonian H_0 which is taken to be a sum $H_0 = H_0^a + H_0^b + \ldots$ of terms (6.113) with the energies $E_a(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_a^2}$, $E_b(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_b^2}$, ... - hence the subscript 0 in $|\alpha_0\rangle$. In the infinite volume V of the space the only normalizable eigenvector of H_0 is the vector $|\text{void}\rangle$; the other eigenvectors $|\alpha_0\rangle$ of H_0 are non-normalizable.[5] Because the vector $|\text{void}\rangle$ is also the lowest energy H_0 eigenvector, it will be denoted $|\Omega_0\rangle$ (since in relativistic theories the numbers of particles cannot be conserved by interactions, there is no point to consider, as in nonrelativistic theories, separate H_0 eigenvectors $|\Omega_0^{(N_a,N_b\ldots)}\rangle$ in each $\mathcal{H}^{(N_a,N_b\ldots)}$ subspace). As explained in Section 5.1, even in the finite volume V, when allowed particle momenta form a discrete set (as a result of imposing periodic boundary conditions) and all state-vectors are normalizable, the Hilbert space is not separable - the set of vectors $|\alpha_0\rangle$ which span the big Hilbert space $\mathcal{H}$ is not countable.[6] The separable subspace spanned in the big Hilbert space by

[5] Non-normalizable state-vectors $|\alpha_0\rangle$, called generalized vectors, are in this respect similar to the plane waves $\psi_{\mathbf{p}} = e^{i\mathbf{p}\cdot\mathbf{x}}$ of ordinary nonrelativistic Quantum Mechanics of a single particle which are generalized (non-normalizable) eigenvectors of the $H_0 = \hat{\mathbf{P}}^2/2m$ and $\hat{\mathbf{P}}$ operators.

[6] This follows from the mathematical facts that for integer M and N both limits

$$\lim_{N\to\infty}\lim_{M\to\infty}M^N \qquad \text{and} \qquad \lim_{N\to\infty}2^N\,,$$

the vectors obtained by acting on $|\Omega_0\rangle = |\text{void}\rangle$ with an arbitrary but *finite* number of the creation operators forms the most natural (but not the only one which can be selected) *Fock space*.

The relativistic character of the theory of free particles constructed in Section 6.5, is ensured by the relativistic form of the energies $E_a(\mathbf{p})$, $E_b(\mathbf{p})$, ..., entering the free Hamiltonians H_0^a, H_0^b, ... of the form (6.113) and by the possibility of constructing (as bilinear combinations of the creation and annihilation operators of the "fundamental" particles "put in" into the system) the remaining Poincaré group generators $\mathbf{J}_0$, $\mathbf{P}_0$, and $\mathbf{K}_0$, acting in $\mathcal{H}$ and satisfying together with H_0 the rules (6.21). Of course the manifestly relativistic character of such a theory of free particles is lost when it is considered not in the infinite space; nevertheless, considering the the system of particles as enclosed in a finite volume V is necessary for example to consider thermodynamical properties of a gas of free relativistic (in the sense of their energy-momentum relation) particles.

The theory of interacting relativistic particles (which in this approach *is* the quantum field theory) is constructed by adding to the free Hamiltonian H_0 an interaction operator V_{int} acting in the big Hilbert space $\mathcal{H}$ spanned by the vector $|\text{void}\rangle = |\Omega_0\rangle$ and all vectors (7.1). Whether the resulting theory of the "fundamental" particles "put in" into the system and now allowed to interact with each other is still relativistic, that is, whether it is possible to construct in $\mathcal{H}$ new Poincaré group generators $\mathbf{J}$, $\mathbf{P}$, and $\mathbf{K}$ which together with $H = H_0 + V_{\text{int}}$ would satisfy the commutation rules (6.21) and additional physical requirements which will be formulated below depends, of course, on the form of V_{int}.

Assuming that the theory obtained by replacing H_0 by $H = H_0 + V_{\text{int}}$, is still a theory of particles (that is, assuming that the Hamiltonian $H = H_0 + V_{\text{int}}$ has still eigenvectors which can be interpreted as representing some kinds of particles - see below), the question what interactions V_{int} allow for constructing the Poincaré group generators requires formulating the theory in the continuum and is, for this reason, most conveniently investigated within the framework of the scattering theory:[7] S-matrices characterizing interactions of

(relevant for counting bosonic and fermionic basis states) are equal to the power of the continuum. It is precisely the nonseparability of the Hilbert space $\mathcal{H}$ which is at the origin of the mentioned insensitivity of the ultimate formalism to the "ontology" underlying the theory.

[7]Although from the fundamental perspective it should be regarded as matter of pure calculational convenience that quantum field theory (or, more generally, any quantum theory) is formulated in the infinite space volume - there is a strong conviction (which in some particular cases can be rigorously justified) that the essential physics of considered systems must be the same, whether they are considered in the infinite space or as confined to a (large) finite volume and that in the latter case their measurable characteristics (if properly defined) do not depend on the size (if it is sufficiently large) of this volume that is, tend to well defined limits as $V \to \infty$ - the proper formulation of the scattering theory requires considering the theory in the infinite space. The point is that in the finite volume, when all eigenvectors of the Hamiltonian are normalizable, scattering processes cannot be sharply distinguished from the general time evolution of the system: all reactions would occur multiply as time goes and it would not make sense to appeal to the infinite time limits in order to define measurable quantities characterizing what in the real world is observed as scattering processes. Therefore, to meaningfully define state-vectors representing scattering reactions the infinite volume is crucial.

relativistic particles, that is the set of amplitudes allowing to compute probabilities (rates) of particle reactions - see Section 10.2 - should transform in a well defined way when the inertial reference frame in which they occur is changed - they should be Lorentz covariant. Therefore in this chapter we will first formulate the scattering theory which in its general form applies to the ordinary nonrelativistic as well as to relativistic quantum mechanics. In fact, despite some important differences between its simplest version - the theory of scattering by an external potential based on the nonrelativistic quantum mechanics of a single particle and the scattering theory applied to the relativistic quantum mechanics of interacting particles (that is quantum field theory) developed here,[8] keeping in mind the former is helpful in understanding also the latter one. Thus we will first derive general formulae, applicable in relativistic and in nonrelativistic theories, expressing S-matrix elements in different ways useful in discussing its various aspects and will work out various approximate and iterative ways of computing them in addition to the basic one based on the Dyson expansion and the Wick theorem. This one will be illustrated here on the example of the elastic scattering of nonrelativistic particles which will serve us to derive the result used in Section 5.5.

We will then investigate the requirement of Lorentz covariance of the S-matrix and will formulate sufficient conditions under which the Hamiltonian $H = H_0 + V_{\rm int}$ leads to a Lorentz covariant S-matrix. It will be seen that if the S-matrix is Poincaré covariant (which is the case if $\mathbf{J}_0$, $\mathbf{P}_0$, and $\mathbf{K}_0$ commute with the S_0 operator which will be introduced in Section 7.1) it is also possible to construct the generators $\mathbf{J}$, $\mathbf{P}$, and $\mathbf{K}$ having the required properties. As will turn out, these sufficient conditions are not satisfied in some theories of physical interest and the ultimate Poincaré covariance of their S-matrices must be ensured by additional special features of these theories; nevertheless, the conditions formulated here constitute a useful reference point for further constructions. Finally, we will discuss in some details general properties of S-matrices in relativistic theories such as unitarity, partial wave expansion and its various possible symmetries.

Of course the fact that the relativistic character of the constructed theory is investigated by appealing to the infinite volume limit and the scattering theory does not mean that the S-matrix exhaust all the physically interesting information which can be obtained from it! Once it is formulated as a relativistic theory, various other properties of the underlying system, like for example its thermal properties which require keeping the volume finite, can be investigated by various other methods and means of general quantum mechanics (e.g. by the Rayleigh-Schrödinger stationary perturbative expansion).

[8]There exist, of course, intermediate level theories based on nonrelativistic quantum mechanics of many particles which, similarly to the relativistic quantum field theories allow to consider multichannel scattering process. (Full Galilean invariance, which forbids changes of the sums of masses of particles of definite kinds, must in such theories be somewhat violated.)

7.1 Time evolution, the S-matrix and the S_0 operator

In developing the scattering theory within the relativistic (or nonrelativistic) quantum mechanics of interacting particles formulated along the lines sketched above we make the following important but physically motivated assumptions. Firstly, we assume that the Hamiltonian $H = H_0 + V_{\rm int}$ is still a Hamiltonian of a system of particles, by which we mean that it possesses particle-like generalized eigenvectors, which in the sense which will be made precise below have properties similar to the multi-particle generalized eigenvectors (7.1) of a free Hamiltonian $\tilde{H}_0$, not necessarily identical with H_0 we start with, that is of a Hamiltonian $\tilde{H}_0$ which in the relativistic case is a sum of (a finite number of) terms $\tilde{H}_0^{\tilde{a}}$, $\tilde{H}_0^{\tilde{b}}, \ldots$ of the form (6.113) with some relativistic energies $E_{\tilde{a}}(\mathbf{p})$, $E_{\tilde{b}}(\mathbf{p}), \ldots$ (in the nonrelativistic case $\tilde{H}_0$ is the sum of terms of the form 5.61)) and with the original operators $a^\dagger_\sigma(\mathbf{p})$, $a_\sigma(\mathbf{p})$ replaced by some other operators $\tilde{a}^\dagger_\sigma(\mathbf{p})$, $\tilde{a}_\sigma(\mathbf{p})$ (constructed out of the original ones by means of some sort of a - perhaps very complicated compared to the one used in Section 5.5 - Bogolyubov transformation) satisfying analogous commutation relations.[9] We assume that all operators $\tilde{a}_\sigma(\mathbf{p})$ annihilate some normalizable (in the continuum) vector $|\tilde{\Omega}_0\rangle$ which is the ground state-vector of $\tilde{H}_0$ and that the $\tilde{H}_0$ generalized eigenvectors $|\tilde{\alpha}_0\rangle$ obtained by acting on $|\tilde{\Omega}_0\rangle$ with the operators $\tilde{a}^\dagger_\sigma(\mathbf{p})$ also span the original Hilbert space (or the relevant Fock space). In a relativistic theory (or in a theory invariant with respect to the transformations forming the Galileo group) this in particular means that $H = H_0 + V_{\rm int}$ possesses generalized (in the infinite space) eigenvectors which with respect to the transformations generated by the operators H, $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K}$ satisfying the rules (6.21) transform in the same way as do (with respect to transformations generated by $\tilde{H}_0$, $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{J}}_0$ and $\tilde{\mathbf{K}}_0$) the corresponding eigenvectors of $\tilde{H}_0$. Furthermore we will assume that, similarly to H_0 (and to $\tilde{H}_0$), the Hamiltonian $H = H_0 + V_{\rm int}$ has (in the infinite space volume) only a single (at least in the Fock space built on the vacuum vector $|\tilde{\Omega}_0\rangle$) normalizable ground-state eigenvector denoted $|\Omega\rangle$ and called the vacuum, that the particle-like non-normalizable eigenvectors of H, which will be introduced in Section 7.3, together with $|\Omega\rangle$ span the whole Hilbert space (or at least the Fock space built on the vacuum vector $|\tilde{\Omega}_0\rangle$) and, finally, that the spectra of the Hamiltonians $H = H_0 + V_{\rm int}$ and of $\tilde{H}_0$ are identical.[10] We therefore postulate that there is a *strict one-to-one correspondence* between all eigenvectors of $H = H_0 + V_{\rm int}$ and the eigenvectors of $\tilde{H}_0$ and that energies of the corresponding eigenvectors (with respect to the respective Hamiltonians, H and $\tilde{H}_0$) are equal. The physical motivation for these assumptions is that if $H = H_0 + V_{\rm int}$ is the Hamiltonian of a system of particles, its non-normalizable (in the continuum) eigenvectors should all represent (as is the case in ordinary nonrelativistic quantum mechanical scat-

[9] It is therefore clear that at least formally, the Poincaré (or the Galileo) group generators $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{J}}_0$ and $\tilde{\mathbf{K}}_0$ satisfying together with $\tilde{H}_0$ the commutation relations (6.21) (the relations (4.50) in the nonrelativistic case) can also be built as operators bilinear in the creation and annihilation operators $\tilde{a}^\dagger_\sigma(\mathbf{p})$, $\tilde{a}_\sigma(\mathbf{p})$.

[10] This is not always true in the scattering theory based on (nonrelativistic) quantum mechanics of a single particle in which the potential $V_{\rm int} = V_{\rm pot}(\mathbf{r})$ added to H_0 can lead to the existence of normalizable eigenvectors of H (i.e. bound states) but seems to be quite a natural assumption in a many-particle quantum theory formulated in the infinite space.

tering on a fixed potential which does not admit bound states) collision-type processes in which long before and long after the reaction particles look as (mutually) noninteracting. Therefore, it should be possible to associate with a given scattering process the Hilbert space (Heisenberg picture) state-vectors which, in a well defined way, correspond, as far as their transformation properties and energies are concerned, to eigenvectors of some $\tilde{H}_0$.

The assumptions formulated above could of course be checked if the theory could be solved exactly. Unfortunately, in most cases one has to rely on some sort of approximations which usually hinge on the second assumption we are going to make. A method with the help of which the true spectrum of H could, at least in principle, be investigated will be outlined in Chapter 13.

In their abstract form the assumptions formulated above do not allow to go too far[11] within the approach to the quantum field theory developed in this and in the two following chapters. Therefore, in order to construct theories in which practical calculations (based on a systematic Dyson perturbative expansion of Section 5.8) can be performed we will make a "technical", simplifying assumption that $V_{\rm int}$ is "small" in the sense that the spectrum of $H = H_0 + V_{\rm int}$ is the same as the spectrum of H_0, i.e. the full H eigenvectors have the properties of the eigenvectors $|\alpha_0\rangle$ of H_0. In other words, we will assume that $\tilde{H}_0 = H_0$ and that the strict one-to-one correspondence holds between the H_0 and H eigenvectors (including the equality of the corresponding eigenvalues) which form alternative bases of the same Fock space. In the considerations of this chapter, however, although we will use the notation H_0, and $|\alpha_0\rangle$, one can treat them as $\tilde{H}_0$ and $|\tilde{\alpha}_0\rangle$.

It is important to realize that these assumptions are neither a priori obvious, nor are they always fulfilled. It could happen that $H = H_0 + V_{\rm int}$ does not possess particle-like eigenstates at all (or not all of its eigenvectors can be interpreted as representing states of particles). This is indeed so in conformal field theory models or theories of "unparticles" discussed in the literature,[12] so that there are theories to which even the general, seemingly well motivated assumption does not apply. Furthermore, even if all $H = H_0 + V_{\rm int}$ eigenvectors represent states of particles, they can be in one-to-one correspondence with eigenvectors of a free-particle Hamiltonian $\tilde{H}_0$ which is very different from H_0 used to build H. The most prominent example of such a theory is Quantum Chromodynamics (QCD) - the theory of strong interactions in which the H_0 eigenvectors represent states of free spin $\frac{1}{2}$ coloured (i.e. transforming nontrivially under the action of the colour $SU(3)_c$ symmetry group) quarks, antiquarks and spin 1, massless coloured gluons, whereas the true H (and, therefore, also $\tilde{H}_0$) eigenvectors represent colourless, i.e. $SU(3)_c$ singlets, baryons, antibaryons and mesons. The approach exploiting the "technical" assumption, which (if no bound states can be formed) is naturally satisfied in nonrelativistic theories (essentially owing to the normal ordering of the interation term with respect to the vector $|{\rm void}\rangle$ and the bsence of aniparticles), was largely shaped by the historical development of

[11] At least in fully relativistic theories formulated in three space domensions - there are simplified nonrelativistic models, like e.g. the Lee model, in which the Hamiltonian $\tilde{H}_0$ can be explicitly constructed.

[12] Eigenvectors of free Hamiltonians H_0 of such theories represent massless particles.

quantum electrodynamics of electrons, positrons and photons, which as a quantum field theory is very special in that the interaction between charged particles and photons is quite weak and, moreover, all particles of this theory are absolutely stable.[13] The unified theory of weak and electromagnetic interactions (of which quantum electrodynamics is only a part) is also weakly coupled but certainly violates the assumption that there is a strict one-to-one correspondence between the particle-like eigenvectors of H and of H_0: $W^{\pm}$ bosons of spin 1, muons, taons are "put in" into the theory as particles and have the corresponding eigenvectors of H_0 but not being absolutely stable they have no, strictly speaking, their counterparts among the particle-like eigenvectors of H, so $\tilde{H}_0$ is different than H_0.

Thus, the assumptions adopted in the approach to quantum field theory based on relativistic quantum mechanics of particles, which is developed in Chapters 7-9 can be satisfied only in a very special (rather narrow) class of relativistic theories and require in addition a judicious construction of the interaction operator $V_{\rm int}$. This will become clear in Section 9.7, where it will turn out that observance of these assumptions (by appropriately adjusting $V_{\rm int}$) is crucial for avoiding some type of ill defined contributions to the transition amplitudes (S-matrix elements) that would otherwise occur in perturbative calculations.

With the two assumptions clearly spelled out as above it becomes possible to formulate the scattering theory based on relativistic (or nonrelativistic - the considerations of this and the following two sections apply equally well to interactions of nonrelativistic particles and also to the ordinary theory of a single nonrelativistic particle moving in an external potential) Quantum Mechanics of particles in the Fock space spanned by the H_0 eigenvectors. To the proper Hilbert space which we will consider initially belong all possible *normalizable* state-vectors $|\Psi\rangle$ that can be constructed out of the Fock space of generalized H_0 eigenvectors as

$$|\Psi\rangle = \int d\alpha \, |\alpha_0\rangle\langle\alpha_0|\Psi\rangle \equiv \int d\alpha \, |\alpha_0\rangle \, \psi(\alpha) \, , \tag{7.6}$$

with integrable profiles $\psi(\alpha)$: $\int d\alpha |\psi(\alpha)|^2 = 1$. One can then consider such states prepared at $t = 0$ and their time evolution generated either by H or H_0. Guided by the physical intuition and in line with the general framework adopted, we assume that the Schrödinger picture counterparts (we set $\hbar = c = 1$)

$$|\Psi(t)\rangle = e^{-iHt}|\Psi\rangle \equiv U(t,0)|\Psi\rangle \, , \tag{7.7}$$

(in the notation of Chapter 1) of normalizable Heisenberg picture state-vectors $|\Psi\rangle$ which represent reactions between particles converge as $t \to \mp\infty$ (in the sense of convergence in

[13] Positronium - a bound state of electron and positron - is unstable. However already electrodynamics of electrons and muons (which are stable in the absence of weak interactions) does not fully fit into the assumed scheme (although the interaction is still weak): the bound state of electron and antimuon (or of positron and muon) is stable and therefore the non-normalizable eivenvectors of the full Hamiltonian of such electrodynamics should correspond, strictly speaking, to the eigenvectors of $\tilde{H}_0$ which is the free Hamiltonian of electrons positrons, muons, antimuons, and of $e^-\mu^+$ and $e^+\mu^-$ bound states treated as elementary particles.

the Hilbert space of sequences of vectors) to some state-vectors[14]

$$|\Psi_{\rm as}^{\rm in/out}(t)\rangle = e^{-iH_0t}|\Psi_{\rm as}^{\rm in/out}\rangle \equiv U_0(t,0)|\Psi_{\rm as}^{\rm in/out}\rangle\,, \tag{7.8}$$

because in experiments one prepares states representing particles which before the collision are well localized and separated in space and are therefore from the practical point of view non-interacting with each other; likewise, long after the collision particles are again well separated and again look as mutually non-interacting. Thus, any state $|\Psi\rangle$ representing a scattering process can be written (employing the notation of Chapter 1) as

$$|\Psi\rangle = \lim_{t\to\mp\infty} U^\dagger(t,0)U_0(t,0)|\Psi_{\rm as}^{\rm in/out}\rangle = U_I^\dagger(t,0)|\Psi_{\rm as}^{\rm in/out}\rangle\,. \tag{7.9}$$

As any smooth, normalized superposition $|\Psi_{\rm as}^{\rm in/out}\rangle$ constructed as in (7.6) with some profile $\psi_{\rm as}^{\rm in/out}(\alpha)$ out of the H_0 eigenvectors $|\alpha_0\rangle$ should represent a possible initial or a possible final state of particles which are going to participate of have participated in some reaction (in agreement with the assumption that the H_0 - in general $\tilde{H}_0$ - span the whole relevant Hilbert or Fock space), one assumes that on the whole proper Hilbert space the operators

$$\Omega(t) \equiv e^{iHt}e^{-iH_0t} = U_I^\dagger(t,0)\,, \tag{7.10}$$

do have the limits[15]

$$\lim_{t\to\mp\infty}\Omega(t) = \Omega(\mp\infty) \equiv \Omega_\pm\,, \tag{7.11}$$

on any normalizable smooth superposition of the $|\alpha_0\rangle$ vectors. $\Omega_\pm$ are called Møller operators. Since

$$\frac{d}{dt}\,\Omega(t) = \frac{d}{dt}\left(e^{iHt}e^{-iH_0t}\right) = i\,U^\dagger(t,0)V_{\rm int}U_0(t,0) \equiv i\,U_I^\dagger(t,0)V_{\rm int}^I(t)\,,$$

and since $\Omega(0) = \hat{1}$, the operator $\Omega(t)$ can alternatively be defined by the integral relation

$$\Omega(t) = \hat{1} + i\int_0^t dt'\,U^\dagger(t',0)V_{\rm int}U_0(t',0)\,. \tag{7.12}$$

[14]The states $|\Psi_{\rm as}^{\rm in/out}\rangle$ considered here should not be identified with the *in* and *out* states analogous to the ones introduced in Section 1.3; the states playing the roles of the *in* and *out* states in the present context will be defined in Section 7.3.

[15]Since $\Omega_\pm$ clearly correspond to the $\varepsilon\to 0$ limits of the operators $U_I^\varepsilon(0,\mp\infty)$ considered in Section 1.2, this amounts to assuming that these regularized interaction picture evolution operators do have finite $\varepsilon\to 0$ limits on all smooth, normalized superpositions of the H_0 eigenvectors. In particular, one assumes here that the operators $U_I^\varepsilon(0,\mp\infty)$ acting on the ground-state eigenvector $|\Omega_0\rangle$ give in the limit $\varepsilon\to 0$ directly the state-vectors $|\Omega_\pm\rangle$, which are normalized lowest energy H eigenvectors and can differ one from another only by a phase factor (recall that in the Gell-Mann - Low construction one obtains the same eigenvector of H, whether one considers the $t\to-\infty$ or $t\to\infty$ limit). As discussed, and as will be seen, this can be true only if the interaction $V_{\rm int}$ is judiciously adjusted.

Furthermore, as the operators $\Omega(t)$ are unitary for any fixed t, that is satisfy $\Omega^\dagger(t)\Omega(t) = \hat{1}$ (and also $\Omega(t)\Omega^\dagger(t) = \hat{1}$), the Møller operators $\Omega_\pm$ are at least isometric, which means that similarly to $\Omega(t)$ they are defined on the whole Hilbert space $\mathcal{H}$ and preserve the norm: $(\Omega_\pm\Psi|\Omega_\pm\Psi) = (\Psi|\Psi)$ and, therefore, also the scalar products of normalizable states:

$$(\Omega_\pm\Phi|\Omega_\pm\Psi) = (\Phi|\Psi)\,, \tag{7.13}$$

that is they satisfy the relations

$$\Omega_+^\dagger\Omega_+ = \Omega_-^\dagger\Omega_- = \hat{1}\,,$$

(but, in general, not necessarily the relations $\Omega_+\Omega_+^\dagger = \Omega_-\Omega_-^\dagger = \hat{1}$). In relativistic quantum mechanics of particles (i.e. in QFT) one assumes that[16] $\Omega_\pm\mathcal{H} = \mathcal{H}$ (and not that only $\Omega_\pm\mathcal{H} \subset \mathcal{H}$), that is that any $\mathcal{H}$ space state-vector can be represented as the image of the action of Ω_+ and Ω_- on some states $|\Psi_{\rm as}^{\rm in/out}\rangle$:[17]

$$|\Psi\rangle = \Omega_+|\Psi_{\rm as}^{\rm in}\rangle = \Omega_-|\Psi_{\rm as}^{\rm out}\rangle\,. \tag{7.14}$$

(This implies that the relations $\Omega_+\Omega_+^\dagger = \Omega_-\Omega_-^\dagger = \hat{1}$ also do hold.)

Already at this point one can introduce the S_0 operator. As usually in a quantum theory, one is interested in scalar products $S_{\Phi\Psi} \equiv \langle\Phi|\Psi\rangle$ of normalized states. Expressing $|\Psi\rangle$ as the Ω_+ image of the appropriate $|\Psi_{\rm as}^{\rm in}\rangle$ and $|\Phi\rangle$ as the Ω_- image of $|\Phi_{\rm as}^{\rm out}\rangle$ one gets

$$S_{\Phi\Psi} \equiv \langle\Phi|\Psi\rangle = \langle\Phi_{\rm as}^{\rm out}|S_0|\Psi_{\rm as}^{\rm in}\rangle\,, \tag{7.15}$$

where the S_0 operator is defined as the product (compare the formula (1.73))

$$S_0 \equiv \Omega_-^\dagger\Omega_+\,. \tag{7.16}$$

It maps asymptotic free states (states of noninteracting particles) corresponding (in the sense specified by 7.9)) to "incoming" states of interacting particles onto free states corresponding to "outgoing" states of interacting particles: $|\Psi_{\rm as}^{\rm out}\rangle = S_0|\Psi_{\rm as}^{\rm in}\rangle$. The scalar product $S_{\Phi\Psi}$ - which is equal to the appropriate matrix elements of S_0 - has the natural interpretation of the probability amplitude of finding the system in the state $|\Phi\rangle$, which, if evolved in time,[18] would become in the far future indistinguishable from an appropriately (i.e. with H_0) evolved state $|\Phi_{\rm as}^{\rm out}\rangle$ which has direct interpretation in terms of noninteracting (and spatially separated in the far future) particles, if it is prepared as the state $|\Psi\rangle$ which, if evolved in time, has in the far past a similar free-particle interpretation, being indistinguishable from the evolved state $|\Psi_{\rm as}^{\rm in}\rangle$. The scalar products $S_{\Phi\Psi}$ thus contain

[16] This is not necessarily true in nonrelativistic Quantum Mechanics of a single particle. See Appendix E.

[17] Again, this is in line with the intuition that if there exists a stable bound state, it can be prepared in the far past and registered in the far future.

[18] Notice that the states are always identified at $t = 0$; that is we implicitly work in the Heisenberg picture (see Section 1.1) which in a relativistic theory allows to keep its Poincaré covariance as manifest as it is possible.

answers to a dominant amount of experimentally accessible questions which usually are formulated in the form "what is the probability that the detectors will register a given free-particle state produced as a result of an interaction of particles which long before interaction were prepared (in the accelerator) as another free-particles state?"

It is also convenient to introduce an alternative notation (corresponding to a slightly different labeling of states) and to call $|\Psi_+\rangle$ and $|\Psi_-\rangle$ the two different states which are the images of the same state $|\Psi\rangle$ under Ω_+ and Ω_-, respectively. Thus, in this notation $|\Psi_\pm\rangle = \Omega_\pm|\Psi\rangle$ which means that

$$\lim_{t\to\mp\infty} U(t,0)|\Psi_\pm\rangle = \lim_{t\to\mp\infty} U_0(t,0)|\Psi\rangle\,. \tag{7.17}$$

Scalar products (7.15) can be now written as $\langle\Phi_-|\Psi_+\rangle = \langle\Phi|S_0|\Psi\rangle$ and since

$$|\Psi_+\rangle = \Omega_+\Omega_-^\dagger|\Psi_-\rangle\,, \qquad |\Psi_-\rangle = \Omega_-\Omega_+^\dagger|\Psi_+\rangle\,, \tag{7.18}$$

they can also be expressed as the matrix elements

$$S_{\Phi\Psi} \equiv \langle\Phi_-|\Psi_+\rangle = \langle\Phi_-|S|\Psi_-\rangle = \langle\Phi_+|S|\Psi_+\rangle\,, \tag{7.19}$$

of the S operator

$$S \equiv \Omega_+\Omega_-^\dagger\,. \tag{7.20}$$

The S operator (which has been used in Section 1.3 - c.f. the formula (1.74)) which is different from the S_0 one[19] will be of little use in the approach developed in Chapters 7-9 (it becomes of relevant only in the approach based on Green's functions, when the structure of the asymptotic states is reconstructed from the poles of these functions). However the notation $|\Psi_\pm\rangle$ will be useful.

Under the assumptions underlying the considerations of this section the operators $H = H_0 + V_{\rm int}$ and H_0 satisfy (c.f. the formula (1.29) and its discussion) the important *intertwining relation*

$$H\,\Omega_\pm = \Omega_\pm H_0\,, \tag{7.21}$$

which in particular implies[20] that $\Omega_\pm^\dagger H\Omega_\pm = H_0$. This relation can be quickly derived by writing

$$e^{iHt}\,\Omega_\pm = e^{iHt}\lim_{\tau\to\mp\infty}\left(e^{iH\tau}e^{-iH_0\tau}\right) = \lim_{\tau\to\mp\infty}\left(e^{iH(\tau+t)}e^{-iH_0(\tau+t)}\right)e^{iH_0t} = \Omega_\pm\,e^{iH_0t}\,.$$

[19]The difference between the S_0 and S operators is particularly sharp in the nonrelativistic potential scattering theory if H possesses bound states: while S_0 acts nontrivially on the whole Hilbert space $\mathcal{H}$, S annihilates the whole subspace $\mathcal{H}_{\rm bound}$ - see Appendix E.

[20]If the operators $\Omega_\pm$ are truly unitary, these relations imply that the spectra of H and H_0 are identical (which is one of our assumptions adopted here). This again shows that in the case of ordinary potential scattering $\Omega_\pm$ cannot be unitary if H has bound states (normalizable eigenvectors) because the spectrum of H_0 is continuous. In such a case $\Omega_\pm$ are only isometric operators.

Differentiating this equality with respect to t at $t = 0$ yields the relation (7.21). The comparison with the rigorously derived formula (1.29) shows, however, that (7.21) can hold only if the interaction $V_{\rm int}$ is very special. The intertwining relations (7.21) combined with the relations (7.14) mean, in particular, that

$$\langle\Psi|H|\Psi\rangle = \langle\Psi^{\rm in}_{\rm as}|H_0|\Psi^{\rm in}_{\rm as}\rangle = \langle\Psi^{\rm out}_{\rm as}|H_0|\Psi^{\rm out}_{\rm as}\rangle\,.$$

Furthermore, exploiting (7.21) it is easy to show that

$$[S_0,\ H_0] = 0\,. \tag{7.22}$$

Indeed:[21]

$$S_0H_0 = \Omega^\dagger_-\Omega_+H_0 = \Omega^\dagger_-H\,\Omega_+ = \Omega^\dagger_-H\,\Omega_-\Omega^\dagger_-\Omega_+ = H_0\,\Omega^\dagger_-\Omega_+ = H_0S_0\,. \tag{7.23}$$

Since any normalizable state-vector $|\Psi\rangle$ can be written as a superposition of non-normalizable generalized (i.e. not belonging to the proper Hilbert space) H_0 eigenvectors $|\alpha_0\rangle$, one can write

$$\langle\beta_0|\Psi^{\rm out}_{\rm as}\rangle = \langle\beta_0|S_0|\Psi^{\rm in}_{\rm as}\rangle = \int d\alpha\,\langle\beta_0|S_0|\alpha_0\rangle\langle\alpha_0|\Psi^{\rm in}_{\rm as}\rangle\,.$$

One is thus led to consider the matrix elements $S_{\beta\alpha} \equiv \langle\beta_0|S_0|\alpha_0\rangle$. From (7.22) it follows that

$$0 = \langle\beta_0|[H_0,\ S_0]|\alpha_0\rangle = (E_\beta - E_\alpha)\langle\beta_0|S_0|\alpha_0\rangle\,,$$

which shows that $\langle\beta_0|S_0|\alpha_0\rangle \propto \delta(E_\beta - E_\alpha)$ (because $x\delta(x) = 0$). Furthermore, because for $V_{\rm int} = 0$ the S_0 operator reduces to the unit operator, it is convenient to write

$$S_0 = \hat{1} - iT_0\,, \tag{7.24}$$

introducing in this way the reaction operator T_0. Thus

$$S_{\beta\alpha} \equiv \langle\beta_0|S_0|\alpha_0\rangle = \delta_{\alpha\beta} - 2\pi i\,\delta(E_\beta - E_\alpha)\,t_{\beta\alpha}(E_\alpha)\,, \tag{7.25}$$

where $2\pi\delta(E_\beta - E_\alpha)\,t_{\beta\alpha}(E_\alpha) = \langle\beta_0|T_0|\alpha_0\rangle$. As will be shown in Chapter 10, it is precisely the quantity $t_{\beta\alpha}(E_\alpha)$ which is needed to compute the rate of the process $\alpha \to \beta$. In the case of the nonrelativistic potential scattering the quantity $t_{\beta\alpha} \equiv t(\mathbf{p}', \mathbf{p})$ is directly related to the standard scattering amplitude $f(\theta)$ - see Appendix E. All measurable characteristics of scattering processes predicted by a given theory defined by the Hamiltonian $H = H_0 + V_{\rm int}$ can be extracted from the matrix elements $S_{\beta\alpha}$ of the corresponding S_0 (or T_0) operator of this theory. One useful representation of this operator will be (in the end of Section

[21] We assume here that the Møller operators are unitary, i.e. that $\Omega_\pm\Omega^\dagger_\pm = \hat{1}$; see (E.2) in Appendix E for a justification in the case they are only isometric.

7.3) derived directly from the differential equation satisfied by the (interaction picture) evolution operator

$$U_I(t_2,t_1) = e^{iH_0t_2}e^{-iH(t_2-t_1)}e^{-iH_0t_1} = \Omega_-^\dagger(t_2)\Omega_+(t_1)\,, \tag{7.26}$$

introduced in Section1.1 of which S_0 is the double limit:

$$S_0 = \Omega_-^\dagger\Omega_+ = \lim_{t_2\to+\infty}\lim_{t_1\to-\infty} U_I(t_2,t_1)\,. \tag{7.27}$$

Using this representation it will be possible to evaluate S-matrix elements using the Dyson expansion of Section 5.8 (applied in conjunction with the Wick theorem of Section 5.9). Before exploiting this representation of the S_0 operator one has, however, to introduce the non-normalizable *in* and *out* (generalized) H eigenvectors and the rezolvent operators which allow to relate these to the H_0 eigenvectors $|\alpha_0\rangle$.

7.2 Rezolvents and the T operator representation of the S-matrix

A very important role in the formal scattering theory is played by the *rezolvent* operators[22]

$$G(z) \equiv (z-H)^{-1}\,, \qquad \text{and} \qquad G_0(z) \equiv (z-H_0)^{-1}\,. \tag{7.28}$$

Their matrix elements between normalizable states are analytic functions on the complex z plane except for isolated poles corresponding to normalizable H (H_0) eigenstates and a branch cut along the continuous part of the H (H_0) spectrum.

Substituting for A and B in the obvious operator identity

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{B}\,(B-A)\,\frac{1}{A}\,,$$

the operators $z-H$ and $z-H_0$ (the operators $z-H_0$ and $z-H$) one obtains two relations

$$\begin{aligned} G(z) &= G_0(z) + G_0(z)\,V_{\rm int}\,G(z)\,,\\ G(z) &= G_0(z) + G(z)\,V_{\rm int}\,G_0(z)\,, \end{aligned} \tag{7.29}$$

It is also easy to see that because $H = H^\dagger$ ($H_0 = H_0^\dagger$),

$$G(z^*) = [G(z)]^\dagger\,, \qquad G_0(z^*) = [G_0(z)]^\dagger\,. \tag{7.30}$$

[22] To relate $G(z)$ to the (Schrödinger picure) evolution operator $U(t,t_0)$ satisfying the equation (1.3) definine $iG^{\rm ret}(t,t_0) \equiv \theta(t-t_0)U(t,t_0)$. Because $\delta(t-t_0)U(t,t_0) = \delta(t-t_0)\hat{1}$, the (retarded Green's) function $iG^{\rm ret}(t,t_0)$ is a particular the solutions of the equation

$$\left(i\hbar\,\frac{d}{dt} - H\right) iG(t,t_0) = i\hbar\,\delta(t-t_0)\,\hat{1}\,.$$

The time Fourier transform of the solution is $i\tilde{G}^{\rm ret}(\omega) = i\hbar/(\hbar\omega - H)$ (the retarded function is selected by defining approprately the way of handling the singularity at $\hbar\omega = H$). This shows that $\tilde{G}(\omega) = \hbar G(\hbar\omega)$.

Matrix elements of the resolvent operator $G_0(z)$ between the non-normalizable H_0 eigenvectors are explicitly given by

$$\langle\beta_0|G_0(z)|\alpha_0\rangle = \delta_{\beta\alpha}\,\frac{1}{z-E_\alpha}\,. \tag{7.31}$$

Another very important operator is the $T(z)$ operator defined as

$$T(z) \equiv V_{\rm int} + V_{\rm int}\,G(z)\,V_{\rm int}\,. \tag{7.32}$$

It has the same analytic properties as $G(z)$ and satisfies the following relations

$$\begin{aligned} G_0(z)\,T(z) &= G(z)\,V_{\rm int}\,, \\ T(z)\,G_0(z) &= V_{\rm int}\,G(z)\,, \end{aligned} \tag{7.33}$$

which readily follow from the relations (7.29). They allow to express $G(z)$ through $T(z)$: replacing in (7.29) $V_{\rm int}\,G(z)$ (or $G(z)\,V_{\rm int}$) using (7.33) one gets

$$G(z) = G_0(z) + G_0(z)\,T(z)\,G_0(z)\,. \tag{7.34}$$

Using (7.33) in (7.32) leads instead to

$$T(z) = V_{\rm int} + V_{\rm int}\,G_0(z)\,T(z)\,, \tag{7.35}$$

which is known as the Lippman-Schwinger equation for $T(z)$. Iterating it yields the series

$$T(z) = V_{\rm int} + V_{\rm int}\,G_0(z)\,V_{\rm int} + V_{\rm int}\,G_0(z)\,V_{\rm int}\,G_0(z)\,V_{\rm int} + \ldots \tag{7.36}$$

Matrix elements (7.25) of the S_0 operator between generalized H_0 eigenstates can be expressed through the operator $T(z)$. To this end, instead of representing S_0 as the double limit of the $U_I(t_2,t_1)$ operator, as in (7.27), it is written as the single limit

$$S_0 = \Omega_-^\dagger\,\Omega_+ = \lim_{\tau\to\infty} U_I(\tau,-\tau) = \lim_{\tau\to\infty} e^{iH_0\tau}e^{-2iH\tau}e^{iH_0\tau}\,.$$

Differentiating with respect to τ the operator $U_I(\tau,-\tau)$ one obtains the differential equation satisfied by it, which, together with the obvious boundary condition at $\tau=0$, allows to write for this operator an integral expression, similar to (1.6). Using it, matrix element of the S_0 operator between normalizable states can be written as

$$\langle\Phi|S_0|\Psi\rangle = \langle\Phi|\Psi\rangle - i\int_0^\infty dt\,e^{-\varepsilon t}\langle\Phi|e^{iH_0t}\,V_{\rm int}\,e^{-2iHt}e^{iH_0t} + e^{iH_0t}e^{-2iHt}\,V_{\rm int}\,e^{iH_0t}|\Psi\rangle\,.$$

The factor $e^{-\varepsilon t}$ is not necessary when the matrix element is taken between two normalizable states, but when introduced,[23] it allows to replace $|\Psi\rangle$ and $|\Phi\rangle$ by the generalized

[23] It is usually (incorrectly) introduced from the beginning as a factor ensuring "adiabatic" switching on and off the interaction - something which certainly does not happen in Nature! Notice also that if $V_{\rm int}$ were defined with such a factor, the evolution operator $U(t,t_0)$ corresponding to the Hamiltonian H (explicitly time dependent then) would have to have the form (1.9) instead of $e^{-iH(t-t_0)}$.

H_0 eigenvectors $|\alpha_0\rangle$ and $|\beta_0\rangle$:

$$\langle\beta_0|S_0|\alpha_0\rangle = \delta_{\beta\alpha} - i\int_0^\infty dt\,\langle\beta_0|V_{\rm int}\,e^{i(E_\beta+E_\alpha-2H+i0)t} + e^{i(E_\beta+E_\alpha-2H+i0)t}\,V_{\rm int}|\alpha_0\rangle$$
$$= \delta_{\beta\alpha} + \frac{1}{2}\langle\beta_0|V_{\rm int}\,G\left(\frac{E_\beta+E_\alpha}{2}+i0\right) + G\left(\frac{E_\beta+E_\alpha}{2}+i0\right)V_{\rm int}|\alpha_0\rangle\,.$$

Using the operator identities (7.33) one can replace here the operators $G(z)$ by the $G_0(z)$ ones which can act directly on the states $|\alpha_0\rangle$ and $\langle\beta_0|$. The second term can be then cast in the form

$$\left\{\frac{1}{E_\beta - E_\alpha + i0} + \frac{1}{E_\alpha - E_\beta + i0}\right\}\langle\beta_0|T\left(\frac{E_\beta+E_\alpha}{2}+i0\right)|\alpha_0\rangle\,,$$

which, upon using the Sochocki formula (C.2), leads to

$$\langle\beta_0|S_0|\alpha_0\rangle = \delta_{\beta\alpha} - 2\pi i\,\delta(E_\beta - E_\alpha)\,\langle\beta_0|T(E_\alpha+i0)|\alpha_0\rangle\,. \tag{7.37}$$

This shows that the matrix element $t_{\beta\alpha}$ of the T_0 operator introduced in (7.25) is given by the special limit $z \to E_\alpha + i0$ of the general matrix element of the $T(z)$ operator. Combining this with the truncated to its first term iterative solution (7.36) of the Lippmann-Schwinger equation (7.35) for $T(z)$, one immediately obtains the formula known as the *Born approximation*

$$t_{\beta\alpha}(E_\alpha) \approx \langle\beta_0|V_{\rm int}|\alpha_0\rangle\,. \tag{7.38}$$

If it is known how to compute the action of $V_{\rm int}$ on free particle states (which is precisely the case, when $V_{\rm int}$ is expressed in terms of the creation and annihilation operators of free particles) this formula provides the simplest working approximation to amplitudes of particle reactions.[24]

7.3 *In* and *out* state-vectors

One can now define the *in* and *out* generalized state-vectors $|\alpha_\pm\rangle$ by the formula

$$|\alpha_\pm\rangle \equiv \Omega_\pm|\alpha_0\rangle\,. \tag{7.39}$$

On particular, (7.39) defines the two vacuum states[25] $|\Omega_\pm\rangle = \Omega_\pm|\Omega_0\rangle$. Owing to the intertwining relations (7.21), $|\alpha_\pm\rangle$ turn out to be just the generalized (non-normalizable) eigenvectors of the Hamiltonian $H = H_0 + V_{\rm int}$:

$$H|\alpha_\pm\rangle = E_\alpha|\alpha_\pm\rangle\,, \tag{7.40}$$

[24]However, frequently in relativistic theories of interacting particles $t_{\beta\alpha}(E_\alpha) = 0$ in this approximation. In Quantum Field Theory the name "Born approximation" is sometimes also used to denote what otherwise is called the tree-level approximation (see Chapter 9); it coincides with (7.38) only for very special interactions $V_{\rm int}$.

[25]Do not confuse the $|\Omega_\pm\rangle$ vacua with the Møller operators $\Omega_\mp$. The vacua $|\Omega_+\rangle$ and $|\Omega_-\rangle$ of closed systems, i.e. systems, the Hamiltonians H of which do not depend on time, differ only by a phase factor.

with the eigenvalue E_α equal to the energy (w.r.t. H_0) of the corresponding $|\alpha_0\rangle$ states. With the assumption that the spectra of H and H_0 are identical, the generalized vectors $|\alpha_0\rangle$, $|\alpha_+\rangle$ and $|\alpha_-\rangle$, related to each other in the same way as are the vectors $|\Psi\rangle$, $|\Psi_+\rangle$ and $|\Psi_-\rangle$ in (7.17), form three equivalent bases of the theory Hilbert space $\mathcal{H}$ (or, rather, of its dual $\mathcal{H}^*$). From this point of view the S-matrix elements

$$S_{\beta\alpha} = \langle\beta_-|\alpha_+\rangle = \langle\beta_0|S_0|\alpha_0\rangle\,, \tag{7.41}$$

form a collection of numbers, such that

$$|\alpha_+\rangle = \int d\beta\,|\beta_-\rangle\,S_{\beta\alpha}\,, \qquad \langle\beta_-| = \int d\alpha\,\langle\alpha_+|\,S_{\beta\alpha}\,. \tag{7.42}$$

As a matrix connecting two complete sets of orthonormal states (it is just the matrix of the change of bases) $S_{\beta\alpha}$ must be unitary:

$$\int d\beta\,S^*_{\beta\gamma}S_{\beta\alpha} = \int d\beta\,\langle\gamma_+|\beta_-\rangle\langle\beta_-|\alpha_+\rangle = \langle\gamma_+|\alpha_+\rangle = \delta_{\gamma\alpha}\,. \tag{7.43}$$

This reflects also the unitarity of the S_0 operator: $S_0^{-1} = S_0^\dagger$. The state-vectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are, in turn, connected by the S operator defined in (7.20):

$$S|\alpha_-\rangle = |\alpha_+\rangle\,, \qquad \text{or} \qquad \langle\beta_+|S = \langle\beta_-|\,, \tag{7.44}$$

so that, in analogy to (7.19),

$$S_{\beta\alpha} = \langle\beta_+|S|\alpha_+\rangle = \langle\beta_-|S|\alpha_-\rangle\,. \tag{7.45}$$

From the practical point of view (7.39) establishes a strict *one-to-one correspondence* between the *in* and *out* eigenvectors of H and the eigenvectors of H_0 on which the formulation of the perturbative calculation of the S-matrix elements will be based. (This strict correspondence will be relaxed only in Chapter 13 where a more flexible, nonperturbative in its essence, way of accessing S-matrix elements will be formulated).

Since any normalizable state $|\Psi\rangle$ can be decomposed into the generalized H_0 eigenvectors $|\alpha_0\rangle$: $|\Psi\rangle = \int d\alpha\,|\alpha_0\rangle\,\psi(\alpha)$, from the relation $|\Psi_\pm\rangle = \Omega_\pm|\Psi\rangle$ one gets

$$|\Psi_\pm\rangle = \Omega_\pm\int d\alpha\,|\alpha_0\rangle\,\psi(\alpha) = \int d\alpha\,|\alpha_\pm\rangle\,\psi(\alpha)\,, \tag{7.46}$$

That is, normalizable state-vectors $|\Psi_\pm\rangle$ decompose onto the generalized H eigenvectors $|\alpha_\pm\rangle$ with the same profiles $\psi(\alpha)$ as do their $\Omega_\pm^\dagger$ images onto the generalized H_0 eigenvectors $|\alpha_0\rangle$. Moreover, from the fact that the $\Omega_\pm$ operators preserve the scalar product of normalizable states (cf. (7.13)) it follows that

$$\langle\beta_\pm|\alpha_\pm\rangle = \langle\beta_0|\alpha_0\rangle = \delta_{\beta\alpha}\,. \tag{7.47}$$

Since the *in* and *out* state-vectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are in the one-to-one correspondence with the free particle vectors $|\alpha_0\rangle$, in addition to the operators $a(\mathbf{k},\sigma)$, $a^\dagger(\mathbf{k},\sigma)$ (which build the states $|\alpha_0\rangle$ out of $|\Omega_0\rangle$) one can define also the *in* and *out* creation and annihilation operators $a_{\rm in}(\mathbf{k},\sigma)$, $a^\dagger_{\rm in}(\mathbf{k},\sigma)$ and $a_{\rm out}(\mathbf{k},\sigma)$, $a^\dagger_{\rm out}(\mathbf{k},\sigma)$ which acting on the corresponding vacua $|\Omega_\pm\rangle = \Omega_\pm|\Omega_0\rangle$ build the *in* and *out* states. These operators satisfy the same commutation relations as do the original operators $a(\mathbf{k},\sigma)$, $a^\dagger(\mathbf{k},\sigma)$ and have the same transformation properties (in the relativistic case with respect to the full Poincaré symmetry group generated by $H = H_0 + V_{\rm int}$, $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K} = \mathbf{K}_0 + \mathbf{W}$ - see Section 7.5) as do the operators creating and annihilating the free-particle states $|\alpha_0\rangle$ (with respect to the Poincaré symmetry group generated by H_0, $\mathbf{P}_0$, $\mathbf{J}_0$ and $\mathbf{K}_0$). From (7.44) it then follows (cf. (1.60)) that

$$S^\dagger a^\dagger_{\rm in}(\mathbf{k},\sigma)S = a^\dagger_{\rm out}(\mathbf{k},\sigma)\,, \qquad S^\dagger a_{\rm in}(\mathbf{k},\sigma)S = a_{\rm out}(\mathbf{k},\sigma)\,. \tag{7.48}$$

Finally, it should be stressed that by themselves the vectors $U(t,0)|\alpha_\pm\rangle = e^{-iE_\alpha t}|\alpha_\pm\rangle$ do not converge to $U_0(t,0)|\alpha_0\rangle = e^{-iE_\alpha t}|\alpha_0\rangle$ in the limits $t \to \mp\infty$. The convergence holds only for normalizable states built as smooth superpositions of such states. Nevertheless, (7.39) stay true in the literal sense because the operators $\Omega_\pm$ are well defined on the whole (proper) Hilbert space.

The operator identities established above allow to derive useful representations of the *in* and *out* states $|\alpha_\pm\rangle$ either in terms of the rezolvent $G(z)$ or in terms of the $T(z)$ operator. To this end we consider first the action of $\Omega_\pm$ on a normalizable state-vector $|\Psi\rangle$. One gets then the scattering states $|\Psi_\pm\rangle$ which, using the formula (7.12) can be written as

$$|\Psi_\pm\rangle = \Omega_\pm|\Psi\rangle = |\Psi\rangle + i\int_0^{\mp\infty} dt'\, e^{-\varepsilon|t'|}\, U^\dagger(t',0)V_{\rm int}U_0(t',0)|\Psi\rangle\,. \tag{7.49}$$

Again the factor $e^{-\varepsilon|t'|}$ (the limit $\varepsilon \to 0^+$ is understood) is not necessary for convergence when $|\Psi\rangle$ is a normalizable state, but is necessary when $|\Psi\rangle$ is decomposed into generalized H_0 eigenstates $|\alpha_0\rangle$:

$$\begin{aligned} |\Psi_\pm\rangle &= |\Psi\rangle + i\int d\alpha \int_0^{\mp\infty} dt\, e^{-i(E_\alpha - H \pm i\varepsilon)t}\, V_{\rm int}|\alpha_0\rangle\langle\alpha_0|\Psi\rangle \\ &= |\Psi\rangle + \int d\alpha\, G(E_\alpha \pm i0)\, V_{\rm int}|\alpha_0\rangle\langle\alpha_0|\Psi\rangle\,. \end{aligned} \tag{7.50}$$

To obtain the representations of the *in* and *out* states $|\alpha_\pm\rangle$ one rewrites (7.50), decomposing $|\Psi\rangle$ onto the $|\alpha_0\rangle$ states, in the form

$$|\Psi_\pm\rangle = \int d\alpha \left(|\alpha_0\rangle + G(E_\alpha \pm i0)\, V_{\rm int}|\alpha_0\rangle\right)\psi(\alpha)\,.$$

Comparing this with (7.46) one gets the representation

$$|\alpha_\pm\rangle = |\alpha_0\rangle + G(E_\alpha \pm i0)\, V_{\rm int}|\alpha_0\rangle\,. \tag{7.51}$$

Yet another representation can be obtained using the identity

$$T(E_\alpha \pm i0)|\alpha_0\rangle = V_{\rm int}[\hat{1} + G(E_\alpha \pm i0)V_{\rm int}]|\alpha_0\rangle = V_{\rm int}|\alpha_\pm\rangle\,, \tag{7.52}$$

which follows from the definition (7.32) of the $T(z)$ operator and (7.51). This relation, combined with the result (7.37), immediately allows to write the element $t_{\beta\alpha}(E_\alpha)$ in the S_0 matrix element (7.25) as[26]

$$t_{\beta\alpha}(E_\alpha) = \langle\beta_0|T(E_\alpha + i0)|\alpha_0\rangle = \langle\beta_0|V_{\rm int}|\alpha_+\rangle\,. \tag{7.53}$$

The identity (7.52) applied to (7.51) after trading in this formula the product $G(E_\alpha \pm i0)\, V_{\rm int}$ for $G_0(E_\alpha \pm i0)\, T(E_\alpha \pm i0)$ in agreement with (7.33), leads to the Lippmann-Schwinger equation for $|\alpha_\pm\rangle$:

$$|\alpha_\pm\rangle = |\alpha_0\rangle + G_0(E_\alpha \pm i0)\, V_{\rm int}|\alpha_\pm\rangle \equiv |\alpha_0\rangle + \frac{1}{E_\alpha - H_0 \pm i0}\, V_{\rm int}|\alpha_\pm\rangle\,, \tag{7.54}$$

or[27]

$$|\alpha_\pm\rangle = |\alpha_0\rangle + \int d\beta\, |\beta_0\rangle\, \frac{\langle\beta_0|V_{\rm int}|\alpha_\pm\rangle}{E_\alpha - E_\beta \pm i0}\,. \tag{7.55}$$

Notice that the formula (7.54) agrees with the identification of the $|\alpha_\pm\rangle$ vectors as the eigenvectors of H, if the relation (7.40) is rewritten in the form

$$(E_\alpha - H_0)|\alpha_\pm\rangle = V_{\rm int}|\alpha_\pm\rangle\,.$$

The $\pm i0$ prescription specifies the way of inverting the operator $(E_\alpha - H_0)$ which has $|\alpha_0\rangle$ as its zero eigenvector.[28] Iterating the Lippmann-Schwinger equation (7.54) e.g. for $|\alpha_+\rangle$ gives the series

$$\begin{aligned} |\alpha_+\rangle = |\alpha_0\rangle &+ G_0(E_\alpha + i0)\, V_{\rm int}|\alpha_0\rangle \\ &+ G_0(E_\alpha + i0)\, V_{\rm int} G_0(E_\alpha + i0)\, V_{\rm int}|\alpha_0\rangle + \dots\,, \end{aligned} \tag{7.56}$$

When closed from the left with $\langle\beta_0|V_{\rm int}$, it reproduces the Born series for $t_{\beta\alpha}(E_\alpha) = \langle\beta_0|T(E_\alpha + i0)|\beta_0\rangle$ which can be obtained from (7.36); the latter series, truncated to the first term, gives the Born approximation (7.38).

[26] Similar representation of $t_{\beta\alpha}(E_\alpha)$ in terms of the *out* state is obtained by taking the Hermitian conjugation of $T(E_\alpha - i0)|\beta_0\rangle = V_{\rm int}|\beta_-\rangle$ and using the property $T^\dagger(z) = T(z^*)$. This leads to $t_{\beta\alpha}(E_\alpha) = \langle\beta_-|V_{\rm int}|\alpha_0\rangle$.

[27] Using (7.53) the formula for $|\alpha_+\rangle$ can be equivalently written as

$$|\alpha_\pm\rangle = |\alpha_0\rangle + \int d\beta\, |\beta_0\rangle\, \frac{t_{\beta\alpha}(E_\alpha)}{E_\alpha - E_\beta \pm i0}\,.$$

[28] Weinberg in his book derives the formula (7.55) directly from this equality. His derivation (quicker than the one given here) suffers, however, from the not fully convincing application of the residue method to the integral over E_β implicit in (7.55): it does not extend to the whole real axis as requires this method, but is restricted to $E_\beta > M_{\rm min} \geq 0$ (energy of the states $|\alpha_0\rangle$ representing particles is never negative).

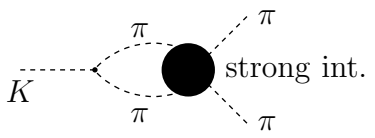

Figure 7.1: Strong interaction induced rescattering of pions produced in the decay of Kaon.

Another useful approximation to $t_{\beta\alpha}(E_\alpha)$ can be obtained if the interaction $V_{\rm int}$ consists of two parts: $V_{\rm int} = V_{\rm strong} + V_{\rm weak}$ of which one is "strong" and the other one "weak". One is then interested in accounting for the strong interactions exactly, while the effects of the weak ones can be treated in the simplest approximation. To this end, in addition to the *in* and *out* eigenstates $|\alpha_\pm\rangle$ of the full Hamiltonian $H = H_0 + V_{\rm strong} + V_{\rm weak}$ one defines also the *in* and *out* states with respect to the strong interaction

$$|\beta_\pm^{\rm strong}\rangle = |\beta_0\rangle + \frac{1}{E_\beta - H_0 \pm i0} V_{\rm strong} |\beta_\pm^{\rm strong}\rangle \,, \tag{7.57}$$

so that

$$\langle\beta_0| = \langle\beta_\pm^{\rm strong}| - \langle\beta_\pm^{\rm strong}| V_{\rm strong} \frac{1}{E_\beta - H_0 \mp i0} \,. \tag{7.58}$$

The full matrix $t_{\beta\alpha}(E_\alpha)$ (7.53) can be then written in the form

$$\begin{aligned} t_{\beta\alpha} &= \left[\langle\beta_-^{\rm strong}| - \langle\beta_-^{\rm strong}| V_{\rm strong} \frac{1}{E_\beta - H_0 + i0} \right] (V_{\rm strong} + V_{\rm weak}) |\alpha_+\rangle \\ &= \langle\beta_-^{\rm strong}| V_{\rm weak} |\alpha_+\rangle + \langle\beta_-^{\rm strong}| V_{\rm strong} |\alpha_0\rangle \,, \end{aligned} \tag{7.59}$$

where the formula (7.54) with $V_{\rm int}$ replaced by $V_{\rm strong} + V_{\rm weak}$ has been used (in the denominator E_β can be replaced by E_α because we need $t_{\beta\alpha}$ for $E_\beta = E_\alpha$) to replace the product $[E_\beta - H_0 + i0]^{-1}(V_{\rm strong} + V_{\rm weak})|\alpha_+\rangle$ by $|\alpha_+\rangle - |\alpha_0\rangle$. This (exact) formula is most useful if the strong interaction cannot induce the $\alpha \to \beta$ transition. The second term, which is just $t_{\beta\alpha}$ in the absence of weak interactions (just set $V_{\rm weak}$ to zero in the formula given in the footnote related to the formula (7.53) to see it!), that is corresponds to all possible transitions $\alpha \to \beta$ induced by $V_{\rm strong}$ alone, is then zero and, moreover, since the effects of $V_{\rm weak}$ are small, one can approximate the full Hamiltonian *in* state $|\alpha_+\rangle$ in the first term by $|\alpha_+^{\rm strong}\rangle$. The resulting formula $t_{\beta\alpha} \approx \langle\beta_-^{\rm strong}| V_{\rm weak} |\alpha_+^{\rm strong}\rangle$ is used e.g. in nuclear physics to compute rates of nuclear weak beta decays ($|\alpha_+^{\rm strong}\rangle$ and $|\alpha_-^{\rm strong}\rangle$ are then the initial and final nucleon states). Furthermore, using the property (7.42) of the S-matrix, this formula can be rewritten as

$$t_{\beta\alpha} = \int d\gamma \, S_{\beta\gamma}^{\rm strong} \, \langle\gamma_+^{\rm strong}| V_{\rm weak} |\alpha_+^{\rm strong}\rangle \,. \tag{7.60}$$

In this form it is used to account for the strong interaction re-scattering effects (shown graphically in figure 7.1) in hadronic weak decays; such effects are crucial for the possibility of experimental detection of CP violation e.g. in the Kaon system.

The Born formula (7.38) is the first term of the entire perturbative series which is obtained either by sandwiching the series (7.36) between the states $\langle\beta_0|$ and $|\alpha_0\rangle$ and evaluating it for $z = E_\alpha + i0$, or by using the Lippmann-Schwinger formula (7.55) for $|\alpha_+\rangle$ in the exact expression (7.53) for $t_{\beta\alpha}$:

$$t_{\beta\alpha} \equiv \langle\beta_0|V_{\rm int}|\alpha_+\rangle = V_{\beta\alpha} + \int d\gamma\, \frac{V_{\beta\gamma}t_{\gamma\alpha}(E_\alpha)}{E_\alpha - E_\gamma + i0}\,, \tag{7.61}$$

where $V_{\beta\alpha} \equiv \langle\beta_0|V_{\rm int}|\alpha_0\rangle$. Iterating this equation yields the series:

$$\begin{aligned} t_{\beta\alpha} = V_{\beta\alpha} &+ \int d\gamma\, \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha - E_\gamma + i0} \\ &+ \int d\gamma \int d\gamma' \frac{V_{\beta\gamma}V_{\gamma\gamma'}V_{\gamma'\alpha}}{(E_\alpha - E_\gamma + i0)(E_\alpha - E_{\gamma'} + i0)} + \ldots\,, \end{aligned} \tag{7.62}$$

This is the so-called "old-fashioned" perturbative expansion. While in some situations it is convenient to investigate some specific issues, its main drawback in relativistic theories is the lack of manifest Lorentz covariance.

An alternative, more satisfactory in this respect, approach to the perturbative computation of S-matrix elements is developed by starting directly from the formulae (7.27) and (7.26) which allow to write the S_0 operator using the expression (1.23):

$$S_0 = U_I(+\infty, -\infty) = {\rm T}\exp\left(-i\int_{-\infty}^{+\infty} dt\, V_{\rm int}^I(t)\right), \tag{7.63}$$

where the interaction operator in the Dirac picture reads

$$V_{\rm int}^I(t) \equiv e^{iH_0 t}\, V_{\rm int}\, e^{-iH_0 t}\,. \tag{7.64}$$

The form (7.63) of the S_0 operator, sandwiched between concrete initial $|\alpha_0\rangle$ and final $|\beta_0\rangle$ states and expanded in the power series, allows to compute the S-matrix element $S_{\beta\alpha}$ with the help of the application to the successive terms of the series the Wick theorem discussed in Section 5.9. The resulting expansion is analogous to the Dyson expantion of Green's functions discussed in Section 5.8.

The equivalence of the formula (7.63) and the formulae (7.25) and (7.62) should be clear from the construction (at least at the formal level), but can also be seen directly by making use of the identity

$$\frac{1}{E_\alpha - E_\beta + i0} = \frac{1}{i}\int_0^{+\infty} d\tau\, e^{i(E_\alpha - E_\beta + i0)\tau}\,, \tag{7.65}$$

to represent the energy denominators in (7.62). For example, the first terms in the expansion of (7.63) give

$$S_{\beta\alpha} = \langle\beta_0|S_0|\alpha_0\rangle = \langle\beta_0|1 - i\int_{-\infty}^{+\infty} dt\, V_{\rm int}^I(t) + \ldots|\alpha_0\rangle$$

$$
\begin{aligned}
&= \delta_{\beta\alpha} - i \int_{-\infty}^{+\infty} dt\, e^{-i(E_\alpha - E_\beta)t}\, V_{\beta\alpha} + \ldots \\
&= \delta_{\beta\alpha} - 2\pi i\, \delta(E_\alpha - E_\beta)\, V_{\beta\alpha} + \ldots \\
&\quad + \frac{(-i)^2}{2!} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, \langle \beta_0 | \mathrm{T}\left(V^I_{\mathrm{int}}(t_1) V^I_{\mathrm{int}}(t_2) \right) | \alpha_0 \rangle + \ldots
\end{aligned}
\tag{7.66}
$$

and so on. The advantage of the time-dependent perturbative expansion based on the formula (7.63) lies in the fact that in relativistic theories it allows to keep Lorentz invariance manifest at each stage of the calculations.

7.4 Scattering of nonrelativistic particles

Before discussing general properties any relativistic S-matrix should possess (as an indication how to construct particle interactions V_{int} leading to relativistic theories), it is instructive to see how the developed formalism applies to the simpler nonrelativistic case. Its application to the scattering of a single spinless particle on a fixed potential - the simplest possible case which can be treated within the framework of the ordinary nonrelativistic quantum mechanics - is discussed in Appendix E. Here we apply it to the problem of the elastic scattering of two nonrelativistic particles (which can be different or identical, be fermions or bosons and can have arbitrary spins). We first recall the usual treatment of this problem in the framework of the two-body Schrödinger equation and then reformulate it using the formalism of second quantization of Chapter 5. The purpose of this is twofold: firstly we want to show how the Dyson expansion (Section 5.8) and the Wick theorem (Section 5.9) provide an efficient and flexible mean to compute scattering amplitudes; we also want to establish the relation between elements $t_{\beta\alpha}(E_\alpha)$ (7.24) of the T_0 operator (7.25) to the ordinary scattering amplitude $f(k,\theta)$ known from the approach based on the Schrödinger equation. Secondly, we want to derive the result (5.102) used in the discussion of the ground state of a system of interacting bosons. This will also give the opportunity to go beyond the first nontrivial order of the Dyson expansion and to have the first encounter with the problem of divergences and their treatment (within the relativistic theory this problem will be discussed in Chapter 14).

In the ordinary formulation of nonrelativistic Quantum Mechanics the amplitude of the elastic scattering of two distinguishable particles of masses m_1 and m_2 (and arbitrary spins) interacting with one another through a potential $V_{\mathrm{pot}}(\mathbf{r}_1 - \mathbf{r}_2)$ which does not depend on their spins can be obtained by solving the two-body stationary Schrödinger equation

$$
\left(-\frac{\hbar^2}{2m_1} \boldsymbol{\nabla}^2_{\mathbf{r}_1} - \frac{\hbar^2}{2m_2} \boldsymbol{\nabla}^2_{\mathbf{r}_2} + V_{\mathrm{pot}}(\mathbf{r}_1 - \mathbf{r}_2) \right) \Psi(\mathbf{r}_1, \mathbf{r}_2) = E\, \Psi(\mathbf{r}_1, \mathbf{r}_2)\,. \tag{7.67}
$$

In the variables $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ it takes the form

$$
\left(-\frac{\hbar^2}{2M} \boldsymbol{\nabla}^2_{\mathbf{R}} - \frac{\hbar^2}{2m_{\mathrm{red}}} \boldsymbol{\nabla}^2_{\mathbf{r}} + V_{\mathrm{pot}}(\mathbf{r}) \right) \Psi(\mathbf{R}, \mathbf{r}) = E\, \Psi(\mathbf{R}, \mathbf{r})\,,
$$

in which $M = m_1 + m_2$ and $m_{\rm red} = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the two-particle system. Writing then $\Psi(\mathbf{R}, \mathbf{r}) = \psi(\mathbf{r}) \exp(i\mathbf{P}\cdot\mathbf{R}/\hbar)$ reduces the problem to the one of scattering in an external potential $V_{\rm pot}(\mathbf{r})$ of a single fictitious particle of mass $m_{\rm red}$:

$$\left(-\frac{\hbar^2}{2m_{\rm red}} \boldsymbol{\nabla}_{\mathbf{r}}^2 + V_{\rm pot}(\mathbf{r}) \right) \psi(\mathbf{r}) = E' \psi(\mathbf{r}) \,. \tag{7.68}$$

Here $E' = E - \mathbf{P}^2/2M$ is the energy of the two scattering particles in their center of mass system (CMS). In this system, going over to which means just setting $\mathbf{P} = \mathbf{0}$, the momenta of the two particles are $\hbar\mathbf{k}$ and $-\hbar\mathbf{k}$ and the energy $E' \equiv E$ ascribed to the fictitious particle is the total energy of the two colliding particles:

$$E' = \frac{1}{2} m_1 \mathbf{v}_1^2 + \frac{1}{2} m_2 \mathbf{v}_2^2 = \frac{\hbar^2}{2} \left(m_1 \frac{\mathbf{k}^2}{m_1^2} + m_2 \frac{\mathbf{k}^2}{m_2^2} \right) = \frac{\hbar^2 \mathbf{k}^2}{2m_{\rm red}} \,.$$

Therefore, the vector $\mathbf{k}$ playing the role of the wave vector of the fictitious particle of mass $m_{\rm red}$ must be identified with the wave vector of one of the scattering particles. The scattering amplitude $f(k, \theta)$, in which $k \equiv |\mathbf{k}|$, is defined in terms of the asymptotic, for $r \to \infty$, form ($|\mathbf{k}'| = |\mathbf{k}|$, $\mathbf{k}'\cdot\mathbf{k} = \mathbf{k}^2 \cos\theta$)

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) \approx e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{f(\mathbf{k}', \mathbf{k})}{r} \, e^{ikr} \equiv e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{f(k, \theta)}{r} \, e^{ikr} \,, \tag{7.69}$$

of the solution of the Schrödinger equation (7.68) (with $E' = \hbar^2\mathbf{k}^2/2m_{\rm red}$). Since the interaction is independent of spin, spins of the scattering paricles play no role if these are distinguishable (the spin projection of each of the particles remains unchanged) and the differential scattering cross section $d\sigma/d\Omega$ is in such a case is simply given by $|f(k, \theta)|^2$.

If the scattering particles are indistinguishable and both have spin s (integer or half-integer) but the interaction potential is still spin-independent, the complete wave function of the system can be written as a product $\Psi(\mathbf{r}_1, \mathbf{r}_2)\chi(\sigma_1, \sigma_2)$ in which $\sigma_{1,2} = -s, \ldots, +s$, of the spin part and of the space part. According to the principles of Quantum Mechanics, the complete wave function of indistinguishable particles moving in the three-dimensional space must be either totally symmetric, $\Psi(\mathbf{r}_1, \mathbf{r}_2)\chi(\sigma_1, \sigma_2) = +\Psi(\mathbf{r}_2, \mathbf{r}_1)\chi(\sigma_2, \sigma_1)$, or totally antisymmetric, $\Psi(\mathbf{r}_1, \mathbf{r}_2)\chi(\sigma_1, \sigma_2) = -\Psi(\mathbf{r}_2, \mathbf{r}_1)\chi(\sigma_2, \sigma_1)$, depending on whether s is integer or half-integer (this is the celebrated spin-statistics connection underlying the formalism of second quantization; it will be given a justification in Chapter 8). The total spin S of the two-particle system and its projection S_z onto the z-axis are in this situation preserved separately (from the orbital angular momentum) and the spin part $\chi(\sigma_1, \sigma_2)$ of the wave function can be chosen to be symmetric or antisymmetric. The function $\psi(\mathbf{r})$ in the decomposition $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}) \exp(i\mathbf{P}\cdot\mathbf{R}/\hbar)$ must then be[29] either even, $\psi(-\mathbf{r}) = \psi(\mathbf{r})$, or odd, $\psi(-\mathbf{r}) = -\psi(\mathbf{r})$, in order that $\Psi(\mathbf{r}_1, \mathbf{r}_2)\chi(\sigma_1, \sigma_2)$ has the appropriate symmetry corresponding to the spin s of the indistinguishable particles. The general rule (see e.g. the Landau & Lifschitz textbook Vol. III, par. 137) is that if the

[29] The factor $\exp(i\mathbf{P}\cdot\mathbf{R}/\hbar)$, in which in this case $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$, is obviously symmetric.

total spin S of the system of two particles is an even number ($S = 0, 2, \ldots$), $\psi(\mathbf{r})$ must be even too and the asymptotic solution of the Schrödinger equation must be taken in the form

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + e^{-i\mathbf{k}\cdot\mathbf{r}} + [f(k,\theta) + f(k,\pi-\theta)]\,\frac{e^{ikr}}{r}\,,$$

whereas when the total spin S is an odd number ($S = 1, 3, \ldots$), $\psi(\mathbf{r})$ must also be odd and the asymptotic solution of the Schrödinger equation should be constructed as

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - e^{-i\mathbf{k}\cdot\mathbf{r}} + [f(k,\theta) - f(k,\pi-\theta)]\,\frac{e^{ikr}}{r}\,.$$

Indeed, if the particles are spinless, $s = 0$, then necessarily $S = 0$ (so it is even) and $\psi(\mathbf{r})$ must be even; similarly, if $s = \frac{1}{2}$ but $S = 0$ (again S is even), which means that $\chi(\sigma_1, \sigma_2) = -\chi(\sigma_2, \sigma_1)$, $\psi(\mathbf{r})$ must again be even, etc. Thus the proper scattering amplitude, the modulus squared of which gives the differential cross section, is

$$f(k,\theta) + f(k,\pi-\theta)\,,$$

when $S = 0, 2, \ldots$ and

$$f(k,\theta) - f(k,\pi-\theta)\,,$$

when $S = 1, 2, \ldots$ The differential scattering cross section is, therefore, given either by[30] $|f(k,\theta) + f(k,\pi-\theta)|^2$ or by $|f(k,\theta) - f(k,\pi-\theta)|^2$.

To relate the ordinary scattering amplitude $f(|\mathbf{k}|,\theta)$ to the matrix element $t_{\beta\alpha}$ of the operator T_0 defined in (7.24) and to the amplitude $\mathcal{A}$ (the formalism based on the Dyson expansion naturally, as will be seen, yields $\mathcal{A}$), which is obtained from the element $t_{\beta\alpha}$ defined by (7.25) after factoring out from it (as is always possible in translationally invariant theories - see Section 7.5 for a general argument) $(2\pi)^3\delta^{(3)}((\mathbf{P}_\beta - \mathbf{P}_\alpha)/\hbar)$ we consider the two-body spin-independent interaction (written in the formalism of second quantization)

$$V_{\rm int} = \frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{y}\,\psi^\dagger_\alpha(\mathbf{x})\psi^\dagger_\beta(\mathbf{y})\,V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)\,\psi_\beta(\mathbf{y})\psi_\alpha(\mathbf{x})\,, \tag{7.70}$$

with a general two-body translationally and rotationally invariant interaction potential $V_{\rm pot}(|\mathbf{r}|)$ and the field operators constructed as in (5.46). The first nontrivial term of the expansion of the formula (7.63) with the two-particle initial and final states $|\alpha_0\rangle$ and $|\beta_0\rangle$

$$\begin{aligned} |\alpha_0\rangle &= a^\dagger_{\sigma_2}(\mathbf{k}_2)\,a^\dagger_{\sigma_1}(\mathbf{k}_1)|\text{void}\rangle \equiv a^\dagger_2 a^\dagger_1|\text{void}\rangle\,,\\ |\beta_0\rangle &= a^\dagger_{\sigma'_2}(\mathbf{k}'_2)\,a^\dagger_{\sigma'_1}(\mathbf{k}'_1)|\text{void}\rangle \equiv a^\dagger_{2'} a^\dagger_{1'}|\text{void}\rangle\,, \end{aligned} \tag{7.71}$$

[30]The total cross section is then given by half of the integral $\int d\Omega\,(d\sigma/d\Omega)$.

then reads[31]

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \frac{i}{2\hbar}\int dt \int d^3\mathbf{x} \int d^3\mathbf{y}\, V_{\rm pot}(\mathbf{x}-\mathbf{y}) \langle a_{1'} a_{2'} \mathrm{T}[\psi^{\dagger I}_{\sigma}(t,\mathbf{x}) \psi^{\dagger I}_{\bar\sigma}(t,\mathbf{y}) \psi^{I}_{\bar\sigma}(t,\mathbf{y}) \psi^{I}_{\sigma}(t,\mathbf{x})] a^\dagger_2 a^\dagger_1 \rangle,$$

where $\langle \ldots \rangle$ stands for the expectation value of $\ldots$ in the state $|\mathrm{void}\rangle$. As the chronological product is irrelevant here (all operators under it are taken at the same instant), the matrix element can be easily worked out (after moving all the c-number factors and integrals involved in the field operators outside the brackets) by just (anti)commuting the creation operators to the left and the annihilation operators to the right, so that they ultimately act on the $|\mathrm{void}\rangle$ vectors giving zeroes. These operations produce the Dirac delta functions depending on the wave vectors and the Kronecker deltas in the spin labels (The Wick theorem (5.184) can be also applied to arrive at the result). The integral over the time t can be readily performed giving (after accounting for the obtained delta functions depending on the wave vectors) the delta function expressing the conservation of the sum of the frequencies $\omega_{\mathbf{k}}$ (energies) of the particles taking part in the reaction. Furthermore, representing the potential $V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)$ as a Fourier transform as in (5.64) allows to explicitly perform also the integrals over the positions $\mathbf{x}$ and $\mathbf{y}$. This produces two additional three-dimensional Dirac delta functions depending on the wave vectors. Finally using the Dirac deltas all integrals over the wave vectors originating from the field operators and from the Fourier transform of $V_{\rm pot}(\mathbf{x}-\mathbf{y})$ can be eliminated and one obtains[32]

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \frac{i}{\hbar}\,(2\pi)^4 \delta(\omega_{\mathbf{k}'_1} + \omega_{\mathbf{k}'_2} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, \delta^{(3)}(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1 - \mathbf{k}_2)\, \mathcal{A}_{\beta\alpha}\,, \qquad (7.72)$$

with

$$\mathcal{A}_{\beta\alpha} = \delta_{\sigma_1\sigma'_1}\delta_{\sigma_2\sigma'_2}\, \tilde V_{\rm pot}(|\mathbf{k}'_1 - \mathbf{k}_1|) \pm \delta_{\sigma_1\sigma'_2}\delta_{\sigma_2\sigma'_2}\, \tilde V_{\rm pot}(|\mathbf{k}'_2 - \mathbf{k}_1|)\,, \qquad (7.73)$$

where the + sign applies to bosons and the − sign to fermions.[33] The two terms arise as follows: moving the two annihilation operators arising from $\psi^I_{\bar\sigma}\psi^I_\sigma$ to the right, past the operators $a^\dagger_2 a^\dagger_1$ one obtains the sum or the difference of two terms; similarly, moving

[31] $V^I_{\rm int}(t)$ defined in (7.64) is obtained by simply inserting in (7.70) the field operators taken in the interaction picture (instead of the Schrödinger picture ones) which amounts to replacing in the formulae (5.46) $\pm i\mathbf{k}\cdot\mathbf{x}$ by $\mp i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})$ where $\omega_{\mathbf{k}} = \hbar\mathbf{k}^2/2m$ (we now display explicitly all the $\hbar$ factors).

[32] The fact that the four-dimensional delta function can be factorized from the amplitude $t_{\beta\alpha}$ follows also from the translational invariance of the considered theory - see Section 7.5.

It is instructive to check the dimensions: since the wave vectors have dimension $[L]^{-1}$,

$$\delta_{\alpha\beta} = (2\pi)^6[\delta_{\sigma'_1\sigma_1}\delta_{\sigma'_2\sigma_2}\delta^{(3)}(\mathbf{k}'_1 - \mathbf{k}_1)\delta^{(3)}(\mathbf{k}'_2 - \mathbf{k}_2) \pm \delta_{\sigma'_1\sigma_2}\delta_{\sigma'_2\sigma_1}\delta^{(3)}(\mathbf{k}'_1 - \mathbf{k}_2)\delta^{(3)}(\mathbf{k}'_2 - \mathbf{k}_1)]\,,$$

has dimension $[L]^6$ and the second term has dimension (the dimension of ω is $[T]^{-1}$, that of $\tilde V_{\rm pot}$ is $[L]^3\times$energy) $\hbar^{-1}[T][L]^3[L]^3\times$energy $= (\hbar c)^{-1}c[T][L]^6\times$energy which is also $[L]^6$ because the dimension of $\hbar c$ is energy$\times[L]$.

[33] If the potential is of the Yukawa form $V_{\rm pot}(\mathbf{x}) = \frac{g^2}{|\mathbf{x}|}\, e^{-M_\phi c|\mathbf{x}|/\hbar}$ with the coupling constant g (g^2 has the physical dimension of energy times length and M_ϕ has the mass dimension - such a potential arises

the two creation operators arising from $\psi^{\dagger I}_\sigma \psi^{\dagger I}_{\bar\sigma}$ to the left past $a_{1'} a_{2'}$ one obtains the sum or the difference of another two terms. Multiplying these one gets four terms which are, however, pairwise equal. This doubling removes the factor of $1/2$ present in the interaction (7.70). Using next the CMS kinematics (in the four-vector notation):

$$\begin{aligned} k_1^\mu &= (\omega_{\mathbf{k}},\, 0,\, 0,\, |\mathbf{k}|)\,, \\ k_2^\mu &= (\omega_{\mathbf{k}},\, 0,\, 0,\, -|\mathbf{k}|)\,, \\ (k_1')^\mu &= (\omega_{\mathbf{k}},\, 0,\, |\mathbf{k}|s_\theta,\, |\mathbf{k}|c_\theta)\,, \\ (k_2')^\mu &= (\omega_{\mathbf{k}},\, 0,\, -|\mathbf{k}|s_\theta,\, -|\mathbf{k}|c_\theta)\,, \end{aligned}$$

one gets $|\mathbf{k}_1 - \mathbf{k}_1'| = 2|\mathbf{k}|\sin(\theta/2)$, $|\mathbf{k}_1 - \mathbf{k}_2'| = 2|\mathbf{k}|\cos(\theta/2) = 2|\mathbf{k}|\sin((\pi-\theta)/2)$. The amplitude can be thus written in the form

$$\mathcal{A}_{\beta\alpha} = \delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,\tilde V_{\rm pot}(|\mathbf{k}|\sin(\theta/2)) \pm \delta_{\sigma_1'\sigma_2}\delta_{\sigma_2'\sigma_1}\tilde V_{\rm pot}(|\mathbf{k}|\sin((\pi-\theta)/2)\,.$$

The two terms of $\mathcal{A}_{\beta\alpha}$ must be therefore proportional to the first terms in the expansions of the nonrelativistic amplitudes $f(k,\theta)$ and $f(k,\pi-\theta)$.

One can now check the quoted rule of Landau & Lifschitz for $f(k,\theta) \pm f(k,\pi-\theta)$. If the (identical) scattering particles are spinless bosons, the amplitude $\mathcal{A}_{\beta\alpha}$ (7.73) comes with the plus sign between its two terms and the rule is obviously satisfied (the total spin $S=0$). Consider now the scattering of spin $s=1/2$ fermions. How it happens that when the total spin S is even (that is $S=0$), the two terms of the amplitude depending on θ and $\pi-\theta$ respectively combine with the plus sign (despite the minus sign between the two terms in (7.73))? Denote $S_\theta \equiv \tilde V_{\rm pot}(|\mathbf{k}|\sin(\theta/2))$ and $C_\theta \equiv \tilde V_{\rm pot}(|\mathbf{k}|\cos(\theta/2))$ and consider the scattering amplitudes (7.73) with different spin configurations. One finds:

$$\mathcal{A}(\uparrow\uparrow \;\longrightarrow\; \uparrow\uparrow) = \mathcal{A}(\downarrow\downarrow \;\longrightarrow\; \downarrow\downarrow) = S_\theta - C_\theta\,.$$

In these two cases both terms contribute because all the spin dependent Kronecker deltas in (7.73) are nonzero. In contrast,

$$\begin{aligned} \mathcal{A}(\uparrow\downarrow \;\longrightarrow\; \uparrow\downarrow) &= \mathcal{A}(\downarrow\uparrow \;\longrightarrow\; \downarrow\uparrow) = S_\theta\,, \\ \mathcal{A}(\uparrow\downarrow \;\longrightarrow\; \downarrow\uparrow) &= \mathcal{A}(\downarrow\uparrow \;\longrightarrow\; \uparrow\downarrow) = -C_\theta\,, \end{aligned}$$

because when the initial (and therefore also final) spins are opposite, only one of the two terms contributes. Now, if the two initial fermions are in the $S=0$ total spin state the

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as a low energy limit of a relativistic interaction mediated by a boson of mass M_ϕ) then

$$\mathcal{A}_{\beta\alpha} = 4\pi\, g^2 \left[\frac{\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}}{(\mathbf{k}_1' - \mathbf{k}_1)^2 + M_\phi^2 c^2/\hbar^2} \pm \frac{\delta_{\sigma_1'\sigma_2}\delta_{\sigma_2'\sigma_1}}{(\mathbf{k}_2' - \mathbf{k}_1)^2 + M_\phi^2 c^2/\hbar^2} \right].$$

The amplitude of the elastic scattering of two different particles can be obtained in the same way, the only difference being that in this case the interaction (7.70) and the initial and final states (7.71) are built out of operators of two different particles (the interaction term in this case does not have the factor of $1/2$) and, in consequence, the resulting amplitude (7.73), similarly as $\delta_{\beta\alpha}$, has only one term.

final ones must be in the same spin state too, because spin is separately (independently of the orbital angular momentum) preserved and to obtain the corresponding scattering amplitude one must combine the four amplitudes as follows:

$$\mathcal{A}\left(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \longrightarrow \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}\right) = \frac{1}{2}\left[\mathcal{A}(\uparrow\downarrow\longrightarrow\uparrow\downarrow) - \mathcal{A}(\uparrow\downarrow\longrightarrow\downarrow\uparrow)\right.$$
$$\left. -\mathcal{A}(\downarrow\uparrow\longrightarrow\uparrow\downarrow) + \mathcal{A}(\downarrow\uparrow\longrightarrow\downarrow\uparrow)\right] = S_\theta + C_\theta\,.$$

And indeed the quoted rule is obeyed. Instead, if the two fermions were in the $S = 1$ total spin state with $S^z = 0$, the amplitude would be

$$\mathcal{A}\left(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}} \longrightarrow \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}\right) = \frac{1}{2}\left[\mathcal{A}(\uparrow\downarrow\longrightarrow\uparrow\downarrow) + \mathcal{A}(\uparrow\downarrow\longrightarrow\downarrow\uparrow)\right.$$
$$\left. +\mathcal{A}(\downarrow\uparrow\longrightarrow\uparrow\downarrow) + \mathcal{A}(\downarrow\uparrow\longrightarrow\downarrow\uparrow)\right] = S_\theta - C_\theta\,,$$

again in agreement with the general rule. In the similar manner one can check that the amplitudes corresponding to transitions between $S = 0$ and $S = 1$ states all vanish as well as the working of the Landau & Lifschitz rule in the scattering of say two identical spin 1 bosons.

Finally using the considered lowest order approximation to the complete amplitude, it is straightforward to establish the (generally valid) relation between the amplitude $\mathcal{A}$ in (7.72) and the ordinary scattering amplitude $f(k,\theta)$. Taking opposite directions of spins of the two scattering particles (i.e. $\sigma_1' = \sigma_1 = +s$ and $\sigma_2' = \sigma_2 = -s$ so that the scattering particles are effectively distinguishable - we assume here that $s > 0$; equivalently one can consider scattering of two different particles), one obtains

$$\mathcal{A} = \tilde{V}_{\rm pot}(|\mathbf{k}_1' - \mathbf{k}|)\,.$$

The corresponding scattering amplitude $f(k,\theta)$ (without the spin factors) defined by the asymptotic form (7.69) of the solution of the Schrödinger equation (7.68) can be computed using the Born approximation which gives

$$f_{\rm Born}(k,\theta) = -\frac{m_{\rm red}}{2\pi\hbar^2}\int d^3\mathbf{r}\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\, V_{\rm pot}(\mathbf{r}) = -\frac{m_{\rm red}}{2\pi\hbar^2}\,\tilde{V}_{\rm pot}(|\mathbf{k}_1' - \mathbf{k})\,.$$

Comparing the two amplitudes one finds that the rule is

$$f(k,\theta) = -\frac{m_{\rm red}}{2\pi\hbar^2}\,\mathcal{A}(k,\theta)\,. \tag{7.74}$$

(If the two scattering particles are identical and both have mass m, then $m_{\rm red} = m/2$.) It should be clear that higher orders of the Dyson expansion of the formula (7.63) will yield higher order terms of the Born expansion of the scattering amplitude. The great advantage of the approach based on the second quantization formalism is that symmetry requirements are automatically taken into account (they are encoded in the properties of

the field operators). One is also not bound to the center of mass frame - particles in the initial state (7.71) can have arbitrary momenta; if the scattering particles are different, the reduced mass $m_{\rm red}$ will come out automatically from the kinematics of the process. Moreover one can easily consider also spin dependent interactions (the arguments based on separate spin conservation are then invalid and the analysis in the language of the wave functions becomes more complicated).

One can now derive the relation (5.102) used in computing in Section 5.5 the energy of the ground state of N bosons. It has been argued that if properties of the interacting system are dominated by low energy excitations of its unperturbed ground state $|\Omega_0\rangle$, the exact spatially nonlocal interaction $V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)$ of particles can be replaced by the effective local interaction $g\,\delta^{(3)}(\mathbf{x}-\mathbf{y})$ the couplig g of which should be adjusted in such a way that the effective interaction reproduces the low energy form of the elastic CMS scattering amplitude $f(k,\theta)$ of two particles produced by the "fundamental" interaction $V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)$. In general, the elastic scattering amplitude $f(k,\theta)$ defined by (7.69) can be expressed through the (real) partial wave shifts $\delta_\ell(k)$ as

$$\begin{aligned} f(k,\theta) &= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_\ell(k)}\sin\delta_\ell(k)\,P_\ell(\cos\theta) \\ &= \sum_{\ell=0}^{\infty}(2\ell+1)\,\frac{1}{k\cot\delta_\ell - ik}\,P_\ell(\cos\theta)\,, \end{aligned}$$

where $P_\ell(x)$ are the Legendre polynomials. By solving the single spinless particle scattering problem with the interaction potential $V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)$, it is possible in principle to find the phase shifts $\delta_\ell(k)$. This would allow also to determine the scattering lengths a_ℓ, effective ranges r_ℓ, and other parametres which are defined as the coefficients in the small k expansions

$$k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2}\,r_0k^2 + \ldots, \qquad k\cot\delta_1 = -\frac{3}{k^2a_1^3} + \ldots, \tag{7.75}$$

of the phase shifts $\delta_l(k)$. These expansions translate into the following low momentum expansion

$$f(k,\theta) = -a_0\left[1 - ia_0k + \left(\frac{1}{2}\,a_0r_0 - a_0^2\right)k^2 + \ldots\right] - a_1^3k^2\cos\theta + \ldots, \tag{7.76}$$

of the elastic scattering amplitude. In the low energy limit $kR \ll 1$, where R is a length scale characterizing the interaction potential $V_{\rm pot}$ (one assumes that $a_\ell \sim R$ and $r_\ell \sim R$) the amplitude becomes insensitive to the precise form of the potential $V_{\rm pot}$ and in the limit $k \to 0$ depends only on its single "global" characteristic a_0; for slightly larger k it becomes sensitive also to a_1, r_0 and so on. It follows that instead of specifying explicitly the functional form of $V_{\rm pot}(|\mathbf{x}|)$ it is sufficient to represent it by the s-wave scattering length a_0 it gives rise to and, if a greater precision is required, also by a_1 and r_0. Therefore the

couplings like g in the low energy effective Hamiltonian of the system of particles should be adjusted so that the effective Hamiltonian gives rise to the same s-wave scattering length a_0. (In general, the interaction term of the effective Hailtonian consists of an infinite set of operators of decreasing lenght dimension - the lenght dimension of the term proportional to g is -6 - the coefficients of which can be adjusted to give the same a_1, $r_0, \ldots$ as the potential $V_{\rm pot}(|\mathbf{x}|)$.) As will be seen, computing, using the effective Hamiltonian with local interaction(s), the scattering amplitude in higher orders of the Dyson expansion one encounters divergences originating from integrations over wave vectors which can have arbitrarily large lengths. Because of this these divergences are called ultraviolet (UV). Since the effective interaction is not valid if wave vectors of the interacting particles exceed a certain scale Λ, it is natural, as in the computation done in Section 5.5, to impose Λ as an ultraviolet cutoff on divergent integrals. The coupling g reproducing the scattering length a_0 will then depend explicitly on this cutoff Λ. This dependence will precisely have the form (5.102) and will make the ground state energy of the system of N bosons a finite (in the limit $\Lambda \to \infty$) quantity. It is frequently argued that expressing g, and therefore also the energy density E_Ω/V of the gas of N interacting bosons, in terms of the scattering length a_0 is motivated by the fact that the properties of such a system, if it is sufficiently diluted, are determined primarily by (rare) binary collisions of particles having low energies. In fact, the procedure adopted is general - it consitutes the essence of the renormalization - and is not related to these particular, largely heuristic, arguments.

We thus take as the effective Hamiltonian of interacting particles (which can be bosons or fermions and can have arbitrary spin - the resulting dependence of g on the cutoff Λ will not depend on this) the infinite space counterpart of the expression (5.101) and reconsider the elastic scattering of two identical particles. The first order (in the coupling g) term of the S-matrix element is immediately obtained by replacing in (7.73) the Fourier transforms $\tilde{V}_{\rm pot}(\mathbf{k}_1' - \mathbf{k}_1)$ and $\tilde{V}_{\rm pot}(\mathbf{k}_2' - \mathbf{k}_1)$ by g. Thus, to this order $\mathcal{A}_{\beta\alpha} = g$ and the relation (7.74) together with the expansion (7.76) give $g = (4\pi\hbar^2/m)a_0$.

The second order term in the Dyson expansion of the S-matrix element is given by the expression

$$\frac{1}{2!}\left(\frac{g}{2i\hbar}\right)^2 \int d^4x_1 \int d^4x_2 \, \langle a_{1'} a_{2'} \mathrm{T}[(\psi^\dagger_{\alpha_1}\psi^\dagger_{\beta_1}\psi_{\beta_1}\psi_{\alpha_1})(x_1)\,(\psi^\dagger_{\alpha_2}\psi^\dagger_{\beta_2}\psi_{\beta_2}\psi_{\alpha_2})(x_2)]\, a_2^\dagger a_1^\dagger\rangle \,. \quad (7.77)$$

The chronological product of the two interactions $V^I_{\rm int}(x_1)$ and $V^I_{\rm int}(x_2)$ can be written using the Wick theorem as a sum of several terms consisting of normally ordered products of the field operators and the c-number contractions of the operators. Here, because the chronological product stands between the two-particle states (7.71), of the entire sum (5.184) relevant are only terms in which left uncontracted (but normally ordered) are two $\psi^\dagger$ and two ψ-operators which as in the first order term have to be next moved to the left (past the two operators $a_{1'}a_{2'}$) and to the right (past the two operators $a_2^\dagger a_1^\dagger$), respectively. The terms of (5.184) in which there are more uncontracted normally ordered operators will give zero and the terms with less uncontracted operators will vanish as will be explained shortly. Of the contractions (5.187) appearing in the Wick formula (5.184)

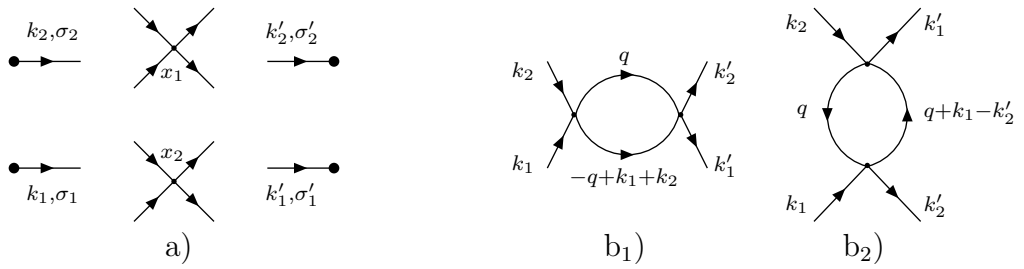

Figure 7.2: a) Graphical illustration of the matrix element in (7.77): the lines marked with the momentum and spin variables represent the initial and the final state while the two interaction terms are marked by their space-time variables $x_i \equiv (t_i, \mathbf{x}_i)$, $i = 1, 2$. b) Two frequency-momentum space order g^2 Feynman diagrams contributing to the amplitude $\mathcal{A}_{\beta\alpha}$ resulting from applying the Wick theorem.

nonzero are only those of $\psi_\alpha(x)$ and $\psi^\dagger_\beta(y)$. Applying the formula (5.186) one obtains

$$\begin{aligned}\overbrace{\psi_\alpha(x)\psi^\dagger_\beta}(y) &= \langle \text{void}|\mathrm{T}\psi_\alpha(x)\psi^\dagger_\beta(y)|\text{void}\rangle \\ &= \theta(x^0-y^0)\langle \text{void}|\psi_\alpha(x)\psi^\dagger_\beta(y)|\text{void}\rangle \pm \theta(y^0-x^0)\langle \text{void}|\psi^\dagger_\beta(y)\psi_\alpha(x)|\text{void}\rangle\,.\end{aligned}$$

Because of the structure of the field operators, nonzero is only the first term which gives the following explicit form of the contraction called also the free propagator (in passing to the second form exploited is the formula (5.138))

$$\begin{aligned}\overbrace{\psi_\alpha(x)\psi^\dagger_\beta}(y) &= \delta_{\alpha\beta}\,\theta(x^0-y^0)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\omega_\mathbf{k}(x^0-y^0)}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \delta_{\alpha\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{i}{\omega-\omega_\mathbf{k}+i0}\, e^{-i\omega(x^0-y^0)}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,. \qquad (7.78)\end{aligned}$$

Evaluation of the expression (7.77) can be most conveniently represented graphically: to the Wick expansion (5.184) and the subsequent (anti)commuting the uncontracted field operators with the creation and annihilation operators building the initial and final states corresponds connecting together lines in Figure 7.2a respecting the directions of arrows and excluding connecting to one another two lines emerging from one and the same interaction vertex (because the interactions V_{int} are normally ordered - this rule will not apply in relativistic theories). There are several possibilities of doing this. The nonzero contributions to the computed S-matrix element correspond to connecting all lines representing the initial and final states to the lines of the two interacting vertices (this corresponds in (5.184) to the term with uncontracted two ψ's and two $\psi^\dagger$'s). (Anti)Commuting the uncontracted field operators with the creation and annihilation operators produces four Dirac delta functions depending on the wave vectors and Kronecker deltas in spin indices which simply substitute the wave vectors and spin labels of the initial and final states into the contracted operators. Writing the Wick contractions of the operators in the form of the Fourier transforms as in (7.78) allows to perform explicitly the integrals over d^4x_1

and d^4x_2. This yields two (four-dimensional, that is in the frequency and wave vector variables) Dirac delta functions. Performing then integrations one finally ends up with $(2\pi)^4\delta(\omega_{\mathbf{k}'_1}+\omega_{\mathbf{k}'_2}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2})$ times $(\delta_{\sigma'_1,\sigma_1}\delta_{\sigma'_2,\sigma_2}\pm\delta_{\sigma'_2,\sigma_1}\delta_{\sigma'_1,\sigma_2})$ times the sum of the integrals:

$$\Big(\frac{g}{i\hbar}\Big)^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{i^2}{[-\omega+k_1^0+k_2^0-\omega_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{q}}+i0][\omega-\omega_{\mathbf{q}}+i0]}\,,$$

and

$$\Big(\frac{g}{i\hbar}\Big)^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{i^2}{[\omega+k_1^0-k'^0_{1,2}-\omega_{\mathbf{k}_1-\mathbf{k}'_{1,2}+\mathbf{q}}+i0][\omega-\omega_{\mathbf{q}}+i0]}\,,$$

The first expression arises from connecting the two "incoming" lines (representing particles in the initial state) with the lines of one of the two interaction vertices, while the two "outgoing" lines with the lines of the other vertex. There are two possible choices of the order of these vertices but owing to the integrations over d^4x_1 and d^4x_2 they lead to the equal contributions and, as explained, simply remove the factor 1/2! in front of (7.77); there are then two possible ways of connecting the incoming lines with the lines of the first vertex, two ways of connecting the outgoing lines with the lines of the second vertex and, finally two ways of sewing together the lines connecting the two vertices. Of these 8 possibilities 4 lead to equal contributiona with $\delta_{\sigma'_1,\sigma_1}\delta_{\sigma'_2,\sigma_2}$ and the other 4 to equal contributiona with $\delta_{\sigma'_2,\sigma_1}\delta_{\sigma'_1,\sigma_2}$.

The second expression arises from connecting the two incoming lines to different vertices, say the one marked k_1,σ_1 to the vertex labeled x_1 (again the terms differing by interchanging x_1 and x_2 give equal contributions). To the same vertex can be then connected either the outgoing lie marked k'_1,σ'_1 or the one marked k'_2,σ'_2.

The integrals over the frequencies in both these expressions can be easily evaluated by the residue method. The one in the second expression vanishes because both poles of the integrand are on the same side of the real ω axis. After performing the integral in the first expressions it is most convenient to assume that $\mathbf{k}_1=-\mathbf{k}_2$ (in other words, to consider the scattering process in the CMS frame). This leads to

$$\Big(\frac{g}{i\hbar}\Big)^2\Big(\frac{m}{i\hbar}\Big)\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{1}{\mathbf{q}^2-\mathbf{k}^2-i0}\,.$$

The remaining integral is linearly divergent. Imposing the cutoff Λ on $|\mathbf{q}|$ one finds (using the Sochocki formula (C.2))

$$\Big(\frac{g}{i\hbar}\Big)^2\Big(\frac{m}{i\hbar}\Big)\left(\frac{i}{4\pi}\,|\mathbf{k}|+\frac{1}{2\pi^2}\,\Lambda-\frac{1}{2\pi^2}\,\frac{\mathbf{k}^2}{\Lambda}+\ldots\right)\equiv\Big(\frac{g}{i\hbar}\Big)^2\Big(\frac{m}{i\hbar}\Big)\,I_0.$$

Up to the second order the S-matrix element is isotropic (independent of the scattering angle θ) and can be written in the "four-dimensional" notation in which $k_1\equiv(\omega_{\mathbf{k}_1},\mathbf{k}_1)$ etc.)

$$S_{\beta\alpha}=\delta_{\beta\alpha}-\frac{i}{\hbar}\,(2\pi)^4\delta^{(4)}(k'_1+k'_2-k_1-k_2)\,(\delta_{\sigma'_1,\sigma_1}\delta_{\sigma'_2,\sigma_1}-\delta_{\sigma'_2,\sigma_1}\delta_{\sigma'_1,\sigma_1})\mathcal{A}\,,$$

with

$$\mathcal{A} = g - g^2 \frac{m}{\hbar^2} \left(\frac{i}{4\pi} |\mathbf{k}| + \frac{1}{2\pi^2} \Lambda - \frac{1}{2\pi^2} \frac{\mathbf{k}^2}{\Lambda} + \dots \right).$$

Inserting this with $g \equiv g_{\rm B}(\Lambda) = (4\pi\hbar^2/m)a_0 + \delta g$ into the formula (7.74) relating $\mathcal{A}$ to the ordinary scattering amplitude $f(k,\theta)$ expanded as in (7.76), adjusting δg so as to cancel only the term diverging linearly with Λ and working up to terms of order a_0^2 one immediately recovers the result (5.102) which was used to express the energy density E_Ω/V of the ground state of N bosons directly in terms of the measurable scattering length a_0.

It is possible to go further and to include an infinite set of contributions to $\mathcal{A}$ coming from "sausage" diagrams each of which consist of several diagrams like the diagram b_1 shown in Figure 7.2 connected to one another to form a chain (a chain diagram consisting of n "sausages" contributes at the order g^{n+1}). This gives the opportunty to demonstrate usig this simple example the freedom in the procedure of removing the UV divergences. Instead of expressing $g_{\rm B}$ as above directly in terms of the measurable length a_0, one can introduce the (finite) renormalized coupling $g_{\rm R}$ by writing $g_{\rm B} = g_{\rm R} + \delta^{(1)}g + \delta^{(2)}g + \dots$, with $\delta^{(n)}g \propto g^{n+1}$. One then obtains

$$\frac{\mathcal{A}}{i\hbar} = \frac{g_{\rm R} + \delta^{(1)}g + \delta^{(2)}g + \dots}{i\hbar} + \left(\frac{g_{\rm R} + \delta^{(1)}g + \dots}{i\hbar} \right)^2 \left(\frac{m}{i\hbar} I_0 \right) + \left(\frac{g_{\rm R} + \dots}{i\hbar} \right)^3 \left(\frac{m}{i\hbar} I_0 \right)^2 + \dots$$

Assuming that $\delta^{(n)}g \propto g_{\rm R}^{n+1}$ and introducing (to simplify the formulae) $x \equiv m/4\pi\hbar^2$ this can be conveniently rewritten in the form[34]

$$\begin{aligned} x\mathcal{A} = x\, g_{\rm R} &+ \left[-(x\, g_{\rm R})^2 \left(ik + \frac{2}{\pi}\Lambda + \dots \right) + x\, \delta^{(1)}g \right] \\ &+ \left[(x\, g_{\rm R})^3 \left(-k^2 - \frac{8}{\pi^2} k^2 + i\frac{4}{\pi} k\Lambda + \frac{4}{\pi^2}\Lambda^2 + \dots \right) \right. \\ &\left. -2(x\, g_{\rm R})\, (x\, \delta^{(1)}g) \left(ik + \frac{2}{\pi}\Lambda - \frac{2}{\pi}\frac{k^2}{\Lambda} + \dots \right) + x\, \delta^{(2)}g \right]. \end{aligned}$$

The counterterms $\delta^{(1)}g$ and $\delta^{(2)}g$ can be now specified. They must certainly cancel the terms with positive powers of Λ but apart from this can be arbitrary. This is the mentioned freedom in the renormalization procedure. Thus one can set

$$x\, \delta^{(1)}g = (x\, g_{\rm R})^2 \left(\frac{2}{\pi}\Lambda + b_1 \right), \qquad x\, \delta^{(2)}g = (x\, g_{\rm R})^3 \left(\frac{4}{\pi^2}\Lambda^2 + \frac{4}{\pi} b_1 \Lambda + b_2 \right), \qquad (7.79)$$

[34] It is crucial to systematically expand the integral I_0 keeping also the terms proportional to inverse powers of the cutoff Λ and rejecting them only after the final expression is obtained: for instance the term $-(8/\pi^2)k^2$ in the middle line arises from the multiplication of a term $\propto k^2/\Lambda$ and a term $\propto \Lambda$ in squaring I_0.

with completely arbitrary factors b_1 and b_2 (having the dimension of Λ). The resulting finite (in the limit $\Lambda \to \infty$) amplitude reads

$$\mathcal{A} = g_{\rm R} + g_{\rm R}^2\, x\, b_1 + g_{\rm R}^3\, x^2 b_2 - (g_{\rm R}^2 x + 2 g_{\rm R}^3 x^2\, b_1) ik - g_{\rm R}^3 x^2 k^2 + \ldots \tag{7.80}$$

and the "bare", cutoff dependent coupling $g_{\rm B}(\Lambda) = g_{\rm R} + \delta^{(1)} g + \delta^{(2)} g + \ldots$ is given by

$$g_{\rm B}(\Lambda) = g_{\rm R} + g_{\rm R}^2\, x \left(\frac{2}{\pi}\, \Lambda + b_1 \right) + g_{\rm R}^3\, x^2 \left(\frac{4}{\pi^2}\, \Lambda^2 + \frac{4}{\pi}\, b_1 \Lambda + b_2 \right) + \ldots \tag{7.81}$$

Thus $g_{\rm B}(\Lambda)$ and, therefore, also the ground state energy density E_Ω/V of the gas of N interacting bosons as well as the scattering amplitude $\mathcal{A}$ (and $f(k,\theta)$ computed using this formalism) - two measurable quantities characterizing the interaction of bosons, can be expressed in terms of the renormalized coupling $g_{\rm R}$ which itself is not a directly measurable quantity. Moreover, $g_{\rm R}$ depends implicitly on the arbitrary factors b_1, b_2 (in higher orders it would depend on even more such arbirary factors) because its actual numerical value reproducing the measured scattering length a_0 depends on the chosen values of b_1 and b_2. Indeed, matching (7.80), using (7.74) onto the expansion[35] (7.76)

$$\mathcal{A} = -\frac{4\pi\hbar^2}{m}\, f(|\mathbf{k}|,\theta) = \frac{1}{x} \left(a_0 - i a_0^2 k - a_0^3 k^2 + \ldots \right),$$

one sees that $a_0 = g_{\rm R} x + (g_{\rm R} x)^2 b_1 + (g_{\rm R} x)^3 b_2 + \ldots$ Inverting this relation allows to explicitly express the renormalized coupling $g_{\rm R}$ in terms of the measurable s-wave scattering length and b_1 and b_2:

$$g_{\rm R} = \frac{1}{x} \left[a_0 - a_0^2 b_1 + a_0^3 (2 b_1^2 - b_2) + \ldots \right]. \tag{7.82}$$

Of course, expressing $g_{\rm B}(\Lambda)$ in terms of a_0 gives the result independent of b_1 and b_2:

$$g_{\rm B}(\Lambda) = \frac{1}{x} \left(1 + \frac{2}{\pi}\, \Lambda a_0 + \frac{4}{\pi^2}\, \Lambda^2 a_0^2 + \ldots \right), \tag{7.83}$$

so E_Ω/V expressed through a_0 is also independent of b_1 and b_2. Nevertheless, the possibility of breaking this fixed relation between directly measurable quantities by expressing them in terms of an unmeasurable but finite parameter like $g_{\rm R}$ proves useful in some applications. In particular this constitutes the commonly used way of applying the ideas of the *renormalization group* to high energy physics and statistical physics problems.

7.5 S-matrix in relativistic quantum mechanics

We now ask the question, what are the conditions the interaction $V_{\rm int}$ should satisfy in order to lead to a relativistically covariant S-matrix. Obviously, the first (but by no means

[35] The terms depending on a_1 and r_0 require including in the effective Hamiltonian terms of length dimension equal -8.

sufficient) requirement is that the vectors $|\alpha_0\rangle$, which in the sense already elucidated can be viewed as representing states of nointeracting particles long before and long after reactions, transform as established in Chapter 6 when the entire system is transformed (boosted, rotated and translated in space-time). Thus, in the Hilbert space there should act a representation of the Poincaré group (or of its universal covering) by unitary operators $U_0(\Lambda, a)$ the action of which on the multi-particle H_0 generalized eigenvectors constructed as tensor products of the 1-particle states follows from the rules (6.111) with $U_0(\Lambda, a)$ and reads[36]

$$U_0(\Lambda,a)|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_0\rangle = e^{-ia\cdot({}^\Lambda p_1+\ldots+{}^\Lambda p_N)} \times \sum_{\bar\sigma_1,\ldots,\bar\sigma_N} \tag{7.84}$$
$$|({}^\Lambda\mathbf{p}_1\bar\sigma_1,\ldots,{}^\Lambda\mathbf{p}_N\bar\sigma_N)_0\rangle\, D^{(s_1)}_{\bar\sigma_1\sigma_1}(W(\Lambda,p_1))\ldots D^{(s_N)}_{\bar\sigma_N\sigma_N}(W(\Lambda,p_N))\,.$$

The subscript "0" denotes the free particle state-vectors $|\alpha_0\rangle$ and ${}^\Lambda\mathbf{p}_i$ stand for the spatial components of the four-vectors $\Lambda^\mu{}_\nu p_i^\nu$. As explained at the end of Section 6.5, the operators P_0^μ and $J_0^{\mu\nu}$, that is, H_0, $\mathbf{P}_0$, $\mathbf{J}_0$ and $\mathbf{K}_0$ generating $U_0(\Lambda, a)$ satisfying the commutation rules (6.21) can be constructed given the creation and annihilation operators of free particles (provided the energy operator H_0 has the appropriate relativistic form).

Furthermore, in a relativistic theory of interacting particles there should exist another set of unitary operators $U(\Lambda, a)$, also furnishing a representation of the Poincaré group, and acting on the introduced in Section 7.3 *in* and *out* state-vectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$, the labels of which are in the one-to-one correspondence with the momentum and spin labels of the state-vectors representing free particle, according to the rule:

$$U(\Lambda,a)|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_\pm\rangle = e^{-ia\cdot({}^\Lambda p_1+\ldots+{}^\Lambda p_N)} \times \sum_{\bar\sigma_1,\ldots,\bar\sigma_N} \tag{7.85}$$
$$|({}^\Lambda\mathbf{p}_1\bar\sigma_1,\ldots,{}^\Lambda\mathbf{p}_N\bar\sigma_N)_\pm\rangle\; D^{(s_1)}_{\bar\sigma_1\sigma_1}(W(\Lambda,p_1))\ldots D^{(s_N)}_{\bar\sigma_N\sigma_N}(W(\Lambda,p_N))\,.$$

Because the *in* and *out* state-vectors are the two sets of eigenvectors of the same Hamiltonian H, the operators $U(\Lambda, a)$ must act on them in exactly *the same way*. This means that the *in* and *out* creation and annihilation operators $a_{\rm in}(\mathbf{p},\sigma)$, $a^\dagger_{\rm in}(\mathbf{p},\sigma)$ and $a_{\rm out}(\mathbf{p},\sigma)$, $a^\dagger_{\rm out}(\mathbf{p},\sigma)$ introduced in Section 7.3 by the formulae (7.48) should satisfy the rules (6.111) with $U(\Lambda, a)$ (instead of $U_0(\Lambda, a)$). Of course, the Hamiltonian $H = H_0 + V_{\rm int}$ is itself one of the operators generating $U(\Lambda, a)$.

The problem we want to investigate in this section can be reformulated as follows: for what interactions $V_{\rm int}$ it is possible to construct the generators $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K}$ which satisfy the commutation rules (6.21) and which act *the same way* on the *in* and *out* states (thereby ensuring that also the operators $U(\Lambda, a)$ will have the required properties)? Since $H = H_0 + V_{\rm int} \neq H_0$, it follows from the commutation rules

$$\big[K^i,\ P^j\big] = -iH\,\delta^{ij}\,, \qquad \big[K^i,\ H\big] = -iP^i\,,$$
$$\big[K^i_0,\ P^j_0\big] = -iH_0\,\delta^{ij}\,, \qquad \big[K^i_0,\ H_0\big] = -iP^i_0\,,$$

[36] In the general case the eigenvectors $|\alpha_0\rangle$ of H_0 should be replaced by the eigenvectors $|\tilde\alpha_0\rangle$ of $\tilde H_0$ - see the discussion in Section 7.1.

that[37] either $\mathbf{P} \neq \mathbf{P}_0$ or $\mathbf{K} \neq \mathbf{K}_0$ (or both).

Assuming for the moment that such operators $U(\Lambda, a)$ do exist, we can write the condition for the Lorentz covariance of the S-matrix. Let $|\alpha_+\rangle = |(\mathbf{p}_1\sigma_1, \ldots, \mathbf{p}_N\sigma_N)_+\rangle$ and $|\beta_-\rangle = |(\mathbf{p}'_1\sigma'_1, \ldots, \mathbf{p}'_M\sigma'_M)_-\rangle$. Then

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta_-|\alpha_+\rangle = \langle\beta_-|U^\dagger(\Lambda,a)U(\Lambda,a)|\alpha_+\rangle \\ &= e^{-ia\cdot({}^\Lambda p_1+\ldots+{}^\Lambda p_N-{}^\Lambda p'_1+\ldots+{}^\Lambda p'_M)} \times \sum_{\bar\sigma'_1,\ldots,\bar\sigma'_M} \sum_{\bar\sigma_1,\ldots,\bar\sigma_N} D^{(s'_1)*}_{\bar\sigma'_1\sigma'_1}(W(\Lambda,p'_1))\ldots \\ &\quad D^{(s'_M)*}_{\bar\sigma'_M\sigma'_M}(W(\Lambda,p'_M))\; D^{(s_1)}_{\bar\sigma_1\sigma_1}(W(\Lambda,p_1))\ldots D^{(s_N)}_{\bar\sigma_N\sigma_N}(W(\Lambda,p_N)) \\ &\quad \langle({}^\Lambda\mathbf{p}'_M\bar\sigma'_M,\ldots,{}^\Lambda\mathbf{p}'_1\bar\sigma'_1)_-|({}^\Lambda\mathbf{p}_1\bar\sigma_1,\ldots,{}^\Lambda\mathbf{p}_N\bar\sigma_N)_+\rangle\,. \end{aligned} \tag{7.86}$$

(The last line in the above formula is just $S_{\beta^\Lambda\alpha^\Lambda}$). Setting in this formula $\Lambda = I$, we infer that the total four-momentum has to be conserved because otherwise the relation

$$S_{\beta\alpha} = e^{-ia\cdot(p_1+\ldots+p_N-p'_1+\ldots+p'_M)}S_{\beta\alpha} \equiv e^{-ia\cdot(P_\alpha-P_\beta)}S_{\beta\alpha}\,,$$

would imply $S_{\beta\alpha} = 0$. It follows, that the formula (7.25) can be written in the form

$$S_{\beta\alpha} = \delta_{\beta\alpha} + (2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\,(-i\mathcal{A}_{\beta\alpha})\,, \tag{7.87}$$

that is, the delta function expressing the overall three-momentum conservation can be factorized[38] from the reaction matrix $t_{\beta\alpha}$ defined in (7.25 as the matrix element of the T_0 operator defined by (7.24):

$$t_{\beta\alpha} = (2\pi)^3\delta^{(3)}(\mathbf{P}_\beta - \mathbf{P}_\alpha)\mathcal{A}_{\beta\alpha}\,. \tag{7.88}$$

The factor $\mathcal{A}_{\beta\alpha}$ (in most textbooks stupidly denoted $\mathcal{M}_{\beta\alpha}$) is frequently called the *invariant amplitude* (despite of being covariant rather than invariant...).

The condition equivalent to (7.86) can also be derived for the S_0 operator defined by (7.16). Writing

$$S_{\beta\alpha} = \langle\beta_0|S_0|\alpha_0\rangle = \langle\beta_0|U_0^\dagger(\Lambda,a)U_0(\Lambda,a)S_0|\alpha_0\rangle\,, \tag{7.89}$$

we see that with (7.84) the formula (7.86) will be recovered provided

$$[S_0,\; U_0(\Lambda,a)] = 0\,, \tag{7.90}$$

[37] This is different than in the nonrelativistic version of the theory in which (for most typical interaction terms $V_{\rm int}$) the boost generators K^i need not be modified to satisfy the commutation rule $[K^i,\; P^j] = -i\hbar\delta^{ij}\hat{M}$ (with the mass operator $\hat{M}$ which, in contrast to the Hamiltonian including interaction, is bilinear in the field operators, replacing the Hamiltonian H of the interacting system) and the rule $[K^i,\; H] = -iP^i$ is satisfied by K^i_0 which simply commutes with $V_{\rm int}$.

[38] This result follows, of course, only from the invariance with respect to spatial translations of the system and is not specific for relativistic theories only; a similar delta function came out automatically - cf. the formula (7.72) - in the computation of the S-matrix element corresponding to the elastic scattering of nonrelativistic particles in a Galileo invariant theory.

or, equivalently, provided the operator S_0 given by (7.63) commutes with all generators H_0, $\mathbf{P}_0$, $\mathbf{J}_0$ and $\mathbf{K}_0$. This can hold only for special forms of $V_{\rm int}$ and we will see, that the existence of the generators $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K}$ which satisfy the commutation rules (6.21) and act the same way on *in* and *out* state-vectors and the condition that S_0 commutes with H_0, $\mathbf{P}_0$, $\mathbf{J}_0$ and $\mathbf{K}_0$ are equivalent.

Let us now construct the generators $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K}$. Most of the interactions $V_{\rm int}$ are such that they commute with the $\mathbf{P}_0$ and $\mathbf{J}_0$ generators. In such cases we can identify

$$\mathbf{P} = \mathbf{P}_0\,, \qquad \text{and} \qquad \mathbf{J} = \mathbf{J}_0\,. \tag{7.91}$$

Indeed, since $\mathbf{P}_0$ and $\mathbf{J}_0$ all commute with H_0, their commutation with the interaction operator $V_{\rm int}$

$$[\mathbf{P}_0,\ V_{\rm int}] = [\mathbf{J}_0,\ V_{\rm int}] = 0\,, \tag{7.92}$$

implies that they commute also with $H = H_0 + V_{\rm int}$ and, hence, also with the Møller operator $\Omega(t) \equiv e^{iHt}e^{-iH_0t}$ (also for finite t). Trivially then

$$\mathbf{P}|\alpha_\pm\rangle = \mathbf{P}\Omega_\pm|\alpha_0\rangle = \mathbf{P}_0\Omega_\pm|\alpha_0\rangle = \Omega_\pm\mathbf{P}_0|\alpha_0\rangle\,. \tag{7.93}$$

Analogous relation holds with the operators $\mathbf{J}$). This shows that the operators $\mathbf{P}$ and $\mathbf{J}$ act the same way on the *in* and *out* state-vectors. It is also clear that if (7.92) holds, the operators $\mathbf{P}_0$ and $\mathbf{J}_0$ commute with the evolution operator,

$$[\mathbf{P}_0\ , U_I(\tau_2,\tau_1)] = [\mathbf{J}_0\ , U_I(\tau_2,\tau_1)] = 0\,, \tag{7.94}$$

for arbitrary finite τ_1 and τ_2 and, therefore, also with the operator $S_0 = U_I(-\infty,+\infty)$. Since the Poincaré group generator H acts the same way on *in* and *out* state-vectors owing to the relations (7.21) and, as has been shown, $[H_0,\ S_0] = 0$, we conclude that those of the commutation rules (6.21) which involve only H, $\mathbf{P}$ and $\mathbf{J}$ are (in most cases) easy to satisfy.

It remains to construct the operator $\mathbf{K}$ satisfying the commutation rules (6.21) and to prove that $[\mathbf{K}_0,\ S_0] = 0$. This is the most tricky part of the construction. If $\mathbf{P} = \mathbf{P}_0$ but $H \neq H_0$, then from the commutation rules $[\mathbf{K},\ H] = -i\mathbf{P} = -i\mathbf{P}_0$ and $[K^i,\ P^j] = -i\delta^{ij}H$ it follows that $\mathbf{K} \neq \mathbf{K}_0$. Therefore we write

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{W}\,. \tag{7.95}$$

The operator $\mathbf{W}$ has to be such that

$$[\mathbf{K}_0,\ V_{\rm int}] = -\,[\mathbf{W},\ H_0 + V_{\rm int}] \equiv -\,[\mathbf{W},\ H]\,, \tag{7.96}$$

and (because by assumption $[K_0^i,\ P_0^j] = -i\delta^{ij}H_0$)

$$\left[W^i,\ P_0^j\right] = -i\delta^{ij}V_{\rm int}\,. \tag{7.97}$$

The operator $\mathbf{K}_0$ is known (its action on the free multi-particle vectors $|\alpha_0\rangle$ is known) and one has to construct $\mathbf{W}$ satisfying the condition (7.96) and such, that $\mathbf{K}$ act the same way on *in* and *out* state-vectors. The last requirement is the most important. One could try to *define* $\mathbf{W}$ (without imposing any conditions on the interaction $V_{\rm int}$ itself) by simply giving its matrix elements between a complete set of state vectors, say the *in* (or *out* or free multi-particle) state-vectors, e.g. by

$$\langle\beta_+|\mathbf{W}|\alpha_+\rangle := \frac{\langle\beta_+|\,[\mathbf{K}_0\,, V_{\rm int}]\,|\alpha_+\rangle}{E_\beta - E_\alpha}\,, \tag{7.98}$$

(so that it would automatically fulfill the condition (7.96)), but the matrix elements defined in this way would not be smooth functions of energy and $\mathbf{K} = \mathbf{K}_0 + \mathbf{W}$ would not (as will become clear shortly) then act the same way on *in* and *out* state-vectors.

We will first show that if there exist an operator $\mathbf{W}$ satisfying the condition (7.96), and the matrix elements of which are smooth functions of energy, then $\mathbf{K}$ acts the same way on *in* and *out* vectors and, simultaneously, the operator $\mathbf{K}_0$ commutes with the S_0 operator. To this end we consider the commutator $[\mathbf{K}_0,\ U_I(\tau_2,\tau_1)]$ for finite τ_1 and τ_2. From the commutation rule $[\mathbf{K}_0,\ H_0] = -i\mathbf{P}_0$ and the fact that $\mathbf{P}_0$ commute with H_0 one gets the relation

$$\left[\mathbf{K}_0,\ e^{iH_0\tau}\right] = \tau\,\mathbf{P}_0\,e^{iH_0\tau}\,. \tag{7.99}$$

Similarly, from $[\mathbf{K},\ H] = i\mathbf{P} = i\mathbf{P}_0$ and the fact that $\mathbf{P}_0$ commutes also with $V_{\rm int}$ (which means that $\mathbf{P}$ commutes with H) one finds

$$\left[\mathbf{K},\ e^{iH\tau}\right] = \tau\,\mathbf{P}\,e^{iH\tau} = \tau\,\mathbf{P}_0\,e^{iH\tau}\,. \tag{7.100}$$

Therefore,

$$\begin{aligned}[\mathbf{K}_0,\ U_I(\tau_2,\tau_1)] &= \left[\mathbf{K}_0,\ e^{iH_0\tau_2}e^{-iH(\tau_2-\tau_1)}e^{-iH_0\tau_1}\right]\\ &= \tau_2\,\mathbf{P}_0U_I(\tau_2,\tau_1) + e^{iH_0\tau_2}\left[\mathbf{K}_0,\ e^{-iH(\tau_2-\tau_1)}\right]e^{-iH_0\tau_1} - \tau_1\,U_I(\tau_2,\tau_1)\mathbf{P}_0\,,\end{aligned}$$

where (7.99) has been used. In the middle term we set then $\mathbf{K}_0 = \mathbf{K} - \mathbf{W}$ which enables us to make use of (7.100), after which we find that the term obtained from the commutator of $\mathbf{K}$ precisely cancels the two terms containing $\mathbf{P}_0$ (recall that $[\mathbf{P}_0,\ U_I(\tau_2,\tau_1)] = 0$). One is therefore left with

$$[\mathbf{K}_0,\ U_I(\tau_2,\tau_1)] = -\mathbf{W}_I(\tau_2)U_I(\tau_2,\tau_1) + U_I(\tau_2,\tau_1)\mathbf{W}_I(\tau_1)\,, \tag{7.101}$$

where $\mathbf{W}_I(\tau)$ is the operator $\mathbf{W}$ taken in the Dirac picture:

$$\mathbf{W}_I(\tau) = e^{iH_0\tau}\,\mathbf{W}\,e^{-iH_0\tau}\,. \tag{7.102}$$

From the result (7.101) it is clear that $\mathbf{K}_0$ does not commute with the evolution operator for finite times τ_1 and τ_2. However, Lorentz-covariance of the S-matrix requires only

vanishing of the right hand side of (7.101) in the limits $\tau_1 \to -\infty$ and $\tau_2 \to +\infty$. This is ensured if the matrix elements of $\mathbf{W}$ between the state-vectors $|\alpha_0\rangle$ are nonsingular functions of energy, for then, for any two smooth normalizable profiles $\psi(\alpha)$ and $\phi(\beta)$ the expressions

$$\int d\alpha \int d\beta\, \phi^*(\beta)\, \psi(\alpha)\, \langle \beta_0 | \mathbf{W}_I(\tau) | \alpha_0 \rangle = \int d\alpha \int d\beta\, \phi^*(\beta)\, \psi(\alpha)\, e^{i(E_\beta - E_\alpha)\tau}\, \langle \beta_0 | \mathbf{W} | \alpha_0 \rangle \,,$$

which correspond to matrix elements of the operators $\mathbf{W}_I(\tau)$ between normalizable (i.e. belonging to the proper Hilbert space) state-vectors are well defined and, by the Riemann-Lebesgue theorem vanish as $\tau \to \pm\infty$, provided the integrand is smooth enough, in particular provided it does not have poles such as the right hand side of (7.98). Thus, if the matrix elements of $\mathbf{W}$ between the complete set of generalized free multi-particle eigenvectors $|\alpha_0\rangle$ of H_0 are nonsingular functions of energy, all matrix elements of $\mathbf{W}_I(\tau)$ between smooth superpositions of such generalized vectors[39] vanish in the limits $\tau \to \pm\infty$, that is, in these limits the operator $\mathbf{W}_I(\tau)$ vanishes (in the weak sense). The right hand side of (7.101) then vanishes in the double limit $\tau_2 \to \infty$, $\tau_1 \to -\infty$ too, and $\mathbf{K}_0$ commutes with the S_0 operator.

In this case also the operator $\mathbf{K} = \mathbf{K}_0 + \mathbf{W}$ acts the same way on *in* and *out* states. To see this, setting in (7.101) $\tau_2 = 0$ and $\tau_1 = \mp\infty$, we consider the commutator

$$[\mathbf{K}_0,\ \Omega_\pm] \equiv [\mathbf{K}_0,\ U_I(0, \mp\infty)] = -\mathbf{W}_I(0)\Omega_\pm + \Omega_\pm \mathbf{W}_I(\mp\infty)\,. \tag{7.103}$$

If matrix elements of $\mathbf{W}$ are nonsingular functions of energy, the last term in (7.103) vanishes as explained above and, because $\mathbf{W}_I(0) = \mathbf{W}$, one obtains the intertwining relation

$$\mathbf{K}\Omega_\pm = \Omega_\pm \mathbf{K}_0\,, \tag{7.104}$$

analogous to (7.93) satisfied by the generators $\mathbf{P}$ and $\mathbf{J}$ and (7.21) relating the Hamiltonians H and H_0. Note that if the matrix elements of $\mathbf{W}$ were singular functions of energy (as in 7.98), the last term in the commutator (7.103) would be nonzero and the specification of the way of going around the singularities would introduce a difference in $\mathbf{W}_I(-\infty)$ and $\mathbf{W}_I(+\infty)$; $\mathbf{K}$ would then act differently on *in* and *out* state-vectors.

Thus, if the operator $\mathbf{W}$ with the required properties can be constructed, all the generators G_0 of the Poincaré group acting on smooth superpositions of the generalized state-vectors $|\alpha_0\rangle$ commute with the S_0 operator and, simultaneously, the related intertwining relation hold:

$$\begin{array}{lll} i) & [G_0,\ S_0] = 0 & \to \quad [U_0(\Lambda, a),\ S_0] = 0\,, \\ ii) & G\,\Omega_\mp = \Omega_\mp\, G_0 & \to \quad U(\Lambda, a) \text{ act the same way} \\ & & \quad \text{on the } \textit{in} \text{ and } \textit{out} \text{ states.} \end{array} \tag{7.105}$$

[39] Such superpositions form a dense set in the Hilbert space.

Analogous intertwining relations will be also crucial for the operator quantization of theories of non-Abelian gauge fields (Section 20.3) based on the BRST symmetry. The same scheme (7.105) works also for parity and charge conjugation operators $\mathcal{P}$ and $\mathcal{C}$, as well as for generators of various possible internal symmetries like isospin or the "eightfold way" $SU(3)$ (see Chapter 12) etc. Usually all generators Q^a of internal symmetries are such that $Q^a = Q_0^a$ (as is the case with $\mathbf{P}$ and $\mathbf{J}$), because their free particle counterparts Q_0^a commute with the interaction operator $V_{\rm int}$. Slightly more tricky is the the action of the time reversal operator $\mathcal{T}$ because it is antiunitary and interchanges the *in* and *out* vectors. We will explore consequences of these symmetries for the S-matrix in due course.

The question now is, for which interactions $V_{\rm int}$ can the operators $\mathbf{W}$ with the required properties be constructed? It turns out that a rather broad class of such interactions can be easily identified. Belong to it all interactions $V_{\rm int}$ the interaction picture counterparts of which,

$$V_{\rm int}^I(t) = e^{iH_0 t}\, V_{\rm int}\, e^{-iH_0 t}\,, \tag{7.106}$$

appearing in (7.63), can be obtained as the space integral of a local interaction Hamiltonian density $\mathcal{H}_{\rm int}(t,\mathbf{x})$:

$$V_{\rm int}^I(t) = \int d^3\mathbf{x}\, \mathcal{H}_{\rm int}(t,\mathbf{x})\,, \tag{7.107}$$

which is such that (using now the four-dimensional notation $x^\mu = (t,\mathbf{x})$)

$$U_0(\Lambda,a)\, \mathcal{H}_{\rm int}(x)\, U_0^{-1}(\Lambda,a) = \mathcal{H}_{\rm int}(\Lambda\cdot x + a)\,, \tag{7.108}$$

and satisfies the condition

$$[\mathcal{H}_{\rm int}(x),\ \mathcal{H}_{\rm int}(y)] = 0 \qquad \text{if} \qquad (x-y)^2 < 0\,. \tag{7.109}$$

Note that for $\Lambda = I$ and $a^\mu = (t,\mathbf{0})$ the requirement (7.108) is consistent with (7.106). It should also be noted that with the condition (7.109) the formula (7.63), which for $V_{\rm int}^I(t)$ given by (7.107), reads

$$S_0 = \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \int d^4x_1 \ldots \int d^4x_N\, \mathrm{T}\left[\mathcal{H}_{\rm int}(x_1)\ldots\mathcal{H}_{\rm int}(x_N)\right],$$

becomes fully covariant: the time ordering of two space-time points x_1 and x_2 is Lorentz invariant only when $(x_1-x_2)^2 > 0$; but the condition (7.109) makes the time ordering irrelevant whenever $(x_1-x_2)^2 \le 0$.

To these requirements one must also add the one spelled out at the beginning of the Section 7.3, namely that that structure of $\mathcal{H}_{\rm int}(x)$ (i.e. $V_{\rm int}$) must be such that there is a one-to-one correspondence between the eigenvectors of H_0 and the *in* and *out* eigenvectors of H. In Chapter 9 it will be shown how to construct interactions $V_{\rm int}$ satisfying the

conditions (7.107)-(7.109) using field operators which it will be constructed in Chapter 8. The additional condition will be investigated in Section 9.7.

The property (7.109) of $\mathcal{H}_{\rm int}(x)$, called the *local causality condition*, has no counterpart in nonrelativistic quantum mechanics.[40] As it will turn out, it is precisely this property of $\mathcal{H}_{\rm int}(x)$ that is responsible for the fundamental *spin-statistics connection* (i.e. it enforces that the creation/annihilation operators of integer and half-integer spin particles satisfy respectively the commutation and the anticommutation rules, thereby implying that these two classes of particles obey respectively the Bose-Einstein and the Fermi-Dirac statistics), as well as for the non-conservation of the number of particles in relativistic theories and, finally, for the existence of antiparticles.

To check that when $\mathcal{H}_{\rm int}(x)$ satisfies the condition (7.108) the operator $\mathbf{W}$ with the required properties can indeed be constructed, we set in (7.108) $a^\mu = 0$ and write

$$U_0(\Lambda) = \exp\!\left(-\frac{i}{2}\,\omega_{\mu\nu} J_0^{\mu\nu}\right). \tag{7.110}$$

Taking the parameters $\omega_{\mu\nu}$ to be infinitesimal, we get from (7.108) the relation

$$\begin{aligned} -\frac{i}{2}\,\omega_{\mu\nu}\,[J_0^{\mu\nu},\ \mathcal{H}_{\rm int}(x)] &\approx \mathcal{H}_{\rm int}(x+\omega\cdot x) - \mathcal{H}_{\rm int}(x) \\ &\approx \frac{\partial \mathcal{H}_{\rm int}(x)}{\partial x^\lambda}\,\omega^\lambda{}_\nu\, x^\nu = \frac{1}{2}\,\omega_{\mu\nu}\left(g^{\lambda\mu}x^\nu - g^{\lambda\nu}x^\mu\right)\frac{\partial \mathcal{H}_{\rm int}(x)}{\partial x^\lambda}\,, \end{aligned} \tag{7.111}$$

which, in view of the arbitrariness of $\omega_{\mu\nu}$, implies

$$-i\,[J_0^{\mu\nu},\ \mathcal{H}_{\rm int}(x)] = \left(g^{\lambda\mu}x^\nu - g^{\lambda\nu}x^\mu\right)\frac{\partial \mathcal{H}_{\rm int}(x)}{\partial x^\lambda}\,. \tag{7.112}$$

Taking now $J_0^{0i} \equiv K_0^i$ (i.e. $\mu = 0$ and $\nu = i$ in the above equality) we get:

$$-i\left[K_0^i\ , \mathcal{H}_{\rm int}(x)\right] = t\,\nabla_i \mathcal{H}_{\rm int}(x) + x^i\,\frac{\partial \mathcal{H}_{\rm int}(x)}{\partial t}\,. \tag{7.113}$$

Finally, setting $t = 0$, and integrating both sides over $d^3\mathbf{x}$ we find

$$\begin{aligned} \left[K_0^i\ , \int d^3\mathbf{x}\,\mathcal{H}_{\rm int}(0,\mathbf{x})\right] \equiv \left[K_0^i\ , V_{\rm int}\right] &= i\int d^3\mathbf{x}\, x^i \left(\frac{\partial \mathcal{H}_{\rm int}(x)}{\partial t}\right)_{t=0} \\ &= i\int d^3\mathbf{x}\, x^i\,\frac{\partial}{\partial t}\left(e^{iH_0 t}\,\mathcal{H}_{\rm int}(0,\mathbf{x})\,e^{-iH_0 t}\right)\Big|_{t=0} \\ &= \left[H_0,\ -\int d^3\mathbf{x}\, x^i\,\mathcal{H}_{\rm int}(0,\mathbf{x})\right]. \end{aligned} \tag{7.114}$$

[40] This is because the boost generator $\mathbf{K}_0$, as already has been remarked, is not modified when the interaction term is included in the Hamiltonian.

This suggests that one can take

$$\mathbf{W} = -\int d^3\mathbf{x}\,\mathbf{x}\,\mathcal{H}_{\rm int}(0,\mathbf{x})\,. \tag{7.115}$$

Owing to the condition (7.109) then

$$[V_{\rm int},\ \mathbf{W}] = -\int d^3\mathbf{y}\int d^3\mathbf{x}\,\mathbf{x}\,[\mathcal{H}_{\rm int}(0,\mathbf{y}),\ \mathcal{H}_{\rm int}(0,\mathbf{x})] = 0\,, \tag{7.116}$$

and the condition (7.96), $[\mathbf{K}_0,\ V_{\rm int}] = -[\mathbf{W},\ H_0 + V_{\rm int}]$ is satisfied because (7.114) ensures that $[\mathbf{K}_0,\ V_{\rm int}] = -[\mathbf{W},\ H_0]$. It turns out that the condition (7.97) is then also satisfied: from (cf. (7.108))

$$\mathcal{H}_{\rm int}(0,\mathbf{x}+\mathbf{a}) = e^{-i\mathbf{a}\cdot\mathbf{P}_0}\,\mathcal{H}_{\rm int}(0,\mathbf{x})\,e^{i\mathbf{a}\cdot\mathbf{P}_0}\,,$$

it follows that

$$-i\left[P_0^j,\ \mathcal{H}_{\rm int}(0,\mathbf{x})\right] = \frac{\partial\mathcal{H}_{\rm int}(0,\mathbf{x})}{\partial x^j}\,.$$

Hence,

$$\begin{aligned}\left[W^i,\ P_0^j\right] &= -\int d^3\mathbf{x}\,x^i\left[\mathcal{H}_{\rm int}(0,\mathbf{x}),\ P_0^j\right] = i\int d^3\mathbf{x}\,x^i\,\frac{\partial\mathcal{H}_{\rm int}(0,\mathbf{x})}{\partial x^j}\\ &= i\int d^3\mathbf{x}\left\{\partial_j(x^i\mathcal{H}_{\rm int}(0,\mathbf{x})) - \delta^{ij}\mathcal{H}_{\rm int}(0,\mathbf{x})\right\}.\end{aligned}$$

This is just the required result: as the operators like $\mathcal{H}_{\rm int}(0,\mathbf{x})$ are in fact operator valued distributions, the first term which is the integral of a total derivative should be treated as the zero operator. Thus, if the matrix elements of the operator $\mathcal{H}_{\rm int}(0,\mathbf{x})$ between generalized H_0 eigenvectors $|\alpha_0\rangle$ are nonsingular functions of energy, the operators $\mathbf{K} = \mathbf{K}_0 + \mathbf{W}$ act the same way on *in* and *out* generalized eigenvectors of H.

7.6 Unitarity of the S-matrix

As has been argued in Section 7.1, the S_0 operator, which is a (double) limit of the unitary evolution operator $U_I(\tau_2,\tau_1)$ (7.63), is itself unitary, i.e. satisfies $S_0^\dagger S_0 = \hat{1}$. This relation written in terms of the S-matrix elements $S_{\beta\alpha}$ and the amplitudes $\mathcal{A}_{\beta\alpha}$ takes the form

$$S_{\beta\alpha} = \delta_{\beta\alpha} + (2\pi)^4\delta^{(4)}(P_\alpha - P_\beta)\,(-i\mathcal{A}_{\beta\alpha})\,,$$

one can therefore write

$$\begin{aligned}\delta_{\beta\alpha} = \int d\gamma\,S^*_{\gamma\beta}S_{\gamma\alpha} &= \int d\gamma\left[\delta_{\gamma\beta} + (2\pi)^4\delta^{(4)}(P_\gamma - P_\beta)\left(+i\mathcal{A}^*_{\gamma\beta}\right)\right]\\ &\qquad\times\left[\delta_{\gamma\alpha} + (2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\,(-i\mathcal{A}_{\gamma\alpha})\right]\\ &= \delta_{\beta\alpha} + (2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\left[i\mathcal{A}^*_{\alpha\beta} - i\mathcal{A}_{\beta\alpha}\right]\\ &\qquad + \int d\gamma\,(2\pi)^8\delta^{(4)}(P_\gamma - P_\beta)\delta^{(4)}(P_\gamma - P_\alpha)\mathcal{A}^*_{\gamma\beta}\mathcal{A}_{\gamma\alpha}\,.\end{aligned}$$

Hence the matrices $\mathcal{A}_{\beta\alpha}$ satisfy the following important unitarity condition[41]

$$-i\left(\mathcal{A}^*_{\alpha\beta} - \mathcal{A}_{\beta\alpha}\right) = \int d\gamma\,(2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\mathcal{A}^*_{\gamma\beta}\mathcal{A}_{\gamma\alpha}\,, \tag{7.117}$$

in which both sides are to be taken for $P_\beta = P_\alpha$. Recall that in the adopted notation the integral over $d\gamma$ involves also summation over different numbers of particles in the state $|\gamma_0\rangle$ and includes appropriate factors $1/n_i!$ for each set of n_i identical particles of type i in this state.

In the perturbative expansion of the S-matrix, when the amplitudes $\mathcal{A}_{\alpha\beta}$ are computed as power series in some (small) coupling constant(s), the importance of the condition (7.117) stems from the fact that it relates contributions to $\mathcal{A}_{\alpha\beta}$ which are of different orders in the couplings. One is therefore able to say something about higher order contributions to $\mathcal{A}_{\alpha\beta}$ knowing it in lower orders. This will be exploited in Section **??**. Here we discuss some consequences of the condition (7.117) which do not rely on any perturbative expansion and have therefore a general character.

The first useful relation is obtained by setting in (7.117) $\beta = \alpha$. One then gets

$$-2\,\mathrm{Im}\mathcal{A}_{\alpha\alpha} = \int d\gamma\,(2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\,|\mathcal{A}_{\gamma\alpha}|^2\,. \tag{7.118}$$

This can for example be used to argue that in the framework of relativistic quantum field theory the *total* decay widths of a particle and of its antiparticle are equal. To this end we write the condition (7.118) for the CPT transformed state $|\alpha\rangle$ of a single unstable particle at rest[42]

$$-2\,\mathrm{Im}\mathcal{A}_{(\mathcal{CPT}\alpha)(\mathcal{CPT}\alpha)} = \int d\gamma\,(2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\,\left|\mathcal{A}_{\gamma(\mathcal{CPT}\alpha)}\right|^2\,, \tag{7.119}$$

($P^\mu_{\mathcal{CPT}\alpha} = P^\mu_\alpha$ because if the $\mathcal{CPT}$ operator commutes with the Hamiltonian, the states $|\alpha\rangle$ and $\mathcal{CPT}|\alpha\rangle$ have the same energy and the action of the $\mathcal{CPT}$ operator does not reverse the three-momenta), and use the fact that because the CPT operation is a valid symmetry of any relativistic, unitary quantum field theory, one always has

$$S_{(\mathcal{CPT}\alpha)(\mathcal{CPT}\beta)} = S_{\beta\alpha}\,, \tag{7.120}$$

which in turn implies that

$$\mathcal{A}_{(\mathcal{CPT}\alpha)(\mathcal{CPT}\beta)} = \mathcal{A}_{\beta\alpha}\,. \tag{7.121}$$

[41] One can arrive at the same result writing $S_0 = \hat{1} - iT_0$ and taking the matrix elements between the $|\alpha_0\rangle$ and $|\beta_0\rangle$ state-vectors of the operator equality

$$-i(T_0^\dagger - T_0) = T_0^\dagger T_0\,,$$

using the relation $\langle\beta_0|T_0^\dagger|\alpha_0\rangle = (\langle\alpha_0|T_0|\beta_0\rangle)^*$.

[42] Since the S-matrix elements can, strictly speaking, be defined only for absolutely stable particles, the "proof" presented here cannot be considered truly rigorous.

Thus, the left hand sides of (7.118) and (7.119) are equal (for $\alpha = \beta$ all phase factors resulting from the $\mathcal{CPT}$ action mutually cancel out) and so are the RHSs. Up to a multiplicative constant factor (see Section 10.2) the expressions on the right hand sides of (7.118) and (7.119) represent the total decay widths of a particle and its antiparticle with reversed spin, respectively. However, by rotational invariance the full decay rate (integrated over all possible directions of the final state particles and summed over possible projections of their spins or over their helicities) cannot depend on the spin projection of the decaying particle represented by $|\alpha\rangle$. This proves the proposition.

Another general consequence of the unitarity of the S-matrix can be obtained by taking for $|\alpha_0\rangle$ in (7.118) some particular two-particle state. Dividing both sides by the initial state flux factor F (to be defined in Section 10.2) we obtain the *optical theorem* in the form

$$\sigma_{\rm tot}(\alpha \to \text{anything}) = -\frac{2}{F}\,\mathrm{Im}\,\mathcal{A}_{\alpha\alpha}\,. \tag{7.122}$$

Further consequences of the relation (7.118) can be explored by going over to the helicity basis of multiparticle state-vectors introduced (on the example of two-particle state-vectors) in Section 6.4. Consider a collision process of two particles which are represented (in the sense explained in Section 7.1) by the state $|\alpha_0\rangle$ and a two-particle final state represented by $|\beta_0\rangle$ in the center of mass (CM) frame. Inserting in the formula

$$\langle\beta_0|T_0|\alpha_0\rangle = (2\pi)^4\delta^{(4)}(P_\alpha - P_\beta)\mathcal{A}_{\beta\alpha}\,. \tag{7.123}$$

in which $T_0 = i(S_0 - \hat{1})$ (cf. (7.24)), the vectors $|\alpha_0\rangle$ and $\langle\beta_0|$ decomposed as in (6.110) into the states with definite total angular momentum, and denoting the labels of the state $\langle\beta_0|$ with primes, we get

$$\begin{aligned}\langle \mathbf{P}', \mathbf{p}', \lambda_1', \lambda_2'|T_0|\mathbf{0}, \mathbf{p}, \lambda_1, \lambda_2\rangle = \sum_{j'}^{\infty}\sum_{m_j'}\sum_{j}^{\infty}\sum_{m_j} D^{(j')*}_{m_j'\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\, D^{(j)}_{m_j\lambda_1-\lambda_2}(\Omega_{\mathbf{p}}) \\ \times\frac{\sqrt{(2j'+1)(2j+1)}}{4\pi}\,\langle \mathbf{P}', \sqrt{s'}, \lambda_1', \lambda_2', j', m_j'|\,T_0|\mathbf{0}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle\,. \end{aligned}\tag{7.124}$$

Since the total angular momentum, its z-axis projection as well as the total four-momentum of the system are conserved by the interaction of a relativistic theory, the matrix element in (7.124) must take the form[43]

$$\begin{aligned}&\langle \mathbf{P}', \sqrt{s'}, \lambda_1', \lambda_2', j', m_j'|\,T_0|\mathbf{0}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle \\ &\qquad = (2\pi)^4\delta^{(3)}(\mathbf{P}')\delta(P^{0\prime} - \sqrt{s})\,64\pi^2\,\mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s)\,\delta_{j'j}\,\delta_{m_j'm_j}\,,\end{aligned}\tag{7.125}$$

[43]Recall (Section 6.4) that the vectors $|\mathbf{P}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle$ are the H_0 eigenvectors with the eigenvalue $\sqrt{\mathbf{P}^2 + s}$.

with the factor $64\pi^2$ introduced for further convenience. $\mathcal{T}^{(j)}_{\lambda_1',\lambda_2',\lambda_1,\lambda_2}(s)$, defined by (7.125), are called *partial wave amplitudes*. We recall (see the formula (6.107)) that the state-vectors $|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ representing two identical particles with an odd value of j vanish if $\lambda_1=\lambda_2$ and so must do the corresponding partial wave amplitudes (when j is odd and $\lambda_1=\lambda_2$ or $\lambda_1'=\lambda_2'$). Comparing with (7.123) we get

$$\mathcal{A}_{\beta\alpha}=16\pi\sum_j^\infty\sum_{m_j}(2j+1)\,\mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s)\,D^{(j)*}_{m_j\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\,D^{(j)}_{m_j\lambda_1-\lambda_2}(\Omega_{\mathbf{p}}). \tag{7.126}$$

The choice of the angular momentum quantization axis in the direction of the momentum $\mathbf{p}$ reduces $D^{(j)}_{m_j\lambda_1-\lambda_2}(\Omega_{\mathbf{p}})$ to $\delta_{m_j\lambda_1-\lambda_2}$, so that:

$$\mathcal{A}_{\beta\alpha}=16\pi\sum_j^\infty(2j+1)\,\mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s)\,D^{(j)*}_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\,. \tag{7.127}$$

In the following we will take the state $|\beta_0\rangle$ to represent the same two particles as $|\alpha_0\rangle$; the factors $\mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s)$ will therefore be the elastic (with possible spin flip) scattering partial wave amplitudes. We will also need the formula

$$\mathcal{A}_{\gamma\beta}=16\pi\sum_j^\infty\sum_{m_j}(2j+1)\,\tilde{\mathcal{T}}^{(j)}_{\lambda_a,\lambda_b;\lambda_1',\lambda_2'}(s)D^{(j)*}_{m_j\lambda_a-\lambda_b}(\Omega_{\tilde{\mathbf{p}}})D^{(j)}_{m_j\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\,,$$

with the state $|\gamma_0\rangle=|\mathbf{0},\tilde{\mathbf{p}},\lambda_a,\lambda_b\rangle\equiv|\mathbf{P}=\mathbf{0},\tilde{\mathbf{p}},\lambda_1\lambda_2\rangle$ representing two particles a and b (not necessarily the same as those in the states $|\alpha_0\rangle$ and $|\beta_0\rangle$) characterized by the helicities λ_a, λ_b and the momentum $\tilde{\mathbf{p}}$ (in their CM frame); we denote the corresponding partial wave amplitudes by $\tilde{\mathcal{T}}^{(j)}_{\lambda_a,\lambda_b;\lambda_1',\lambda_2'}(s)$. For $|\gamma_0\rangle=|\alpha_0\rangle$, that is, setting $\lambda_a=\lambda_1$, $\lambda_b=\lambda_2$ and $\tilde{\mathbf{p}}=\mathbf{p}$, we get from this formula that

$$\mathcal{A}^*_{\alpha\beta}=16\pi\sum_j^\infty(2j+1)\,\mathcal{T}^{(j)*}_{\lambda_1,\lambda_2;\lambda_1'\lambda_2'}(s)D^{(j)*}_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\,, \tag{7.128}$$

with the same partial wave amplitudes $\mathcal{T}^{(j)}_{\lambda_1,\lambda_2;\lambda_1',\lambda_2'}(s)$ as in (7.127) but with the helicity labels interchanged.

These formulae allow to single out the contribution of two-particle states $|\gamma_0\rangle=|\tilde{\mathbf{p}}_a,\lambda_a,\tilde{\mathbf{p}}_b,\lambda_b\rangle$ to the unitarity condition (7.117) *specified to the elastic scattering amplitude*, i.e. with $|\alpha_0\rangle$ and $|\beta_0\rangle$ states representing the same two particles. The integral $\int d\gamma$ in (7.117) involves the following contribution of a two-particle state $|\gamma_0\rangle$:

$$\begin{aligned}&\sum_{\lambda_a,\lambda_b}N_{\lambda_a,\lambda_b}\int d\Gamma_{\tilde{\mathbf{p}}_a}\int d\Gamma_{\tilde{\mathbf{p}}_b}(2\pi)^4\delta(\tilde{E}_a+\tilde{E}_b-\sqrt{s})\,\delta^{(3)}(\tilde{\mathbf{p}}_a+\tilde{\mathbf{p}}_b)\mathcal{A}^*_{\gamma\beta}\mathcal{A}_{\gamma\alpha}\\&\qquad=\frac{1}{32\pi^2 s}\,\lambda^{1/2}(s,m_a^2,m_b^2)\sum_{\lambda_a,\lambda_b}N_{\lambda_a\lambda_b}\int d\Omega_{\tilde{\mathbf{p}}}\left(\mathcal{A}^*_{\gamma\beta}\mathcal{A}_{\gamma\alpha}\right)_{\text{on shell}}\,,\end{aligned} \tag{7.129}$$

where m_a and m_b are the masses of the particles a and b in the state $|\gamma_0\rangle$, the subscript "on shell" means $\tilde{\mathbf{p}}_a = -\tilde{\mathbf{p}}_b \equiv \tilde{\mathbf{p}}$ with $|\tilde{\mathbf{p}}|$ determined by $\sqrt{s}$ (and m_a and m_b), and the function $\lambda(s, m_a^2, m_b^2)$ (do not confuse it with the helicity labels!) reads

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\,. \tag{7.130}$$

The factor $N_{\lambda_a,\lambda_b} = 1/2$ if $a = b$ (i.e. if the particles are identical) with $\lambda_a = \lambda_b$, and $N_{\lambda_a,\lambda_b} = 1$ otherwise, follows from the explicit form of the completeness relation (7.5) (explained in detail in Section 5.1). Using now the property (6.109) of the D-functions we arrive at

$$\begin{aligned}
-i\sum_j (2j+1)\, D^{(j)*}_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\, \Big[\mathcal{T}^{(j)*}_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}(s) - \mathcal{T}^{(j)}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(s)\Big] \\
= \sum_{(ab)} \sum_j (2j+1)\, D^{(j)*}_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'}) \\
\times \sum_{\lambda_a,\lambda_b} \frac{2}{s}\,\lambda^{1/2}(s, m_a^2, m_b^2)\, N_{\lambda_a\lambda_b}\, \tilde{\mathcal{T}}^{(j)*}_{\lambda_a\lambda_b;\lambda_1'\lambda_2'}(s)\, \tilde{\mathcal{T}}^{(j)}_{\lambda_a\lambda_b;\lambda_1\lambda_2}(s) \\
+ \frac{1}{16\pi}\int d\gamma\,(2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\, \mathcal{A}^*_{\gamma\beta}\mathcal{A}_{\gamma\alpha}\,,
\end{aligned} \tag{7.131}$$

where the sum in the second line is over all kinematically allowed final states with two[44] particles (ab) and the integral in the last line includes all kinematically allowed three- and more particle final states.

If the three- and more-particle channels are kinematically inaccessible (or forbidden by some conservation laws) the last line is absent and integrating both sides of (7.131) over $d\Omega_{\mathbf{p}'}$ with $D^{(j)}_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})$ we obtain the unitarity condition in the form

$$\begin{aligned}
-i\Big[\mathcal{T}^{(j)*}_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}(s) - \mathcal{T}^{(j)}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(s)\Big] \\
= \frac{2}{s}\,\lambda^{1/2}(s, m_1^2, m_2^2) \sum_{\tilde\lambda_1,\tilde\lambda_2} N_{\tilde\lambda_1\tilde\lambda_2}\, \mathcal{T}^{(j)*}_{\tilde\lambda_1\tilde\lambda_2;\lambda_1'\lambda_2'}(s)\, \mathcal{T}^{(j)}_{\tilde\lambda_1\tilde\lambda_2;\lambda_1\lambda_2}(s) \\
+ \sum_{(ab)\neq(12)} \frac{2}{s}\,\lambda^{1/2}(s, m_a^2, m_b^2) \sum_{\lambda_a,\lambda_b} N_{\lambda_a\lambda_b}\, \tilde{\mathcal{T}}^{(j)*}_{\lambda_a\lambda_b;\lambda_1'\lambda_2'}(s)\, \tilde{\mathcal{T}}^{(j)}_{\lambda_a\lambda_b;\lambda_1\lambda_2}(s)\,,
\end{aligned} \tag{7.132}$$

where in the second line we have explicitly singled out the contribution of the elastic channel. In the fully elastic scattering channel *with no change of helicities*, i.e. for $\lambda_1' = \lambda_1$, $\lambda_2' = \lambda_2$, this can be rewritten in the form

$$\Big[\mathrm{Re}\,\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s)\Big]^2 + \left[\mathrm{Im}\,\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s) + \frac{s}{2N_{\lambda_1\lambda_2}\sqrt{\lambda_{12}(s)}}\right]^2 = \frac{s^2}{4N^2_{\lambda_1\lambda_2}\lambda_{12}(s)} - R_j^2(s)\,, \tag{7.133}$$

[44] Pairs (ab) and (ba) must be treated here as one and the same state i.e. only one of them should be included in the sum. Alternatively, one can sum over all a's and all b's, including in the sum both states, (ab) and (ba), but setting $N_{\lambda_a\lambda_b} = \frac{1}{2}$.

in which

$$R_j^2(s) = \sum_{(\lambda_1'\lambda_2')\neq(\lambda_1\lambda_2)} \frac{N_{\lambda_1'\lambda_2'}}{N_{\lambda_1\lambda_2}} \left|\mathcal{T}^{(j)}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(s)\right|^2 + \sum_{(ab)\neq(12)} \sum_{\lambda_a,\lambda_b} \frac{N_{\lambda_a\lambda_b}}{N_{\lambda_1\lambda_2}} \sqrt{\frac{\lambda_{ab}(s)}{\lambda_{12}(s)}} \left|\tilde{\mathcal{T}}^{(j)}_{\lambda_a\lambda_b;\lambda_1\lambda_2}(s)\right|^2, \tag{7.134}$$

and we have used the notation $\lambda_{12}(s) \equiv \lambda(s, m_1^2, m_2^2)$ and $\lambda_{ab}(s) \equiv \lambda(s, m_a^2, m_b^2)$.

It is easy to see that inelastic processes leading to multi-particle final states can also be included in $R_j^2(s)$. To show this without introducing explicit kinematical characterizations of the multi-particle states, we write, using the formula (6.110), the amplitudes of transitions from the $|\alpha_0\rangle$ and $|\beta_0\rangle$ two-particle states with the same particle content into any three- or more-particle state $|\gamma_0\rangle$ as

$$\mathcal{A}_{\gamma\beta} = \sum_{j'}^{\infty} \sum_{m_j'} \sqrt{\frac{2j'+1}{4\pi}}\, \mathcal{T}^{(j')}_{\gamma;\ \lambda_1',\lambda_2',m_j'}(\gamma; s')\, D^{(j')}_{m_j'\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})\,,$$

$$\mathcal{A}_{\gamma\alpha} = \sum_{j''}^{\infty} \sqrt{\frac{2j''+1}{4\pi}}\, \mathcal{T}^{(j'')}_{\gamma;\ \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s)\,, \tag{7.135}$$

where the amplitudes $\mathcal{T}^{(j)}_{\gamma;\ \lambda_1,\lambda_2,m_j}(\gamma; s)$ are defined by the equality

$$(2\pi)^4\delta^{(3)}(\mathbf{P}_\gamma)\delta(P_\gamma^0 - \sqrt{s})\, \mathcal{T}^{(j)}_{\gamma;\ \lambda_1,\lambda_2,m_j}(\gamma; s) \equiv \langle\gamma_0|T_0|\mathbf{0}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle\,. \tag{7.136}$$

The symbol γ used as the argument of $\mathcal{T}^{(j)}$ is to remind that this amplitude depends, apart from s, also on the variables (relative momenta and helicities) needed to specify the multiparticle state $|\gamma_0\rangle$. In $\mathcal{A}_{\gamma\alpha}$, similarly as in (7.127), the equality $D^{(j_\alpha)}_{m_j\lambda_1-\lambda_2}(\Omega_{\mathbf{p}}) = \delta_{m_j\lambda_1-\lambda_2}$ has been used. With these formulae, and setting $\lambda_1' = \lambda_1$, $\lambda_2' = \lambda_2$, the last line of (7.131) takes the form

$$\frac{1}{16\pi} \sum_{j''}^{\infty} \sum_{j'}^{\infty} \sum_{m_j'} \frac{\sqrt{(2j''+1)(2j'+1)}}{4\pi}\, D^{(j')*}_{m_j'\lambda_1-\lambda_2}(\Omega_{\mathbf{p}'}) \times \int d\gamma\, (2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\, \mathcal{T}^{(j')*}_{\gamma;\ \lambda_1,\lambda_2,m_j'}(\gamma; s)\, \mathcal{T}^{(j'')}_{\gamma;\ \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s)\,.$$

Integrating now as previously both sides of (7.131) with $D^{(j)}_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\Omega_{\mathbf{p}'})$ over $d\Omega_{\mathbf{p}'}$ we get in (7.132) an extra term

$$+\frac{1}{16\pi} \sum_{j''}^{\infty} \sqrt{\frac{2j''+1}{2j+1}} \int d\gamma\, (2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha) \times \mathcal{T}^{(j)*}_{\gamma;\ \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s)\, \mathcal{T}^{(j'')}_{\gamma;\ \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s)\,.$$

Figure 7.3: Partial amplitudes of the elastic scattering must lie on the Argand circles: if inelastic channels are closed, i.e. if $R_j^2(s) = 0$, (left) the radius of the circle is $s/2N_{\lambda_1\lambda_2}\sqrt{\lambda(s, m_1^2, m_2^2)}$; if inelastic channels are open (right) it is smaller. Short-dashed lines show possible partial elastic scattering amplitudes in a weakly coupled theory (small corrections in the perturbative expansion) whereas the long-dashed ones illustrate elastic scattering amplitudes typical for a strongly coupled (nonperturbative) theory.

However, angular momentum conservation implies that only $j'' = j$ can contribute to the sum and therefore the last term in (7.131) adds to R_j^2 given by (7.134) a strictly nonnegative contribution proportional to

$$\frac{1}{16\pi}\int d\gamma\,(2\pi)^4\delta^{(4)}(P_\gamma - P_\alpha)\,\left|\mathcal{T}^{(j)}_{\gamma;\;\lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s)\right|^2. \tag{7.137}$$

The relation (7.133) demonstrates that the amplitude $\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s)$ of the elastic scattering with no change of helicities must lie on a circle, called the *Argand circle*, of radius not grater than $s/2N_{\lambda_1\lambda_2}\sqrt{\lambda_{12}(s)}$ (in the limit $\sqrt{s}\to\infty$ this bound tends to the finite value equal $1/2N_{\lambda_1\lambda_2}$) and the center located at the point $(0, -s/2N_{\lambda_1\lambda_2}\sqrt{\lambda_{12}(s)}\,)$ in the complex plane, as shown graphically in Figure 7.3. This shows, that the elastic scattering amplitude must have a nonzero imaginary part which grows (because the radius of the Argand circle shrinks due to increasing of the factor $R_j^2(s)$ in (7.133)) as more and more inelastic channels open up with increasing $\sqrt{s}$ (at high energies elastic scattering amplitudes are therefore predominantly imaginary). From (7.133) it also follows, that the fully elastic (with no change of helicities) scattering partial wave amplitude $\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s)$ can be represented in terms of the (complex) *phase shift* $\delta_j(s) + i\beta_j(s)$

$$\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s) = i\,\frac{s}{2N_{\lambda_1\lambda_2}\sqrt{\lambda_{12}(s)}}\left(e^{2i\delta_j(s)-2\beta_j(s)} - 1\right), \tag{7.138}$$

where $0 < \delta_j(s) < \pi$ and

$$\beta_j(s) = -\frac{1}{4}\ln\left(1 - \frac{4N_{\lambda_1\lambda_2}^2\lambda(s)}{s^2}\,R_j^2(s)\right). \tag{7.139}$$

Of course, if only the elastic channel with no helicity change is open $\beta_j(s) = 0$ and the phase shifts are real numbers.

These results, used in (7.125) and combined with the formula (6.108), allow to write down the S-matrix element corresponding to an elastic (with no change of helicities) scattering in the basis of states with definite angular momentum in the form

$$\begin{aligned}\langle \mathbf{P}', \sqrt{s'}, \lambda_1, \lambda_2, j', m_j' | \hat{1} - iT_0 | \mathbf{P}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j \rangle \qquad (7.140)\\ = (2\pi)^4 \delta^{(4)}(P' - P)\, 64\pi^2\, \delta_{j'j} \delta_{m_j' m_j}\, \frac{s}{2N_{\lambda_1\lambda_2}\sqrt{\lambda_{12}(s)}}\, e^{2i\delta_j(s) - 2\beta_j(s)}\,,\end{aligned}$$

where if the particles in the initial state are identical, the factor $N^{-1}_{\lambda_1\lambda_2}$ should be written as $1 + (-1)^j \delta_{\lambda_1\lambda_2}$ to account for the fact that the state-vectors $|\mathbf{P}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle$ vanish in such a case if j is odd and $\lambda_1 = \lambda_2$. We have also used the fact that $|\mathbf{p}|$ (the length of the momentum of the first particle in the CM frame) in the formula (6.108) is given by

$$|\mathbf{p}| = \frac{1}{2\sqrt{s}}\, \lambda^{1/2}(s, m_1^2, m_2^2)\,. \qquad (7.141)$$

The restriction to the elastic scattering amplitude with no helicity flip i.e. to $\lambda_1' = \lambda_1$, $\lambda_2' = \lambda_2$ can be removed by diagonalizing the elastic scattering amplitude in the spin space

$$\delta_{kl} \mathcal{T}^{(j,k)}(s) = (U^\dagger)_{k,(\lambda_1'\lambda_2')}\, \mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s)\, U_{(\lambda_1\lambda_2)l}\,, \qquad (7.142)$$

with the help of a $(2s_1+1)\times(2s_2+1)$ unitary (s dependent) matrix U and writing the unitarity condition (7.117) in the basis in which $\mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}$ is diagonal (it suffices to sandwich the relation (7.132) between $(U^\dagger)_{k,(\lambda_1'\lambda_2')}$ and $U_{l(\lambda_1\lambda_2)}$). One then gets the representations (7.133), and consequently, also (7.138) with appropriate phase shifts $\delta_k^{(j)}(s) + i\beta_k^{(j)}(s)$, for each of the elastic scattering amplitudes $\mathcal{T}^{(j,k)}(s)$ with $R^2_{jk}(s)$ which now does not include the contribution from the elastic channel.

Since the S-matrix is unitary and its eigenvalues must be complex numbers of unit modulus, it is in principle possible, by appropriately choosing bases of the Hilbert subspaces corresponding to definite values j of the total angular momentum, i.e. by using in each such subspace (originally spanned by state vectors $|\alpha_0^{(j)}\rangle$ representing two-, three- and more particles with the angular momentum j) a unitary s-dependent transformation $|\alpha_0^{(j)}\rangle\, U^{(j)}_{\alpha,\tilde{\alpha}}(s) = |\tilde{\alpha}_0^{(j)}\rangle$, to diagonalize it completely.[45] This would lead to only purely elastic diagonal amplitudes $\mathcal{T}^{(j)}_{(\tilde{\alpha})}(s)$ with purely real phase shifts $\delta_j^{(\tilde{\alpha})}(s)$. In the bases $|\tilde{\alpha}_0^{(j)}\rangle$, the S-matrix would take the simple form

$$S_{\tilde{\beta}^{(j')}\tilde{\alpha}^{(j)}} \equiv \langle \tilde{\beta}^{(j')} | S_0 | \tilde{\alpha}^{(j)} \rangle = \langle \tilde{\beta}^{(j')} | \tilde{\alpha}^{(j)} \rangle\, e^{2i\delta_j^{(\tilde{\alpha})}(s)}\,. \qquad (7.143)$$

[45] The bases $|\tilde{\alpha}_0^{(j)}\rangle$ in which the S-matrix is diagonal may not, for general values of the Mandelstamm variable s, consist of experimentally realizable states: for example in the case of two particle processes they are usually linear combinations of state-vectors representing different pairs of particles.

This makes it clear that the S-matrix eigenvalues (eigenvalues of the S_0 operator) are the (properly defined) phase shift factors $e^{2i\delta_j^{(\tilde{\alpha})}(s)}$ - a fact which at first sight may be obscured by the formula (7.140) which gives the S-matrix elements in the original basis of the Hilbert subspace spanned by the vectors representing definite pairs of particle with fixed helicities.

Since not all elements of the scattering matrix are known (or can be effectively computed, even if the relevant theory is given), symmetries of the interactions can be exploited in practice to diagonalize the scattering amplitudes at least at low energies (small $\sqrt{s}$), when only a limited number of channels can be reached from a given initial state due to kinematical restrictions. A canonical example is provided by the strong interactions of low energy pions (the lightest strongly interacting particles), if the electromagnetic and weak interactions are neglected: due to the isospin symmetry of the strong interactions (see Chapter 12) their S-matrix is diagonal in the isospin basis (pions are spinless)

$$\langle I', I_3', \mathbf{p}, -\mathbf{p}|S_0|I, I_3, \mathbf{k}, -\mathbf{k}\rangle = \delta_{I'I}\delta_{I_3'I_3}\, S^{(I)}(s)\,, \tag{7.144}$$

and in the isospin-angular momentum basis $|I, I_3, j, m\rangle$ the formula (7.143) takes the form

$$\begin{aligned}\langle \mathbf{0}, \sqrt{s'}, I', I_3', j', m_j'|S_0|\mathbf{P}, \sqrt{s}, I, I_3, j, m_j\rangle &= \langle \mathbf{0}, \sqrt{s'}, I', I_3', j', m_j'|\mathbf{P}, \sqrt{s}, I, I_3, j, m_j\rangle\, e^{2i\delta_j^I(s)} \\ &\propto \delta_{I',I}\delta_{I_3',I_3}\, e^{2i\delta_j^I(s)}\,,\end{aligned} \tag{7.145}$$

(the proportionality factor is as in (7.140)) where the (purely real) isospin-angular momentum phase shifts $\delta_j^I(s)$ parametrize the corresponding isospin partial wave amplitudes $\mathcal{T}_I^{(j)}(s)$. The combined formulae (7.145) and (7.60) are the basis for accounting for the final state re-scattering effects in decay processes induced by the weak interactions. They are particularly important in the analysis of the CP violation in the kaon system (see Section 12.4).

From the relation (7.133) or directly from Figure 7.3 one gets two *unitarity bounds* pertaining to elastic scattering (with no change of helicities) partial wave amplitudes:

$$\begin{aligned} N_{\lambda_1\lambda_2}\left|\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s)\right| &\leq \frac{s}{\lambda^{1/2}(s, m_1^2, m_2^2)}\,, \\ N_{\lambda_1\lambda_2}\left|\mathrm{Re}\,\mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s)\right| &\leq \frac{s}{2\lambda^{1/2}(s, m_1^2, m_2^2)}\,. \end{aligned} \tag{7.146}$$

Moreover, since R_j^2 cannot exceed $s^2/4N^2_{\lambda_1\lambda_2}\lambda(s, m_1^2, m_2^2)$ (the right hand side of (7.133) must be positive), one gets also the bounds on partial wave amplitudes of any two body (not necessarily elastic) scattering:[46]

$$\sqrt{N_{\lambda_a\lambda_b}N_{\lambda_1\lambda_2}}\left|\tilde{\mathcal{T}}^{(j)}_{\lambda_a\lambda_b;\lambda_1\lambda_2}(s)\right| \leq \frac{s}{2\lambda^{1/4}(s, \tilde{m}_a^2, \tilde{m}_b^2)\lambda^{1/4}(s, m_1^2, m_2^2)}\,. \tag{7.147}$$

[46]It is clear that the quantities (7.137) related to the contribution of multi-particle production to the total cross section are also bounded by this requirement.

Notice, that at the reaction threshold, where $\lambda^{1/2}(s, m_a^2, m_b^2) = 0$, the bounds (7.147) and (7.146) disappear. If $\sqrt{s}$ is much greater than any of the masses involved, the unitarity bounds become[47]

$$\begin{aligned} N_{\lambda_1\lambda_2} \left| \mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s) \right| &\leq 1\,, \\ N_{\lambda_1\lambda_2} \left| \mathrm{Re}\, \mathcal{T}^{(j)}_{\lambda_1\lambda_2;\lambda_1\lambda_2}(s) \right| &\leq \frac{1}{2}\,, \\ \sqrt{N_{\lambda_a\lambda_b} N_{\lambda_1\lambda_2}} \left| \tilde{\mathcal{T}}^{(j)}_{\lambda_a\lambda_b;\lambda_1\lambda_2}(s) \right| &\leq \frac{1}{2}\,. \end{aligned} \tag{7.148}$$

The bounds (7.146) and (7.147) have been obtained assuming only that the evolution of the quantum system is unitary.[48] In particular they do not rely on any perturbative expansion. Scattering amplitudes derived from local quantum field theory models which (are believed to) give rise to unitary S-matrices should in principle, respect these bounds. Since elastic scattering partial wave amplitudes computed in the lowest order of the perturbative expansion in quantum field theories are (usually) real (i.e. lie on the horizontal axis in Figure 7.3), they cannot satisfy the unitarity relation (7.133). Higher order contributions must therefore bring elastic amplitudes back on the Argand circle. Two distinct situations can be then encountered. If the (absolute value of the) real part of the lowest order amplitude is bounded by $1/2N_{\lambda_1\lambda_2}$, higher order contributions required to restore unitarity can be relatively small (short dashed lines in Figure 7.3) and the perturbative expansion is likely to be reliable. In contrast, if the real part of the lowest order amplitude greatly exceeds $1/2N_{\lambda_1\lambda_2}$, the necessary higher order contributions must be comparable or even larger than the lowest order term and the perturbative expansion evidently fails. In specific quantum field theory models the magnitude of the lowest order amplitudes depends usually on the energy $\sqrt{s}$. In renormalizable theories (see Chapter 14) the lowest order amplitudes are bounded for $\sqrt{s} \to \infty$ by some constants and reliability of the perturbation expansion depends on the magnitude of such a limiting value (whether such a constant is smaller or bigger than $1/2N_{\lambda_1\lambda_2}$). In nonrenormalizable theories the lowest order amplitudes usually grow with $\sqrt{s}$ and above some critical energy the perturbation expansion unavoidably breaks down.

Example of the latter situation is provided by the phenomenological Fermi theory of weak interactions (introduced in Chapter 12). Amplitudes computed in this model in the lowest order grow linearly with s and violate the unitarity bounds (7.148) at $\sqrt{s} \gtrsim 600$ GeV, indicating that at such energies either the theory of weak interaction becomes strongly coupled (and the perturbative expansion cannot be applied to it) or

[47] In the literature it is customary to include the factor $\sqrt{N_{\lambda_a\lambda_b} N_{\lambda_1\lambda_2}}$ in $\mathcal{T}^{(j)}_{\lambda_a\lambda_b;\lambda_1\lambda_2}(s)$. In this way the $N_{\lambda_1\lambda_2}$ factors disappear altogether from the formulae (7.131)-(7.138) and the unitarity bounds (7.148) for distinct and identical particles look the same. We preferred not to do so, in order to keep control over such factors and, moreover, because the factors $\sqrt{N_{\lambda_1\lambda_2}}$ should *not* be included (for the cross section calculation) in the amplitudes of scatterings of identical particles in the initial state.

[48] Another assumption is that the partial wave expansion of amplitudes makes sense. This may not be true in the presence of massless particles which produce long range interactions.

that the Fermi theory is only an effective model which should be replaced by a more fundamental theory in which exchanges of new particles restore unitarity of amplitudes computed in the lowest order. It is the second option that is realized in the Nature - the Fermi theory turned out to provide only an effective, low energy approximation to the results obtained in the Standard Theory of electroweak interactions.

Unitarity bounds derived in this section were also important in discussing (before the year 2012) possible versions of the extension (ultraviolet completion in the modern parlance) of the theory of electroweak interactions in which exchanges of (virtual) massive spin 1 (vector) bosons - particles the existence of which had been already experimentally well established - replace the contact interactions of the Fermi theory. Scattering amplitudes of longitudinally polarized vector bosons computed within this theory in the lowest order grow linearly with s and violate these bounds if the contributions of the sector of the theory responsible for electroweak symmetry breaking is not taken into account. This sector was, before 2012, experimentally unexplored (and remains largely such even now). The Standard Model - the concrete renormalizable realization of such an extension (deviced - using the input provided by works of A. Salam and S. Glashow - by S. Weinberg around 1968), in which the electroweak gauge symmetry is broken by a single doublet of scalar fields - predicted the existence, without fixing its mass, of a single neutral spinless particle, h^0, whose contribution to the discussed scattering amplitudes cuts their rising with s - they reach a constant value proportional to the mass squared of h^0. It could be then concluded that the bounds would be violated if the mass of h^0 were greater than ~ 1 TeV. The discovery of h^0 with mass equal 125 GeV, giving strong support for the mechanism of electrowek symmetry breaking realized in the Standard Model, ended (forever?) the speculations concerning more exotic possibilities of saving unitarity in the scattering of longitudinally polarized electroweak vector bosons.

7.7 Other symmetries of the S-matrix

Discrete symmetries: parity, time reversal and charge conjugation may or may not be exact symmetries of a given model of relativistic quantum mechanics (a quantum field theory model). Even if they are not exact symmetries of the real world (we know they are not), it is still interesting to consider them because in physics, in contrast to pure mathematics in which the statement that some operation is not a symmetry closes the issue, we are interested not only in whether they are symmetries but also, *how* they are violated, i.e. by which type of interactions and in which processes.

The action of the parity operator on state-vectors representing single particles has been discussed in Section 6.3. Eigenvectors of H_0 representing states of several free particles transform of course as tensor products:

$$\mathcal{P}_0|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_0\rangle = \eta_1\ldots\eta_N|(-\mathbf{p}_1\sigma_1,\ldots,-\mathbf{p}_N\sigma_N)_0\rangle \tag{7.149}$$

(we have assumed that all particles are massive; modifications for massless particles are

obvious). The same formula, with $\mathcal{P}$ instead of $\mathcal{P}_0$ applies to the *in* and *out* states.

If $\mathcal{P}_0$ commutes with the interaction operator $V_{\rm int}$ (and with H_0, so in all with H), one can set $\mathcal{P} = \mathcal{P}_0$. The S-matrix satisfies then the following identity

$$S_{\mathbf{p}'_1\sigma'_1,\mathbf{p}'_2\sigma'_2\ldots;\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2\ldots} = \eta^*_{1'}\eta^*_{2'}\ldots\eta_1\eta_2\ldots S_{-\mathbf{p}'_1\sigma'_1,-\mathbf{p}'_2\sigma'_2\ldots;-\mathbf{p}_1\sigma_1-\mathbf{p}_2\sigma_2\ldots}\,. \tag{7.150}$$

This shows that if parity is conserved in elementary processes, then for $\eta^*_{1'}\eta^*_{2'}\ldots\eta_1\eta_2\ldots = +1$ the S-matrix is an even function of the particle momenta, whereas for $\eta^*_{1'}\eta^*_{2'}\ldots\eta_1\eta_2\ldots = -1$ it must be an odd function.

What are the internal parities η of various known elementary particles? Parities η of a small set of particles can be fixed by a convention. This is because in the real world the parity operator $\mathcal{P} = \mathcal{P}_0$ can be redefined

$$\mathcal{P}' = \mathcal{P}\, e^{-ic_Q\hat{Q}-ic_B\hat{B}-ic_L\hat{L}-\ldots}\,, \tag{7.151}$$

where $\hat{Q}$, $\hat{B}$ and $\hat{L}$ are the operators of the electric charge, baryon and lepton numbers and the ellipses stand for other quantum numbers conserved in parity-conserving interactions.[49] If $\mathcal{P}$ commutes with the Hamiltonian so does $\mathcal{P}'$. This freedom in the definition of the parity operator allows to assign (by convention) $\eta = +1$ to p, n and e^- (these particles have all different combinations of $\hat{B}$, $\hat{L}$ and $\hat{Q}$ and one can choose c_1, c_2 and c_3 so that the action of $\mathcal{P}'$ is consistent with this assignment). In the same way using the factor with the strangeness or charm operators S, C one can by convention assign $\eta = 1$ to one strange baryon and one charmed baryon.

Internal parities of the remaining particles should be assigned in such a way that parity is conserved in as broad class of processes as possible. Consider for example the process of radiative capturing of π^- in the "π meson deuteron" i.e. in the $d\,\pi^-$ bound state: $d + \pi^- \to n + n$. The capturing occurs from the orbital $l = 0$ ground state of such an "atom". The initial state has $j = 1$ (deuteron is predominantly an $l = 0$ proton-neutron bound state with the total spin $s = 1$) and, therefore, the final state should also have $j = 1$. Since the final state, being the state of the two identical fermions, must be antisymmetric in their labels, it must have $s = 1$, $l = 1$ (other possibilities consistent with $j = 1$: $s = 0,\ l = 1$ or $s = 1,\ l = 0$ or $s = 1,\ l = 2$ all lead to symmetric final states). Thus, if parity is conserved, we must have

$$\eta_d\eta_{\pi^-} = -\eta_n^2\,. \tag{7.152}$$

Since $\eta_d = +1$ ($\eta_p = \eta_n = +1$ by convention and $l = 0$), it follows that

$$\eta_{\pi^-} = -1\,. \tag{7.153}$$

[49] Even if the lepton and baryon numbers are not strictly conserved - at least as far as the lepton number is concerned, there are strong indications that it is indeed violated by interactions responsible for small, but experimentally well established, neutrino masses - this is irrelevant here, because they are conserved in the same interactions which preserve parity.

From the isospin symmetry (see section 12) it then follows that also

$$\eta_{\pi^+} = \eta_{\pi^0} = -1\,. \tag{7.154}$$

Negative parity of the π mesons has important consequences. One is that if parity is conserved, a particle decaying into three pions must necessarily have negative parity. Indeed, in the rest frame of the decaying particle the S-matrix element corresponding to its decay into three pions can depend only on $\mathbf{p}_1\cdot\mathbf{p}_2$, $\mathbf{p}_1\cdot\mathbf{p}_3$ or $\mathbf{p}_2\cdot\mathbf{p}_3$ (as pions are spinless, no other vectors are available; moreover, since in this frame $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=\mathbf{0}$ the triple product $(\mathbf{p}_1\times\mathbf{p}_2)\cdot\mathbf{p}_3=0$). Hence, parity of the final state is negative and so must be parity of the initial state. Similarly, a particle decaying into two pions must necessarily have positive parity. Non-conservation of parity in weak interactions indicated first by the the observation of K^+ decays into two and three pions became evident around 1950 (see Chapter 12). It is by now firmly established and is one of the cornerstones of the theory of electroweak interactions (Chapter **??**).

The action of the time reversal operator on state-vectors of single particles has been given in Section 6.3. Its action on free multi-particle state-vectors therefore reads

$$\begin{aligned}&\mathcal{T}_0|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_0\rangle\\ &\qquad=\zeta_1(-1)^{s_1-\sigma_1}\ldots\zeta_N(-1)^{s_N-\sigma_N}|(-\mathbf{p}_1\,-\sigma_1,\ldots,-\mathbf{p}_N\,-\sigma_N)_0\rangle\,.\end{aligned} \tag{7.155}$$

(as for parity we have assumed that all particles are massive; modifications for massless particles are obvious). If $\mathcal{T}_0$ commutes with the interaction operator $V_{\rm int}$, we can set $\mathcal{T}=\mathcal{T}_0$. It is also easy to see that the action of $\mathcal{T}$ changes the *in* states into the *out* states and vice-versa:

$$\begin{aligned}&\mathcal{T}|(\mathbf{p}_1\sigma_1,\ldots,\mathbf{p}_N\sigma_N)_\pm\rangle\\ &\qquad=\zeta_1(-1)^{s_1-\sigma_1}\ldots\zeta_N(-1)^{s_N-\sigma_N}|(-\mathbf{p}_1\,-\sigma_1,\ldots,-\mathbf{p}_N\,-\sigma_N)_\mp\rangle\,.\end{aligned} \tag{7.156}$$

This can be seen from the formal expression (7.39):

$$\begin{aligned}\mathcal{T}|\alpha_\pm\rangle&=\lim_{\tau\to\mp\infty}\mathcal{T}_0\;e^{iH\tau}e^{-iH_0\tau}|\alpha_0\rangle=\lim_{\tau\to\mp\infty}e^{-iH\tau}e^{iH_0\tau}\mathcal{T}_0|\alpha_0\rangle\\ &=\lim_{\tau\to\pm\infty}e^{iH\tau}e^{-iH_0\tau}|(\mathcal{T}\alpha)_0\rangle\equiv|(\mathcal{T}\alpha)_\mp\rangle\,,\end{aligned} \tag{7.157}$$

(recall that $\mathcal{T}$ is antilinear!). We have introduced the compact notation $|(\mathcal{T}\alpha)\rangle$ for the state (including its phase factors) appearing on the right hand side of (7.156); thus if $|\alpha_0\rangle=|(\mathbf{p},\sigma\rangle)_0$, then $|(\mathcal{T}\alpha)_0\rangle\equiv\zeta(-1)^{s-\sigma}|(-\mathbf{p},-\sigma)_0\rangle$. The same conclusion follows also from the expression (7.54).

If the time reversal is a symmetry operation, from the properties of antiunitary operators (see Chapter 4) it follows that[50]

$$\begin{aligned}S_{\beta\alpha}&=(\beta_-|\alpha_+)=(\beta_-|\mathcal{T}^{-1}\mathcal{T}\alpha_+)=(\beta_-|\mathcal{T}^\dagger\mathcal{T}\alpha_+)\\ &=(\mathcal{T}\beta_-|\mathcal{T}\alpha_+)^*=(\mathcal{T}\alpha_+|\mathcal{T}\beta_-)=((\mathcal{T}\alpha)_-|(\mathcal{T}\beta)_+)\equiv S_{(\mathcal{T}\alpha)(\mathcal{T}\beta)}\,,\end{aligned} \tag{7.158}$$

[50]We have to abandon for a while the Dirac bra-ket notation. The alternative derivation of this result

where if $S_{\beta\alpha} \equiv S_{\mathbf{p}'_1\sigma'_1,\mathbf{p}'_2\sigma'_2\ldots;\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2\ldots}$, the symbol on the right hand side should be understood as

$$S_{(\mathcal{T}\alpha)(\mathcal{T}\beta)} \equiv \zeta^*_{1'}(-1)^{s_{1'}-\sigma_{1'}}\zeta^*_{2'}(-1)^{s_{2'}-\sigma_{2'}}\ldots\zeta_1(-1)^{s_1-\sigma_1}\zeta_2(-1)^{s_2-\sigma_2}\ldots \\ \times S_{-\mathbf{p}_1-\sigma_1-\mathbf{p}_2-\sigma_2\ldots;-\mathbf{p}'_1-\sigma'_1,-\mathbf{p}'_2-\sigma'_2\ldots} \tag{7.159}$$

Notice, that in general the time reversal does not imply that the rate of the reaction $\alpha \to \beta$ is the same as of the reaction $\mathcal{T}\alpha \to \mathcal{T}\beta$. There are however some special situations in which it does imply this. This is when the S-matrix can be split into two parts ($S^{\text{strong}}_{\beta\alpha}$ can be viewed as the zeroth order term in the expansion of $S_{\beta\alpha}$ in some small parameter)

$$S_{\beta\alpha} = S^{\text{strong}}_{\beta\alpha} + S^{\text{weak}}_{\beta\alpha}\,, \tag{7.160}$$

where $|S^{\text{weak}}_{\beta\alpha}| \ll |S^{\text{strong}}_{\beta\alpha}|$ because S^{strong} is due to the strong interactions, whereas S^{weak} is due to the weak ones. In the first order in $S^{\text{weak}}_{\beta\alpha}$ the unitarity condition for S then reads

$$\hat{1} = S^\dagger S = S^{\text{strong}\dagger}S^{\text{strong}} + S^{\text{strong}\dagger}S^{\text{weak}} + S^{\text{weak}\dagger}S^{\text{strong}}\,. \tag{7.161}$$

So, approximately,

$$S^{\text{weak}} \approx -S^{\text{strong}}S^{\text{weak}\dagger}S^{\text{strong}}\,, \tag{7.162}$$

because $S^{\text{strong}\dagger}S^{\text{strong}} = \hat{1}$ (the zeroth order terms must match). More concretely,

$$\begin{aligned} S^{\text{weak}}_{\beta\alpha} &\approx -\int d\gamma \int d\gamma' S^{\text{strong}}_{\beta\gamma}\left[S^{\text{weak}\dagger}\right]_{\gamma\gamma'} S^{\text{strong}}_{\gamma'\alpha} \\ &= -\int d\gamma \int d\gamma' S^{\text{strong}}_{\beta\gamma}\left[S^{\text{weak}}_{\gamma'\gamma}\right]^* S^{\text{strong}}_{\gamma'\alpha} \\ &= -\int d\gamma \int d\gamma' S^{\text{strong}}_{\beta\gamma}\left[S^{\text{weak}}_{(\mathcal{T}\gamma)(\mathcal{T}\gamma')}\right]^* S^{\text{strong}}_{\gamma'\alpha}\,. \end{aligned} \tag{7.163}$$

This relation is particularly useful if, as in the case of the nuclear β decays, $S^{\text{strong}}_{\beta\alpha} = 0$ for the relevant states α and β (the process cannot occur through the strong interactions). In the basis of states with fixed total angular momentum using the formula (7.143) $S^{\text{strong}}_{\gamma\alpha} = \delta_{\gamma\alpha}\exp(2i\delta_\alpha)$ we get from (7.163)

$$S^{\text{weak}}_{\beta\alpha} \approx -e^{2i(\delta_\beta+\delta_\alpha)}\left[S^{\text{weak}}_{(\mathcal{T}\beta)(\mathcal{T}\alpha)}\right]^*\,. \tag{7.164}$$

which does imply that the rate of the process $\alpha \to \beta$ is approximately[51] the same as the rate of the reaction $(\mathcal{T}\alpha) \to (\mathcal{T}\beta)$.

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is ($\mathcal{T} = \mathcal{T}_0$)

$$S_{\beta\alpha} = (\beta_0|S_0\alpha_0) = (\beta_0|\mathcal{T}^\dagger\mathcal{T}S_0\alpha_0) = (\mathcal{T}\beta_0|S_0^\dagger\mathcal{T}\alpha_0)^* = ((\mathcal{T}\alpha)_0|S_0(\mathcal{T}\beta)_0) = S_{(\mathcal{T}\alpha)(\mathcal{T}\beta)}\,,$$

because $\mathcal{T}S_0 = S_0^\dagger\mathcal{T}$, owing to the antiunitarity of $\mathcal{T}$.

[51] Since the weak interaction is much much weaker than the strong one, this is in fact an almost perfect approximation.

Action of the charge conjugation operator $\mathcal{C}_0$ on free one-particle states was defined in Section 6.3. If the interaction operator $V_{\rm int}$ commutes with $\mathcal{C}_0$, one can take $\mathcal{C} = \mathcal{C}_0$ and the charge conjugation symmetry implies that

$$S_{\mathbf{p}'_1\sigma'_1,\mathbf{p}'_2\sigma'_2,\ldots;\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2,\ldots} = \xi^*_{1'}\xi^*_{2'}\ldots\xi_1\xi_2\ldots S_{\bar{\mathbf{p}}'_1\bar{\sigma}'_1,\bar{\mathbf{p}}'_2\bar{\sigma}'_2,\ldots;\bar{\mathbf{p}}_1\bar{\sigma}_1,\bar{\mathbf{p}}_2\bar{\sigma}_2,\ldots}\,, \tag{7.165}$$

where we have denoted antiparticle momentum and spin labels by bars.

Charge conjugation parities of different particles are assigned in a similar way as the intrinsic parities. First of all, $\mathcal{C}$ can always be redefined

$$\mathcal{C}' = \mathcal{C}\, e^{-ia_1\hat{Q}-ia_2\hat{B}-ia_3\hat{L}}\,, \tag{7.166}$$

so that for three particles that have different Q, B and L charge conjugation parities C can be fixed by a convention. Furthermore, charge conjugation parities of neutral particles, which like photon or neutral pion do not carry any conserved quantum numbers, are uniquely determined. As will become evident in Chapter 8, a state of a (massive) fermion-antifermion pair has negative charge conjugation parity (more precisely, the electromagnetic current operator $J^\mu_{\rm EM}$ is such that $\mathcal{C}J^\mu_{\rm EM}\mathcal{C}^{-1} = -J^\mu_{\rm EM}$). Therefore, the photon which couples to such pairs must also have negative charge conjugation parity (i.e. the photon field operator must have the property $\mathcal{C}A^\mu\mathcal{C}^{-1} = -A^\mu$). Since π^0 decays into two photons, $\xi_{\pi^0} = +1$ and, by isospin symmetry, the same must be also true for $\pi^\pm$. It then follows that the process $\pi^0 \to 3\gamma$ is forbidden (experimentally $\mathrm{Br}(\pi^0 \to 3\gamma) < 3.1\times10^{-8}$).

7.8 The cluster decomposition principle

In this section we shall briefly discuss, without entering into details, the so-called cluster decomposition principle, which any physically sensible S-matrix should satisfy in order the theory predictions for measurements made in spatially remote laboratories be uncorrelated. The cluster decomposition principle imposes a simple but nontrivial constraint on the general structure of quantum field theory Hamiltonians expressed in terms of the creation and annihilation operators of free particles in the momentum representation.

We recall first that, as discussed in Chapter 5, any operator O acting in the Hilbert space spanned by the states (7.1) can be represented in the general form[52] (5.80) which in the context of a relativistic theory can be written as (for simplicity of the notation we consider here only one type of spinless particles)

$$O = \sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\int d\Gamma_{\mathbf{p}_1}\ldots\int d\Gamma_{\mathbf{p}_M}\int d\Gamma_{\mathbf{q}_1}\ldots\int d\Gamma_{\mathbf{q}_N} \tag{7.167}$$
$$C_{MN}(\mathbf{p}_1,\ldots,\mathbf{p}_M;\mathbf{q}_1,\ldots,\mathbf{q}_N)\, a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_M)\, a(\mathbf{q}_1)\ldots a(\mathbf{q}_N)\,,$$

[52] The multi-particle states considered here can be the free multi-particle states but can also be the *in* or the *out* states, with which the corresponding creation and annihilation operators $a^\dagger_{\rm in}(\mathbf{p})$, $a_{\rm in}(\mathbf{p})$ and $a^\dagger_{\rm out}(\mathbf{p})$, $a_{\rm out}(\mathbf{p})$ can also be associated (see Section 8.7).

with appropriately chosen functions $C_{MN}(\mathbf{p}_1,\ldots,\mathbf{p}_M;\mathbf{q}_1,\ldots,\mathbf{q}_N)$. Relativistic invariance will not, however, be relevant in the considerations presented below.

Consider now the S-matrix of a theory written in the position representation

$$S_{\mathbf{y}_1,\ldots,\mathbf{y}_M;\mathbf{x}_1,\ldots,\mathbf{x}_N} = \int d\Gamma_{\mathbf{p}_1}\ldots\int d\Gamma_{\mathbf{p}_M}\int d\Gamma_{\mathbf{q}_1}\ldots\int d\Gamma_{\mathbf{q}_N} \tag{7.168}$$
$$e^{-i\mathbf{p}_1\cdot\mathbf{y}_1}\ldots e^{-i\mathbf{p}_M\cdot\mathbf{y}_M}e^{i\mathbf{q}_1\cdot\mathbf{x}_1}\ldots e^{i\mathbf{q}_N\cdot\mathbf{x}_N}\,S_{\mathbf{p}_1,\ldots,\mathbf{p}_M;\mathbf{q}_1,\ldots,\mathbf{q}_N}\,,$$

and imagine a process $\alpha\to\beta$ (where α and β stand for collections of positions and spins of the initial and final state particles, respectively), in which particles in the initial and final states exhibit some clustering: $\alpha=(\alpha_1)(\alpha_2)(\alpha_3)\ldots(\alpha_n)$, $\beta=(\beta_1)(\beta_2)(\beta_3)\ldots(\beta_n)$. By clustering we mean that the subprocesses $\alpha_1\to\beta_1$, $\alpha_2\to\beta_2$, $\ldots$,$\alpha_n\to\beta_n$ are measured separately in remote laboratories (like e.g. CERN, SLAC and FNAL). The S-matrix satisfies the cluster decomposition principle if it factorizes in such a case, that is, if

$$S_{\beta\alpha}=S_{\beta_1\alpha_1}S_{\beta_2\alpha_2}S_{\beta_3\alpha_3}\ldots S_{\beta_n\alpha_n}\,, \tag{7.169}$$

when the distances $|\mathbf{x}_i-\mathbf{x}_j|$ and $|\mathbf{y}_i-\mathbf{y}_j|$ are large if $\mathbf{x}_i$, $\mathbf{y}_i$ and $\mathbf{x}_j$, $\mathbf{y}_j$ belong to different clusters[53] (large compared to the typical distances in the same cluster). This in turn ensures independence of the theory predictions for experiments performed in different laboratories: if we are interested in the probability of a concrete final state $|\beta_1\rangle$ in an experiment performed e.g. at CERN, we sum over all possible final states which can be found in far-away laboratories:

$$\begin{aligned}P(\alpha_1\to\beta_1) &= \sum_{\beta_2,\beta_3,\ldots} S_{\beta\alpha}S^*_{\beta\alpha} \\ &= S_{\beta_1\alpha_1}S^*_{\beta_1\alpha_1}\sum_{\beta_2}S_{\beta_2\alpha_2}S^*_{\beta_2\alpha_2}\sum_{\beta_3}S_{\beta_3\alpha_3}S^*_{\beta_3\alpha_3}\ldots = S_{\beta_1\alpha_1}S^*_{\beta_1\alpha_1}\,.\end{aligned} \tag{7.170}$$

where in the last step unitarity of the S matrix has been used.

To implement the factorization of the S matrix let us first define its *connected* part $S^C_{\beta\alpha}$ by using a combinatoric trick.[54] We write the S-matrix as the sum

$$S_{\beta\alpha}=\sum_{\text{partitions}}(\pm)S^C_{\beta_1\alpha_1}S^C_{\beta_2\alpha_2}\ldots\,, \tag{7.171}$$

over all possible partitions of the individual particle labels in α and β into clusters $(\alpha_1)(\alpha_2)(\alpha_3)\ldots(\alpha_n)$, $(\beta_1)(\beta_2)(\beta_3)\ldots(\beta_n)$ (not treating as different those partitions which

[53]The factorization property of the S matrix is not in conflict with the phenomenon of entanglement and the well known Einstein Podolski Rosen correlations: entangled (correlated) particles must have interacted (e.g. they originate from a decay of another particle) before the measurements of their properties are made and, hence, must belong to the same cluster β_i.

[54]The term "connected" derives from the form of the Feynman diagrams (to be introduced in Section 9) which contribute to this part of the S-matrix but the trick itself can be implemented without any reference to diagrams.

differ only by a permutation of labels within the same group, or differ by a permutation of the clusters as wholes). The sign $\pm$ depends on whether the number of interchanges of fermionic labels is even or odd. The definition (7.171) is recursive:

$$S_{\beta\alpha} = S^C_{\beta\alpha} + \sum_{\text{partitions}'} (\pm) S^C_{\beta_1\alpha_1} S^C_{\beta_2\alpha_2} \dots , \tag{7.172}$$

where now the sum goes over the partitions in which all clusters (α_i), (β_i) contain less particles than α and β. (We assume here that no one of the clusters (α_i) and/or (β_i) is empty; this requires that in (7.171) all S^C_{00} are to zero). It is easier to understand this on examples. By definition[55]

$$S_{\mathbf{y},\mathbf{x}} \equiv S^C_{\mathbf{y},\mathbf{x}} \propto \int d\Gamma_{\mathbf{p}} \int d\Gamma_{\mathbf{q}}\, e^{-i\mathbf{p}\cdot\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{x}}\, \delta^{(3)}_{\Gamma}(\mathbf{p}-\mathbf{q}) \equiv \delta_{\mathbf{y},\mathbf{x}} \,. \tag{7.173}$$

The elements of the S-matrix corresponding to $2 \to 2$ transitions take then the form

$$S_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2} = S^C_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2} + \delta_{\mathbf{y}_1,\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2} \pm \delta_{\mathbf{y}_1,\mathbf{x}_2}\delta_{\mathbf{y}_2,\mathbf{x}_1} \,. \tag{7.174}$$

Similarly, the ones corresponding to $2 \to 3$ transitions read

$$\begin{aligned} S_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_1\mathbf{x}_2} = {} & S^C_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_1\mathbf{x}_2} + S^C_{\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_2}\delta_{\mathbf{y}_1,\mathbf{x}_1} \pm S^C_{\mathbf{y}_1\mathbf{y}_3;\mathbf{x}_2}\delta_{\mathbf{y}_2,\mathbf{x}_1} + S^C_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_2}\delta_{\mathbf{y}_3,\mathbf{x}_1} \\ & \pm S^C_{\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_1}\delta_{\mathbf{y}_1,\mathbf{x}_2} + S^C_{\mathbf{y}_1\mathbf{y}_3;\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2} \pm S^C_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1}\delta_{\mathbf{y}_3,\mathbf{x}_2} \,, \end{aligned}$$

those corresponding to $3 \to 3$ transitions,

$$\begin{aligned} S_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3} = {} & S^C_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3} \\ & + \delta_{\mathbf{y}_1,\mathbf{x}_1} S^C_{\mathbf{y}_2\mathbf{y}_3;\mathbf{x}_2\mathbf{x}_3} \pm \text{permutations} \\ & + \delta_{\mathbf{y}_1,\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2}\delta_{\mathbf{y}_3,\mathbf{x}_3} \pm \text{permutations} \,, \end{aligned}$$

and

$$\begin{aligned} S_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4} = {} & S^C_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4} \\ & + S^C_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2} S^C_{\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_3\mathbf{x}_4} \pm \text{permutations} \\ & + \delta_{\mathbf{y}_1,\mathbf{x}_1} S^C_{\mathbf{y}_2\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4} \pm \text{permutations} \\ & + \delta_{\mathbf{y}_1,\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2} S^C_{\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_3\mathbf{x}_4} \pm \text{permutations} \\ & + \delta_{\mathbf{y}_1,\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2}\delta_{\mathbf{y}_3,\mathbf{x}_3}\delta_{\mathbf{y}_4,\mathbf{x}_4} \pm \text{permutations} \,. \end{aligned} \tag{7.175}$$

those corresponding to $4 \to 4$ transitions etc.

The main point is that $S_{\beta\alpha}$ (7.172) satisfies the cluster decomposition principle (7.169) if the connected matrices $S^C_{\beta_i\alpha_i}$ vanish when at least one of the particles in the clusters (α_i)

[55] In the presence of interactions $S_{\mathbf{p},\mathbf{q}}$ differs from $\delta^{(3)}_{\Gamma}(\mathbf{p}-\mathbf{q})$ by a phase factor which is not relevant for what follows.

or (β_i) is spatially separated from the other particles in the same cluster. To see it on an example, consider a $4 \to 4$ process and the corresponding S-matrix element (7.175). Let us assume that the positions: initial $\mathbf{x}_1$ and $\mathbf{x}_2$ and final $\mathbf{y}_1$ and $\mathbf{y}_2$ are far away from the positions: initial $\mathbf{x}_3$ and $\mathbf{x}_4$ and final $\mathbf{y}_3$ and $\mathbf{y}_4$. From (7.175), throwing out all vanishing terms, we then get

$$\begin{aligned} S_{\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4} &= S^C_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2} S^C_{\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_3\mathbf{x}_4} \\ &+ (\delta_{\mathbf{y}_1,\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2} \pm \delta_{\mathbf{y}_1,\mathbf{x}_2}\delta_{\mathbf{y}_1,\mathbf{x}_1}) S^C_{\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_3\mathbf{x}_4} \\ &+ (\delta_{\mathbf{y}_3,\mathbf{x}_3}\delta_{\mathbf{y}_4,\mathbf{x}_4} \pm \delta_{\mathbf{y}_3,\mathbf{x}_4}\delta_{\mathbf{y}_4,\mathbf{x}_3}) S^C_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2} \\ &+ (\delta_{\mathbf{y}_1,\mathbf{x}_1}\delta_{\mathbf{y}_2,\mathbf{x}_2} \pm \delta_{\mathbf{y}_1,\mathbf{x}_2}\delta_{\mathbf{y}_1,\mathbf{x}_1})(\delta_{\mathbf{y}_3,\mathbf{x}_3}\delta_{\mathbf{y}_4,\mathbf{x}_4} \pm \delta_{\mathbf{y}_3,\mathbf{x}_4}\delta_{\mathbf{y}_4,\mathbf{x}_3}) \,, \end{aligned}$$

which is just the product $S_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2} S_{\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_3\mathbf{x}_4}$ with $S_{\mathbf{y}_1\mathbf{y}_2;\mathbf{x}_1\mathbf{x}_2}$ and $S_{\mathbf{y}_3\mathbf{y}_4;\mathbf{x}_3\mathbf{x}_4}$ given by (7.174).

To see what form of $S^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots}$ should take in order to have the required property, let us first note that if in the integral

$$\begin{aligned} S^C_{\mathbf{y}_1\mathbf{y}_2\ldots;\mathbf{x}_1\mathbf{x}_2\ldots} = \int d\Gamma_{\mathbf{p}_1} \int d\Gamma_{\mathbf{p}_2} \ldots \int d\Gamma_{\mathbf{q}_1} \int d\Gamma_{\mathbf{q}_2} \ldots \\ e^{-i\mathbf{p}_1\cdot\mathbf{y}_1} e^{-i\mathbf{p}_2\cdot\mathbf{y}_2} \ldots e^{i\mathbf{q}_1\cdot\mathbf{x}_1} e^{i\mathbf{q}_2\cdot\mathbf{x}_2} \ldots \, S^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots} \,, \end{aligned} \tag{7.176}$$

$|S^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots}|/\sqrt{E_{\mathbf{p}_1}\ldots E_{\mathbf{q}_1}\ldots}$ is a Lebesgue integrable function then $S^C_{\mathbf{y}_1\mathbf{y}_2\ldots;\mathbf{x}_1\mathbf{x}_2\ldots}$ vanishes by virtue of the Riemann-Lebesgue theorem if one of the positions $|\mathbf{x}_i|$ and/or $|\mathbf{y}_i|$ is large. This is, however, too strong a condition because $S^C_{\mathbf{y}_1\mathbf{y}_2\ldots;\mathbf{x}_1\mathbf{x}_2\ldots}$ should not vanish if all $\mathbf{x}_i$ and $\mathbf{y}_i$ are simultaneously shifted by the same vector $\mathbf{a}$ (no matter how large $|\mathbf{a}|$ is). By translational invariance $S^C_{\mathbf{y}_1\mathbf{y}_2\ldots;\mathbf{x}_1\mathbf{x}_2\ldots}$ should depend only on differences of the positions of initial and final state particles. This means that $S^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots}$ has to be proportional to the single delta function (therefore it cannot be Lebesgue integrable) expressing the conservation of the total 3-momentum (and, by Lorentz covariance, also to the delta function expressing the conservation of the total energy):

$$\begin{aligned} S^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots} = \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 + \ldots - \mathbf{q}_1 - \mathbf{q}_2 - \ldots) \\ \times\delta(E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \ldots - E_{\mathbf{q}_1} - E_{\mathbf{q}_2} - \ldots) \tilde{S}^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots} \,. \end{aligned} \tag{7.177}$$

The function $\tilde{S}^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots}$ cannot then contain any additional delta functions. If $\tilde{S}^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots}$ was an analytic function of the momenta, $S^C_{\mathbf{y}_1\mathbf{y}_2\ldots;\mathbf{x}_1\mathbf{x}_2\ldots}$ would vanish exponentially fast with growing differences of the particle positions. One can however, allow also for poles (they appear in theories with massless particles) in $\tilde{S}^C_{\mathbf{p}_1\mathbf{p}_2\ldots;\mathbf{q}_1\mathbf{q}_2\ldots}$ which leads to power-like fall-off of $S^C_{\mathbf{y}_1\mathbf{y}_2\ldots,\mathbf{x}_1\mathbf{x}_2\ldots}$ with growing differences of particle positions (such a character of the fall-off always signals the presence of long-range forces).

The final question is what interactions V_{int} lead to S-matrices satisfying the cluster decomposition principle. The answer is simple if V_{int} is built out of the creation and annihilation operators associated with eigenvectors $|\alpha_0\rangle$ of a free-particle Hamiltonian H_0. The cluster decomposition principle is then satisfied if the coefficient functions

$h_{MN}(\mathbf{p}_1, \ldots, \mathbf{p}_M; \mathbf{q}_1, \ldots, \mathbf{q}_N)$ of various terms of the interaction

$$V_{\rm int} = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \int d\Gamma_{\mathbf{p}_1} \ldots \int d\Gamma_{\mathbf{p}_M} \int d\Gamma_{\mathbf{q}_1} \ldots \int d\Gamma_{\mathbf{q}_N} \tag{7.178}$$
$$h_{MN}(\mathbf{p}_1, \mathbf{p}_2, \ldots; \mathbf{q}_1, \mathbf{q}_2, \ldots)\, a^\dagger(\mathbf{p}_1) \ldots a^\dagger(\mathbf{p}_M) a(\mathbf{q}_1) \ldots a(\mathbf{q}_N)\,,$$

contain only a single overall delta function

$$h_{MN}(\mathbf{p}_1, \ldots, \mathbf{p}_M; \mathbf{q}_1, \ldots, \mathbf{q}_N) = \delta^{(3)}(\mathbf{p}_1 + \ldots + \mathbf{p}_M - \mathbf{q}_1 - \ldots - \mathbf{q}_N)$$
$$\times \tilde{h}_{MN}(\mathbf{p}_1, \ldots, \mathbf{p}_M; \mathbf{q}_1, \ldots, \mathbf{q}_N)\,. \tag{7.179}$$

This is automatically ensured when the quantum field theory Hamiltonians are built out of field operators (to be introduced in the next Chapter).

E Potential scattering

In this Appendix we briefly recall the standard scattering theory formulated within the nonrelativistic Quantum Mechanics of a single (for simplicity spinless) particle moving in an external spherically symmetric potential $V_{\rm pot}(r)$ (playing the role of the interaction operator $V_{\rm int}$) on which some restrictions (concerning its behaviour as $r \to 0$ and $r \to \infty$) are usually imposed.

The Hilbert space $\mathcal{H}$ is in this case $L_2(\mathbb{R}^3)$ - the space of all Lebesque square-integrable functions. $H_0 = \mathbf{P}^2/2m$ has in $\mathcal{H}$ no normalizable eigenvectors. In contrast, $H = H_0 + V_{\rm pot}(r)$ can have discrete normalizable eigenvectors $|\phi_n\rangle$ corresponding to bound states of the particle in the considered potential $V_{\rm pot}(r)$. Thus the spectrum of H typically consists of isolated discrete values $E_n < 0$ (we assume $V_{\rm pot}(\infty) = 0$) and of a continuous part starting from $E = 0$.

In this case the convergence of the operators $\Omega(t) = e^{iHt/\hbar}e^{-iH_0t/\hbar}$ can be proved rigorously (for a class of potentials $V_{\rm pot}(r)$). This reduces to showing that the integral term in the formula (7.12) has a well defined limit $t \to \pm\infty$ on any normalizable vector $|\psi\rangle$. This is so, if the sequence of Hilbert space vectors

$$|\psi_t\rangle = \int_0^t d\tau\, U^\dagger(\tau,0)V_{\rm int}U_0(\tau,0)|\psi\rangle\,,$$

is for $t \to \pm\infty$ a Cauchy sequence which in turns is ensured if[1]

$$\| \int_0^{\pm\infty} d\tau\, U^\dagger(\tau,0)V_{\rm int}U_0(\tau,0)|\psi\rangle \| \;\leq \int_0^{\pm\infty} d\tau \parallel U^\dagger(\tau,0)V_{\rm int}U_0(\tau,0)|\psi\rangle \| < \infty\,.$$

Because $U^\dagger(t,0)$ is unitary, i.e. it preserves the norm, and because $\| \int f \| \leq \int \| f \|$, this in turn is ensured if

$$\int_0^{\pm\infty} d\tau \parallel V_{\rm int}U_0(\tau,0)|\psi\rangle \| \; < \infty\,.$$

Since the time evolution generated by H_0 ultimately drives any localized (normalizable) wave packet (the normalizable vector $|\psi\rangle$) outside the domain in which $V_{\rm int} = V_{\rm pot}(r)$ acts (i.e. is nonzero), this condition is (for an appropriate class of potentials) satisfied.

A priori vectors of the form $\Omega_\pm|\psi\rangle$ span two subspaces $\mathcal{H}_\pm$ of the Hilbert space $\mathcal{H}$. One can easily show that normalizable eigenvectors of H are orthogonal to all vectors of this form. Indeed, let $|\phi_n\rangle$ be a vector corresponding to a discrete H eigenvalue E_n and let $|\psi_\pm\rangle = \Omega_\pm|\psi\rangle$. Then

$$\langle\phi_n|\psi_\pm\rangle = \langle\phi_n|U^\dagger(t,0)U(t,0)|\psi_\pm\rangle = e^{iE_nt/\hbar}\langle\phi_n|U(t,0)|\psi_\pm\rangle\,.$$

[1] This is because the usual rule "tails contribute nothing to convergent integrals the integration domains of which extend to ∞ (or to $-\infty$)" applied to this integral turns out to be just the condition for the sequence of vectors to be a Cauchy sequence.

As this holds for any instant of time, t can be taken to $\pm\infty$ in which limit $U(t,0)|\psi_\pm\rangle$ can be replaced by $U_0(t,0)|\psi_{\rm as}^{\rm in/out}\rangle$. The vector $U_0(t,0)|\psi_{\rm as}^{\rm in/out}\rangle$ represents a wave packet moving freely, which ultimately, in the limits $t\to\pm\infty$ leaves the region in which $|\phi_n\rangle$ is localized and the scalar product must be zero. Furthermore, using a reasoning similar to the one applied above, one shows that the operator $\Omega^\dagger(t)=e^{iH_0t}e^{-iHt}$, which can be written in the form

$$\Omega^\dagger(t)=\hat{1}-i\int_0^t dt'\,U_0^\dagger(t',0)V_{\rm int}U(t',0)\,,$$

does not have the $t\to\pm\infty$ limits when acting on normalizable vectors $|\phi_n\rangle$. Thus, while the operator limits $\lim_{t\to\mp\infty}\Omega(t)$ do exist, the limits $\lim_{t\to\mp\infty}\Omega^\dagger(t)$, if H has normalizable eigenstates (bound states) $|\phi_n\rangle$, do not. Notice that this is not in conflict with the fact that $\Omega(t)$ and $\Omega^\dagger(t)$ satisfy for any finite t the relation $\Omega(t)\Omega^\dagger(t)=\hat{1}$, because the existence of the operator limits of $\Omega(t)$ and of $\Omega(t)\Omega^\dagger(t)$ does not imply the same for $\Omega^\dagger(t)$; the operators $\Omega_\pm^\dagger$ (defined below) are then *not* limits of $\Omega^\dagger(t)$.

An important assumption (the validity of which can be rigorously established for some classes of potentials $V_{\rm pot}(r)$) is the one about the asymptotic completeness. It states that $\mathcal{H}_+=\mathcal{H}_-=\mathcal{H}_{\rm scatt}$ and that $\mathcal{H}=\mathcal{H}_{\rm scatt}\oplus\mathcal{H}_{\rm bound}$. Defining the Hermitian conjugations $\Omega_\pm^\dagger$ of $\Omega_\pm$ one has to specify their action on the vectors $|\phi_n\rangle$ belonging to $\mathcal{H}_{\rm bound}$ by applying the rule (Section 4.1)

$$(\psi|\Omega_\pm^\dagger\phi_n)=(\Omega_\pm\psi|\phi_n)=(\psi_\pm|\phi_n)=0\,.$$

As ψ is an arbitrary vector belonging to $\mathcal{H}$, this implies that the operators $\Omega_\pm^\dagger$ must necessarily annihilate the whole subspace $\mathcal{H}_{\rm bound}$ (on $\mathcal{H}_{\rm scatt}$ they act as operators inverse to $\Omega_\pm$: $\Omega_\pm^\dagger|\psi_\pm\rangle=|\psi\rangle$). Thus while $\Omega_\pm^\dagger\Omega_\pm=\hat{1}$,

$$\Omega_\pm\Omega_\pm^\dagger=\hat{1}-\Pi_{\rm bound}\,,\tag{E.1}$$

where $\Pi_{\rm bound}$ is the projector onto $\mathcal{H}_{\rm bound}$. The proof (7.23) that $S_0H_0=H_0S_0$ goes however unmodified:

$$\Omega_-^\dagger H\,\Omega_+=\Omega_-^\dagger H\,(\Omega_-\Omega_-^\dagger+\Pi_{\rm bound})\,\Omega_+=\Omega_-^\dagger H\,\Omega_-\Omega_-^\dagger\Omega_+\,,\tag{E.2}$$

because $\Pi_{\rm bound}\Omega_+=0$ (all scattering states $\Omega_+|\psi\rangle$ belong to the subspace complementary to the one of bound states).

The state-vectors $|\alpha_0\rangle$ in the case considered here are simply the state-vectors $|\mathbf{p}\rangle$, that is the plane waves $\psi_{\mathbf{p}}(\mathbf{x})=e^{i\mathbf{p}\cdot\mathbf{x}}=\langle\mathbf{x}|\mathbf{p}\rangle$, and the formula (7.25) takes the form

$$S_{\mathbf{p}',\mathbf{p}}\equiv\langle\mathbf{p}'|S_0|\mathbf{p}\rangle=(2\pi)^3\delta^{(3)}(\mathbf{p}'-\mathbf{p})-2\pi i\,\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})\,t(\mathbf{p}',\mathbf{p})\,.\tag{E.3}$$

The usual scattering amplitude $f(\mathbf{p}',\mathbf{p})\equiv f(\theta)$ (here θ is the angle between $\mathbf{p}'$ and $\mathbf{p}$) is then related to $t(\mathbf{p}',\mathbf{p})$ by

$$f(\mathbf{p}',\mathbf{p})=-\frac{m}{2\pi\hbar^2}\,t(\mathbf{p}',\mathbf{p})\,.\tag{E.4}$$

This can be justified by analysing scattering of a localized wave packet peaked around a well defined momentum $\mathbf{p}$ - one finds that the elastic scattering differential cross section $\sigma(\theta) \equiv d\sigma/d\Omega$ is just given by $|f(\theta)|^2$.

The formula (7.38) with the definition (E.4) gives the standard Born approximation

$$f(\mathbf{p}', \mathbf{p}) = -\frac{m}{2\pi\hbar^2} \langle \mathbf{p}'|V_{\rm int}|\mathbf{p}\rangle = -\frac{m}{2\pi\hbar^2} \int d^3\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\, V_{\rm pot}(|\mathbf{x}|)\,, \tag{E.5}$$

with $\mathbf{q} \equiv \mathbf{p}' - \mathbf{p}$.

The *in* and *out* state-vectors $|\alpha_\pm\rangle$, denoted $|\mathbf{p}_\pm\rangle$ in the case considered here, satisfy the equation (7.51) which, when written in the standard position representation, reads

$$\langle \mathbf{x}|\mathbf{p}_\pm\rangle \equiv \psi_{\mathbf{p}_\pm}(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}} + \int d^3\mathbf{y}\, \langle \mathbf{x}|G_0(E_\mathbf{p} \pm i0)|\mathbf{y}\rangle\, V_{\rm pot}(|\mathbf{y}|)\, \langle \mathbf{y}|\mathbf{p}_\pm\rangle\,. \tag{E.6}$$

The matrix element $\langle \mathbf{x}|G_0(z)|\mathbf{y}\rangle$ can be found explicitly:

$$\langle \mathbf{x}|G_0(z)|\mathbf{y}\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{z - E_\mathbf{p}} = \frac{im}{2\pi^2\hbar^2|\mathbf{x}-\mathbf{y}|} \int_{-\infty}^{\infty} dp\, \frac{p\, e^{ip|\mathbf{x}-\mathbf{y}|}}{p^2 - 2mz/\hbar^2}\,.$$

The remaining integral can be computed by the residue method. It has two simple poles at $p = \sqrt{2m/\hbar^2}\, w_\pm$, where $w_\pm^2 = z$ and the roots are labeled so that $\mathrm{Im}(w_+) > 0$, $\mathrm{Im}(w_-) < 0$. According to the Jordan lemma, the integration contour must be closed with a large semicircle in the upper half plane. This gives

$$\langle \mathbf{x}|G_0(z)|\mathbf{y}\rangle = -\frac{m}{2\pi\hbar^2}\, \frac{\exp(i\sqrt{2m/\hbar^2}\, w_+\, |\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\,.$$

To find the asymptotic form of $\psi_{\mathbf{p}_\pm}(\mathbf{x})$ for $|\mathbf{x}| \to \infty$, when $V_{\rm pot}(|\mathbf{y}|)$ vanishes sufficiently fast for $|\mathbf{y}| \to \infty$, one can approximate

$$|\mathbf{x}-\mathbf{y}| = r\,(1 - \mathbf{n}\cdot\mathbf{y}/r) + \dots\,,$$

where $r \equiv |\mathbf{x}|$ and $\mathbf{n} \equiv \mathbf{x}/|\mathbf{x}| = \mathbf{x}/r$. For $z = E_\mathbf{p} \pm i0$, so that[2] $\sqrt{2m/\hbar^2}\, w_+ = \pm|\mathbf{p}| + i0$, the formula (E.6) takes then the form

$$\psi_{\mathbf{p}_\pm}(\mathbf{x}) \approx e^{i\mathbf{p}\cdot\mathbf{x}} + \frac{e^{\pm i|\mathbf{p}|r}}{r} \left(-\frac{m}{2\pi\hbar^2} \int d^3\mathbf{y}\, e^{\mp i|\mathbf{p}|\mathbf{n}\cdot\mathbf{y}}\, V_{\rm pot}(|\mathbf{y}|)\, \psi_{\mathbf{p}_\pm}(\mathbf{y}) \right).$$

The factor in the bracket is (c.f. the formula (7.53)) just $-m/2\pi\hbar^2$ times the matrix element

$$\langle \pm\mathbf{p}'|V_{\rm int}|\mathbf{p}_\pm\rangle \equiv t(\pm\mathbf{p}', \mathbf{p})\,,$$

[2]Setting $w_+ = \xi + i\eta$ one solves for ξ the equation $2mw_+^2 = 2m(\xi^2 - \eta^2 + 2i\xi\eta)/\hbar^2 = \mathbf{p}^2 \pm i0$; since w_+ is defined as having $\eta > 0$, it is ξ which must reproduce the sign of the small imaginary part $\pm i0$.

where $\mathbf{p}' \equiv \mathbf{n}|\mathbf{p}|$ which, upon using the definition (E.4) allows to identify $\psi_{\mathbf{p}_\pm}(\mathbf{x})$ (playing here the role of the the $|\alpha_\pm\rangle$ states) as the (generalized) eigenfunctions of H, commonly used in the ordinary stationary scattering theory

$$\psi_{\mathbf{p}_+}(\mathbf{x}) \approx e^{i\mathbf{p}\cdot\mathbf{x}} + \frac{f(\mathbf{p}', \mathbf{p})}{r}\, e^{+i|\mathbf{p}|r}\,,$$
$$\psi_{\mathbf{p}_-}(\mathbf{x}) \approx e^{i\mathbf{p}\cdot\mathbf{x}} + \frac{f(-\mathbf{p}', \mathbf{p})}{r}\, e^{-i|\mathbf{p}|r}\,,$$

and representing asymptotically the incoming plane wave and the outgoing ($\psi_{\mathbf{p}_+}$) or incoming ($\psi_{\mathbf{p}_-}$) spherical wave. The convergence (7.17) of wave packets (well localized in space) built out of the *in* or *out* states to the wave packets built with the same profile $g(\mathbf{k})$ out of the plane waves takes here the form

$$e^{-iHt}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\, g(\mathbf{p})\, \psi_{\mathbf{p}_\pm}(\mathbf{x}) \to e^{-iH_0 t}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\, g(\mathbf{p})\, \psi_{\mathbf{p}}(\mathbf{x})\,.$$

Heuristically, it can be justified by appealing to the fact that as $t \to \mp\infty$, any localized wave packet formed out of $\psi_{\mathbf{p}_\pm}(\mathbf{x})$ is driven by the time evolution far away from $\mathbf{x} = \mathbf{0}$, into a region in which $\psi_{\mathbf{p}_\pm}(\mathbf{x})$ effectively do not differ much from the plane waves $\psi_{\mathbf{p}}(\mathbf{x}) \equiv e^{i\mathbf{p}\cdot\mathbf{x}}$.