

7 Quantum field theory as a theory of interacting particles and the scattering theory

In this chapter we lay foundations for the first approach to formulating relativistic quantum field theories. This approach is close in spirit to the view expressed by Steven Weinberg - one of the founders of the Standard Theory of elementary particle interactions - that quantum field theory is merely a convenient machinery allowing one to systematically construct amplitudes of particle reactions (comprising together the $S$-matrix) satisfying a certain set of physically motivated requirements such as Poincaré covariance, unitarity, cluster decomposition and analyticity (which have been formulated in the historical development of high energy particle physics quite independently of the field theory). This view, while convenient as a starting point for our considerations, seems, however, too restricted. A more balanced view is probably that quantum field theory is just a quantum theory of some physical system. But what this system really is? In other words, what is the “ontology” underlying the quantum field theory? We will see that to some extent the ultimate formalism we will come to know dispenses us of such questions.\(^1\) Nevertheless, in formulating quantum field theory one has to stick to some “ontology”. The two obvious possible choices which lead to the quantum field theory as we know it are particles and fields (but one cannot exclude that the true underlying physical system may ultimately prove to be something else). Therefore, in this and in the two following chapters (8 and 9) our underlying ontology will be particles. Quantum field theory as a quantum theory of a system of fields\(^2\) will be developed in chapter 11. We decided to present both formulations because this allows to better understand different aspects of the developed formalism.

Adopting particles as the basic ontology is natural in condensed matter and solid state physics. Physical systems considered in these areas can certainly be treated as composed of particles\(^3\) whose properties - masses, spins, \(\ldots\)

\(^1\)In this sense quantum field theory seems to favour the view, nowadays widespread, if not prevailing among theoreticians, that only the mathematical formalism matters and the ontology is largely irrelevant; this was most probably the attitude to physical theories of Dirac, but certainly not the one of Bohr!

\(^2\)It is of course possible to formulate quantum field theory as a quantum theory of a mixed system consisting of fields and particles. (This was the approach adopted in section 3.8 in which quantum theory of radiation was presented as a prototype quantum field theory). In fact it seems that this may be the most natural point of view on the physical system underlying the quantum field theory: fermionic fields are Grassman algebra valued mappings which hardly, if at all, can be ascribed any physical reality - fermions most probably should be considered true particles. Bosons, in contrast, are most naturally interpreted as quantum excitations of continuous fields.

\(^3\)That is, the question what these particles are made of and why they have properties
charges etc. - are well known. The Hilbert space $\mathcal{H}$ of a theory constructed adopting this ontology is naturally a multiparticle space of the same kind as spaces built in chapter 5, possessing the vector $|\text{void}\rangle$ from which other vectors can be obtained by the action of an arbitrary number (which can also be infinite) of creation operators corresponding to the kinds of particles which are “put” in the system (as its fundamental constituents); the theory is constructed by adding to the free hamiltonian $H_0$ an interaction operator $V_{\text{int}}$ acting in $\mathcal{H}$. The resulting quantum mechanics of a many particle system with the Hamiltonian similar to (5.67) is a model of (nonrelativistic) quantum field theory and in general properties of excitations of the resulting system, interpreted in terms of quasi-particles, are very different than properties of the “fundamental” particles “put” in the system.

In the approach developed in this and in the two following chapters, relativistic field theories will be formulated in the similar spirit, as quantum theories of interacting relativistic particles. Therefore the starting point will be a relativistic theory of free particles of a finite number of definite kinds, $a, b, \ldots$, constructed on the basis of the second quantization formalism in section 6.5. The “arena” of the latter theory is the big multiparticle Hilbert space which is a direct sum of multiple tensor products of single-particle Hilbert spaces $\mathcal{H}^{(1)}_a, \mathcal{H}^{(1)}_b, \ldots$, and of the one dimensional $\mathcal{H}^{(0)}$ as in section 5.1. The big Hilbert space is therefore spanned by the vector $|\text{void}\rangle$ (which spans $\mathcal{H}^{(0)}$) and all possible multi-particle state-vectors constructed as (appropriately symmetrized/antisymmetrized) tensor products of one-particle state-vectors

$$
|\{p_1 \sigma_1, p_2 \sigma_2, \ldots, p_N \sigma_N\}_0\rangle. \quad (7.1)
$$

In the continuum (i.e. in the infinite space volume) the vectors (7.1) are normalized so that

$$
\langle \{p'_N \sigma'_N, p'_{N-1} \sigma'_{N-1}, \ldots, p'_1 \sigma'_1\}_0|\{p_1 \sigma_1, p_2 \sigma_2, \ldots, p_M \sigma_M\}_0\rangle
= \delta_{NM} \sum_P (-1)^P \delta_{\Gamma}(p'_1 - p_{P(1)}) \ldots \delta_{\Gamma}(p'_N - p_{P(N)}). \quad (7.2)
$$

The sum in (7.2) is over permutations within groups of labels corresponding to identical particles and $(-1)^P$ is the sign of the permutation of fermionic labels in a given permutation $P$. The symbol $\delta_{\Gamma}(p' - p)$ which here is assumed to include also the Kronecker delta of the spin variables $\sigma$, depends they have is entirely irrelevant for problems which are of interest in these areas of physics.

4If the state (7.1) represents $N_a$ particles of type $a$, $N_b$ particles of type $b$, etc. ($N_a + N_b + \ldots = N$), different groups of labels, e.g. $\{p_1 \sigma_1, \ldots, p_{N_a} \sigma_{N_a}\}$ correspond to different types of particles but we do not introduce any additional index to distinguish which labels correspond to which type of particles. Only labels corresponding to identical particles are symmetrized or antisymmetrized as described in section 5. Basis $N$-particle states constructed as appropriate linear combinations of the states (7.1) are also in use (see section 6.4 for examples of such alternative bases in the $N = 2$ case).
on the normalization of the one-particle states; with the one usually adopted in nonrelativistic applications \( \delta_\Gamma(p' - p) = (2\pi)^3 \delta_{\sigma'\sigma} \delta^{(3)}(p' - p) \); in relativistic theories more convenient is the normalization such that 
\[
\delta_\Gamma(p' - p) = (2\pi)^3 2E_p \delta_{\sigma'\sigma} \delta^{(3)}(p' - p),
\]
corresponding to \( 2E_p \) particles in the unit volume (see Section 10.2). Because in general considerations we will be not interested in the detailed composition of the multi-particle state-vectors, it is practical to introduce a compact notation, in which \(|\alpha_0\rangle\) stands for states of the form (7.1) and the scalar product (7.2) is concisely written as
\[
\langle \beta_0 | \alpha_0 \rangle = \delta(\beta - \alpha) \equiv \delta_{\beta\alpha}.
\]

The completeness relation
\[
\hat{\mathcal{I}} = \sum_{N=0}^{\infty} \left( \sum_{N_1} \sum_{N_2} \ldots \right) \delta_{N,(N_1+N_2+\ldots)} \frac{1}{N_1!N_2!\ldots} \sum_{\sigma_1,\ldots,\sigma_N} \int d\Gamma_p \langle (p_1\sigma_1,\ldots,p_N\sigma_N)_{0}\rangle \langle (p_N\sigma_N,\ldots,p_1\sigma_1)_0\rangle,
\]
in which the summation is over different numbers \(N_i\) of distinct types of particles, will be then compactly written as
\[
\hat{\mathcal{I}} = \int d\alpha \ |\alpha_0\rangle \langle \alpha_0|, \quad \text{i.e.} \quad |\Psi\rangle = \int d\alpha \ |\alpha_0\rangle \langle \alpha_0|\Psi\rangle.
\]
where \(|\Psi\rangle\) is any vector of the Hilbert space \(\mathcal{H}\).

The vectors (7.1) are the eigenveectors of the free Hamiltonian \(H_0\) which is taken to be a sum \(H_0 = H_0^a + H_0^b + \ldots\) of terms (6.113) with the energies \(E_a(p) = \sqrt{p^2 + m_a^2}\), \(E_b(p) = \sqrt{p^2 + m_b^2}\), \ldots - hence the subscript 0 in \(|\alpha_0\rangle\). In the continuum (infinite volume \(V\) of the space) the only normalizable eigenvector of \(H_0\) is the vector \(|\text{void}\rangle\); the other eigenvectors \(|\alpha_0\rangle\) of \(H_0\) are non-normalizable.\(^5\) Because the vector \(|\text{void}\rangle\) is also the lowest energy \(H_0\) eigenvector, it will be denoted \(|\Omega_0\rangle\). As explained in section 5.1, even in the finite volume \(V\), when allowed particle momenta form a discrete set (as a result of imposing periodic boundary conditions) and all state-vectors are normalizable, the Hilbert space is not separable - the set of vectors \(|\alpha_0\rangle\) which span the big Hilbert space \(\mathcal{H}\) is not countable.\(^6\) The separable subspace

\(^5\)Non-normalizable state-vectors \(|\alpha_0\rangle\), called generalized vectors, are in this respect similar to the plane waves \(\psi_p = e^{ip\cdot x}\) of ordinary one-particle nonrelativistic Quantum Mechanics which are generalized (non-normalizable) eigenvectors of the \(H_0 = \hat{P}^2/2m\) and \(\hat{P}\) operators.

\(^6\)This follows from the mathematical facts that for integer \(M\) and \(N\) both limits
\[
\lim_{N \to \infty} \lim_{M \to \infty} M^N \quad \text{and} \quad \lim_{N \to \infty} 2^N,
\]

228
spanned in the big Hilbert space by the vectors obtained by acting on \(|\Omega_0\rangle = |\text{void}\rangle\) with an arbitrary but finite number of the creation operators forms the most natural (but not the only one possible) Fock space.

The relativistic character of the theory of free particles constructed in section 6.5, is ensured by the relativistic form of the energies \(E_a(p), E_b(p), \ldots\), entering the free Hamiltonians \(H^0_a, H^0_b, \ldots\) of the form (6.113) and the possibility of constructing out of the creation and annihilation operators of the “fundamental” particles “put” in the system of the remaining Poincaré group generators \(J_0, P_0, K_0\), acting in \(\mathcal{H}\) (which in this case have all the form of sums of terms bilinear in the creation and annihilation operators) satisfying together with \(H_0\) the rules (6.21). Of course the full relativistic character of the dynamics is lost when the theory of free particles is considered not in the continuum, but taking the volume \(V\) finite is necessary for example to consider thermodynamical properties of a gas of free relativistic (in the sense of the energy-momentum relation) particles.

The theory of interacting relativistic particles (which in this approach is the quantum field theory) is constructed by adding to the free Hamiltonian \(H_0\) an interaction operator \(V_{\text{int}}\) acting in the big Hilbert space \(\mathcal{H}\) spanned by the vector \(|\text{void}\rangle = |\Omega_0\rangle\) and all vectors (7.1). Whether the resulting theory of the “fundamental” particles “put” in the system and now allowed to interact with each other is still relativistic, that is, whether it is possible to construct in \(\mathcal{H}\) new Poincaré group generators \(J, P, K\) which together with \(H = H_0 + V_{\text{int}}\) would satisfy the commutation rules (6.21) and additional physical requirements which will be formulated below, depends of course on the form of \(V_{\text{int}}\).

Assuming that the theory obtained by replacing \(H_0\) by \(H = H_0 + V_{\text{int}}\), is still a theory of particles (that is, assuming that the Hamiltonian \(H = H_0 + V_{\text{int}}\) has still eigenvectors which can be interpreted as representing particles - see below), the question what interactions \(V_{\text{int}}\) allow for constructing the Poincaré group generators requires formulating the theory in the continuum and is, for this reason, most conveniently investigated within the framework of the scattering theory: \(^7\) \(S\)-matrix characterizing interactions of relativistic

(relevant for counting bosonic and fermionic basis states) are equal to the power of the continuum. It is precisely the nonseparability of the Hilbert space \(\mathcal{H}\) which is at the origin of the mentioned insensitivity of the ultimate formalism to the “ontology” underlying the theory.

\(^7\) Although from the fundamental perspective it should be regarded as matter of pure calculational convenience that quantum field theory (or, more generally, any quantum theory) is formulated in the continuum - there is a strong conviction that physical systems described by quantum theories can always be confined to a finite space volume and that their measurable characteristics (if properly defined) do not depend on the size of this volume (if it is sufficiently large, that is, tend to well defined limits as \(V \to \infty\)) - the
particles, that is the set of amplitudes allowing to compute probabilities (rates) of particle reactions, should transform in a well defined way when the reference frame is changed - it should be Lorentz covariant. Thus, we will first formulate the scattering theory which in its general form applies to the ordinary nonrelativistic as well as to relativistic quantum mechanics. In fact, despite some important differences between its simplest version - the theory of scattering by an external potential based on the nonrelativistic quantum mechanics of a single particle and the scattering theory applied to the relativistic quantum mechanics of particles (that is quantum field theory) developed here, keeping in mind the former is helpful in understanding also the latter one. We will derive general formulae, applicable in relativistic and in nonrelativistic theories, expressing the \( S \)-matrix elements in different forms useful in discussing its various aspects and will work out various approximate and iterative ways of computing them.

We will then investigate in detail the requirement of Lorentz covariance of the \( S \)-matrix and will formulate sufficient conditions under which the Hamiltonian \( H = H_0 + V_{\text{int}} \) leads to a Lorentz covariant \( S \)-matrix. It will be seen that if the \( S \)-matrix is Lorentz covariant (which is the case if \( J_0, P_0, \) and \( K_0 \) commute with \( V_{\text{int}} \)) it is also possible to construct the generators \( J, P, \) and \( K \) having the required properties. As will turn out, these sufficient conditions are not satisfied in some theories of physical interest and the ultimate Poincaré covariance of their \( S \)-matrices must be ensured by additional special features of these theories; nevertheless, the conditions formulated here constitute a useful reference point for further constructions. Finally, we will discuss in some details general properties of \( S \)-matrices in relativistic theories such as unitarity, partial wave expansion and various symmetries.

Of course the fact that the relativistic character of the constructed theory is investigated by appealing to the infinite volume limit and the scattering theory does not mean that the \( S \)-matrix exhaust all the physically interesting information which can be obtained from it! Once it is formulated as a relativistic theory, various other properties of the underlying system, like for example its thermal properties which require keeping the volume finite, proper formulation of the scattering theory requires considering the theory in the continuum. The point is that in the finite volume, when all eigenvectors of the Hamiltonian are normalizable, scattering processes cannot be sharply distinguished from the general time evolution of the system: all reactions would occur multiply as time goes and it would not make sense to appeal to the infinite time limits in order to define measurable quantities characterizing what in the real world is observed as scattering processes. Therefore, to meaningfully define state-vectors representing scattering reactions the infinite volume is crucial.

\(^8\)There exist, of course, intermediate level theories based on nonrelativistic quantum mechanics of many particles which, similarly to the relativistic quantum field theories allow to consider multichannel scattering process.
can be investigated by various other methods and means of general quantum mechanics (e.g. by the Rayleigh-Schrödinger stationary perturbative expansion).

7.1 Time evolution, $S$-matrix and the $S_0$ operator

In developing the scattering theory within the quantum mechanics of interacting particles formulated along the lines sketched above we make the following important but physically motivated assumptions. Firstly, we assume that the Hamiltonian $H = H_0 + V_{\text{int}}$ is still a Hamiltonian of a system of particles, by which we mean that it possesses particle-like generalized eigenvectors, which in the sense specified below have properties similar to the multi-particle generalized eigenvectors (7.1) of a free Hamiltonian $\tilde{H}_0$, not necessarily identical with $H_0$ we start with, that is of a Hamiltonian $\tilde{H}_0$ which is a sum of terms $\tilde{H}_0^a, \tilde{H}_0^b$ of the form (6.113) with some relativistic energies $E_\tilde{a}(p), E_\tilde{b}(p), \ldots$ and with the original operators $\hat{a}_\sigma^\dagger(p), \hat{a}_\sigma(p)$ replaced by some other operators $\tilde{\hat{a}}_\sigma^\dagger(p), \tilde{\hat{a}}_\sigma(p)$ (constructed out of the original ones by means of some sort of a - perhaps very complicated compared to the ones considered in section ?? - Bogolyubov transformation) satisfying the same commutation relations$^9$ and such that all $\tilde{\hat{a}}_\sigma(p)$ annihilate some normalizable (in the continuum) free vacuum vector $|\tilde{\Omega}_0\rangle$. In a relativistic theory this in particular means that $H = H_0 + V_{\text{int}}$ has, among others generalized (in the continuum) eigenvectors, the eigenvectors which with respect to the transformations generated by $\hat{H}_0, \tilde{\hat{P}}_0, \tilde{\hat{J}}_0$ and $\tilde{\hat{K}}_0$ transform in the same way as do the discussed in section 6 one-particle eigenstates and that the one-particle eigenvectors of $\tilde{H}_0$ have precisely the same properties (with respect to transformations generated by $\hat{H}_0, \tilde{\hat{P}}_0, \tilde{\hat{J}}_0$ and $\tilde{\hat{K}}_0$). Furthermore we will assume that, similarly to $H_0$ (and to $\tilde{H}_0$), the Hamiltonian $H = H_0 + V_{\text{int}}$ has (in the infinite space volume) only a single (at least in the Fock space built on the vacuum vector $|\tilde{\Omega}_0\rangle$) normalizable state denoted $|\Omega\rangle$ and called the vacuum, which is its lowest energy eigenvector, that the particle-like non-normalizable eigenvectors of $H$, which will be introduced in section 7.3, together with $|\Omega\rangle$ span the whole big Hilbert space and, finally, that the spectra of the Hamiltonians $H = H_0 + V_{\text{int}}$ and $\tilde{H}_0$ are identical.$^{10}$ We therefore pose that there is a strict one-to-one correspondence between all eigenvectors of $H = H_0 + V_{\text{int}}$ and $\tilde{H}_0$ and that energies of the corresponding eigenvectors (with respect to the respective Hamiltonians, $H$ and $\tilde{H}_0$) are equal. The physical motivation

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$^9$It is therefore clear that at least formally, the Poincaré group generators $\tilde{\hat{P}}, \tilde{\hat{J}}$ and $\tilde{\hat{K}}$ satisfying together with $\tilde{H}_0$ the commutation relations (6.21) can also be built.

$^{10}$This is not always true in the scattering theory based on (nonrelativistic) quantum mechanics of a single particle in which the potential $V_{\text{int}}$ added to $H_0$ can lead to the existence of normalizable eigenvectors of $H$ (i.e. bound states).
for these assumptions is that if $H = H_0 + V_{\text{int}}$ is a Hamiltonian of a system of particles, its non-normalizable (in the continuum) eigenvectors should all represent (as is the case in ordinary nonrelativistic quantum mechanical scattering on a fixed potential) collision-type processes in which long before and long after the reaction particles look as (mutually) noninteracting. Therefore, it should be possible to associate with a given scattering process the Hilbert space (Heisenberg picture) state-vectors which, in a well defined way, correspond, as far as their transformation properties and energies are concerned to eigenvectors of some $\tilde{H}_0$.

The assumptions formulated above could of course be checked if the theory could be solved exactly. Unfortunately, in most cases one has to rely on some sort of approximations which usually hinge on the second assumption we are going to make. A method with the help of which the true spectrum of $H$ could, at least in principle, be investigated will be outlined in Section 13.

In their abstract form the assumptions formulated above do not allow to go too far\textsuperscript{11} within the approach to the quantum field theory developed in this and in the two following chapters. Therefore, we will make a “technical”, simplifying assumption that $V_{\text{int}}$ is “small” in the sense that the spectrum of $H = H_0 + V_{\text{int}}$ is the same as the spectrum of $H_0$, i.e. the full $H$ eigenvectors have the properties of the eigenvectors $|\alpha_0\rangle$ of $H_0$. In other words, we will assume that $\tilde{H}_0 = H_0$ and that the strict one to one correspondence holds between the the $H_0$ and $H$ eigenvectors (including the equality of the corresponding eigenvalues).

It is important to realize that these assumptions are neither a priori obvious, nor are they always fulfilled. It could happen that $H = H_0 + V_{\text{int}}$ does not possess particle-like eigenstates at all (or not all of its eigenstates are particle-like). This is indeed so in conformal field theory models or theories of “unparticles” discussed in the literature,\textsuperscript{12} so that there are theories to which even the general, seemingly well motivated assumption do not apply. Furthermore, even if $H = H_0 + V_{\text{int}}$ does possess particle-like eigenstates, they can be similar to the eigenvectors of a free-particle Hamiltonian $\tilde{H}_0$ which is very different from $H_0$ used to build $H$. The most prominent example is Quantum Chromodynamics (QCD) - the theory of strong interactions whose $H_0$ describes free spin $\frac{1}{2}$ coloured (i.e. transforming nontrivially under the action of the colour $SU(3)_c$ symmetry group) quarks, antiquarks and spin 1, massless coloured gluons, whereas the true $\tilde{H}$ eigenstates represent colourless, i.e. $SU(3)_c$ singlets, baryons, antibaryons and mesons. The approach exploiting the “technical” assumption was largely shaped by the historical

\textsuperscript{11}At least in fully relativistic theories - there are simplified nonrelativistic models, like e.g. the Lee model, in which the Hamiltonian $\tilde{H}_0$ can be explicitly constructed.

\textsuperscript{12}Free Hamiltonians $H_0$ of such theories describes massless particles.
development of quantum electrodynamics which as a quantum field theory is special in that the interaction between charged particles and photons is quite weak and, moreover, all particles of this theory are absolutely stable.\textsuperscript{13} The unified theory of weak and electromagnetic interactions (of which quantum electrodynamics is only a part) is also weakly coupled but certainly violates the assumption that there is a strict one-to-one correspondence between the particle-like eigenvectors of $H$ and of $H_0$: $W^\pm$ bosons of spin 1, muons, taons are “put” the theory as particles and have the corresponding eigenvectors of $H_0$ but not being absolutely stable they have no, strictly speaking, their counterparts among the particle-like eigenvectors of $H$.

In the approach to quantum field theory based on relativistic quantum mechanics of particles, which is developed in sections 7-9 these assumptions can be satisfied only for interactions $V_{\text{int}}$ that are judiciously constructed. This will become clear in Section 9.7, where it will turn out that observance of these assumptions (by appropriately adjusting $V_{\text{int}}$) is crucial for avoiding some type of ill defined contributions to the transition amplitudes ($S$ matrix elements) that would otherwise occur in perturbative calculations.

With the two assumptions spelled out above it becomes possible to formulate the scattering theory based on relativistic Quantum Mechanics of particles (that is on Quantum Field Theory) in the Fock space of $H_0$ eigenstates. To the proper Hilbert space which we will consider initially belong all possible normalizable state-vectors $|\Psi\rangle$ that can be constructed out of the Fock space of generalized $H_0$ eigenstates as

$$|\Psi\rangle = \int d\alpha \, |\alpha_0\rangle \langle \alpha_0 | \Psi\rangle \equiv \int d\alpha \, |\alpha_0\rangle \psi(\alpha),$$

with integrable profiles $\psi(\alpha)$: $\int d\alpha \, |\psi(\alpha)|^2 = 1$. One can then consider the time evolution of such states defined at $t = 0$ generated either by $H$ or $H_0$. Guided by the physical intuition we expect that those normalizable state-vectors $|\Psi\rangle$ which represent reactions between particles, and whose time evolution is governed by $H$ (we set $\hbar = c = 1$):

$$|\Psi(t)\rangle = e^{-itH} |\Psi\rangle \equiv U(t, 0) |\Psi\rangle,$$

(7.6)

(in the notation of section 1) should converge as $t \to \mp\infty$ (in the sense of

\textsuperscript{13}Positronium - a bound state of electron and positron - is unstable. However already electrodynamics of electrons and muons (which are stable in the absence of weak interactions) does not fully fit into the assumed scheme: the bound state of electron and antimuon (or of positron and muon) is stable and therefore the nonnormalizable eigenvectors of the full Hamiltonian of such electrodynamics should correspond, strictly speaking, to the eigenvectors of $H_0$ which is the free Hamiltonian of electrons positrons, muons, antimuons, and of $e^-\mu^+$ and $e^+\mu^-$ bound states treated as elementary particles.
convergence in the Hilbert space of sequences of vectors) to some state-vectors

\[ |\Psi_{in/out}(t)\rangle = e^{-iH_{0}t} |\Psi_{in/out}\rangle \equiv U_{0}(t, 0) |\Psi_{in/out}\rangle , \]

(7.7)
because in experiments one prepares states representing particles which before the collision are well localized and separated in space and are therefore almost non-interacting with each other; likewise, long after the collision particles are again well separated and again look as mutually non-interacting. As any smooth superposition of the basic vectors should represent an initial or a final state of particles, one assumes that on the whole Hilbert space the family of operators

\[ \Omega(t) \equiv e^{iHt}e^{-iH_{0}t} = U_{0}^{\dagger}(t, 0) , \]

(7.8)
(again in the notation of section 1) does have the limits

\[ \lim_{t \to \pm \infty} \Omega(t) = \Omega(\pm \infty) \equiv \Omega_{\pm} , \]

(7.9)
on any normalizable smooth superposition of the \(|\alpha_{0}\rangle\) states. \(\Omega_{\pm}\) are called Møller operators. Since

\[ \frac{d}{dt} \Omega(t) \frac{d}{dt} e^{iHt}e^{-iH_{0}t} = i U_{0}^{\dagger}(t, 0)V_{\text{int}}U_{0}(t, 0) , \]

and since \(\Omega(0) = \hat{1}\) the operator \(\Omega(t)\) can alternatively be defined by the integral relation

\[ \Omega(t) = \hat{1} + i \int_{0}^{t} dt' U_{0}^{\dagger}(t', 0)V_{\text{int}}U_{0}(t', 0) . \]

(7.10)
Furthermore, as the operators \(\Omega(t)\) are unitary for any fixed \(t\), that is satisfy \(\Omega(t)^\dagger \Omega(t) = 1\), the Møller operators \(\Omega_{\pm}\) are at least isometric, which means that they are defined on the whole Hilbert space \(\mathcal{H}\) and preserve the norm: \((\Omega_{\pm} \Psi | \Omega_{\pm} \Psi) = (\Psi | \Psi)\), that is satisfy the relation

\[ \Omega_{\dagger}^{\pm} \Omega_{\pm} = \hat{1} . \]

They preserve, therefore, also the scalar products of normalizable states:

\[ (\Omega_{\pm} \Phi | \Omega_{\pm} \Psi) = (\Phi | \Psi) . \]

(7.11)

\[^{14}\text{It should be stressed again that in the relativistic theory (QFT) this can be true only for special interactions \(V_{\text{int}}\); in particular for such interactions \(\Omega(t)\) have also limit on \(|\Omega_{0}\rangle\) (which is normalizable by itself) - \(\Omega_{\pm}\) acting on this vector give the state-vectors \(|\Omega_{\pm}\rangle\), which are normalized lowest energy \(H\) eigenvectors and can differ one from another only by a phase factor.}\]
In relativistic quantum mechanics of particles (i.e. in QFT) we assume that \[ \Omega_\pm \mathcal{H} = \mathcal{H} \] (and not \( \Omega_\pm \mathcal{H} \subset \mathcal{H} \)), that is that any \( \mathcal{H} \) space state-vector can be represented as the image of the action of \( \Omega_+ \) and \( \Omega_- \) on the some states \( |\Psi_{\text{in/out}}\rangle \):

\[
|\Psi\rangle = \Omega_+ |\Psi_{\text{in}}\rangle = \Omega_- |\Psi_{\text{out}}\rangle .
\] (7.12)

At this point one can already introduce the \( S_0 \) operator. As usually in a quantum theory one is interested in scalar products \( S_{\Phi\Psi} \equiv \langle \Phi | \Psi \rangle \) of normalized states. Expressing \( |\Psi\rangle \) as the \( \Omega_+ \) image of the appropriate \( |\Psi_{\text{in}}\rangle \) and \( |\Phi\rangle \) as the \( \Omega_- \) image of \( |\Phi_{\text{out}}\rangle \) one gets

\[
S_{\Phi\Psi} \equiv \langle \Phi | \Psi \rangle = \langle \Phi_{\text{out}} | S_0 | \Psi_{\text{in}} \rangle ,
\] (7.13)

where the \( S_0 \) operator is defined as the product

\[
S_0 \equiv \Omega_-^\dagger \Omega_+ .
\] (7.14)

It, among other things, maps the asymptotic “incoming” states onto the corresponding “outgoing” states: \( |\Psi_{\text{out}}\rangle = S_0 |\Psi_{\text{in}}\rangle \). The scalar product \( S_{\Phi\Psi} \) - which is equal to the appropriate matrix elements of \( S_0 \) - has the natural interpretation of the probability amplitude of finding the system in the state \( |\Phi\rangle \), which, if evolved in time,\(^{16}\) would become in the far future indistinguishable from an (appropriately (i.e. with \( H_0 \)) evolved state \( |\Phi_{\text{out}}\rangle \) which has direct interpretation in terms of noninteracting (and spatially separated in the far future) particles, if it is prepared as the state \( |\Psi\rangle \) which, if evolved in time, has in the far past a similar free-particle interpretation, being indistinguishable from the evolved state \( |\Psi_{\text{in}}\rangle \). The scalar products \( S_{\Phi\Psi} \) thus contain answers to a prevailing amount of experimentally accessible questions which usually are formulated in the form “what is the probability that the detectors will register a given free-particle state produced as a result of an interaction of particles which long before interaction were prepared (in the accelerator) as another free-particles state?”

It is also convenient to introduce an alternative notation (corresponding to a slightly different labeling of states) and to call \( |\Psi_+\rangle \) and \( |\Psi_-\rangle \) the two different states which are images of the same state \( |\Psi\rangle \) under \( \Omega_+ \) and \( \Omega_- \), respectively. Thus, in this notation

\[
\lim_{t \to \mp \infty} U(t, 0) |\Psi_\pm\rangle = U_0(t, 0) |\Psi\rangle .
\] (7.15)

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15 This is not necessarily true in nonrelativistic Quantum Mechanics of a single particle. See Appendix C.

16 Notice that the states are always identified at \( t = 0 \); that is we implicitly work in the Heisenberg picture (see section 1.1) which in Quantum Field Theory allows to keep relativistic invariance as manifest as possible.
Matrix elements of the $S_0$ operator in this notation are written as $\langle \Phi_- | \Psi_+ \rangle$ and since

$$|\Psi_+\rangle = \Omega_+ \Omega^\dagger_- |\Psi_-\rangle, \quad |\Psi_-\rangle = \Omega_- \Omega^\dagger_+ |\Psi_+\rangle,$$

they can also be expressed as the matrix elements

$$S_{\Phi \Psi} \equiv \langle \Phi_- | \Psi_+ \rangle = \langle \Phi_- | S | \Psi_- \rangle = \langle \Phi_+ | S | \Psi_+ \rangle,$$

of the $S$ operator

$$S \equiv \Omega_+ \Omega^\dagger_-.$$

The $S$ operator (which has been used in section 1.3) which is different from the $S_0$ one will be of little use in what follows. However the notation $|\Psi_\pm\rangle$ will be useful.

Under the assumptions underlying the considerations of this section the operators $H = H_0 + V_{\text{int}}$ and $H_0$ satisfy the important intertwining relation

$$H \Omega_\pm = \Omega_\pm H_0,$$

which in particular imply that $\Omega_\pm^\dagger H_0 \Omega_\pm = H_0$. Indeed,

$$e^{iHt} \Omega_\pm = e^{iHt} \lim_{\tau \to \mp \infty} (e^{iH_0 \tau} e^{-iH_0 \tau}) = \lim_{\tau \to \mp \infty} (e^{iH(\tau+t)} e^{-iH_0(t+\tau)}) e^{iH_0 t} = \Omega_\pm e^{iH_0 t}.$$

Differentiating this equality with respect to $t$ at $t = 0$ yields the relation (7.19). Exploiting this relation one can write

$$S_0 H_0 = \Omega_\pm^\dagger \Omega_+ H_0 = \Omega_\pm^\dagger H_0 \Omega_+ = H_0 \Omega_\pm^\dagger \Omega_+ = H_0 S_0,$$

that is,

$$[S_0, H_0] = 0.$$

The relation (7.19) means, in particular, that

$$\langle \Psi | H | \Psi \rangle = \langle \Psi^{\text{in}}_{\text{as}} | H_0 | \Psi^{\text{in}}_{\text{as}} \rangle = \langle \Psi^{\text{out}}_{\text{as}} | H_0 | \Psi^{\text{out}}_{\text{as}} \rangle.$$

17 The difference between the $S_0$ and $S$ operators is particularly sharp in the nonrelativistic potential scattering theory, if $H$ possesses bound states: while $S_0$ acts nontrivially on the whole Hilbert space $\mathcal{H}$, $S$ annihilates the whole subspace $\mathcal{H}_{\text{bound}}$.

18 If $\Omega_\pm$ are unitary, these relations imply that the spectra of $H$ and $H_0$ are identical (which is one of our assumptions adopted here). This shows that in the case of ordinary potential scattering $\Omega_\pm$ cannot be unitary if $H$ has bound states (normalizable eigenstates) because $H_0$ does not have such eigenstates. In such a case $\Omega_\pm$ are only isometric operators.

19 See (C.2) for a more precise justification.
The energy is conserved in the scattering process.

Since any normalizable state-vector $|\Psi\rangle$ can be written as a superposition of the non-normalizable generalized (i.e. not belonging to the proper Hilbert space) $H_0$ eigenstates $|\alpha_0\rangle$, one can write

$$
|\beta_0\rangle_{\text{out}} = \langle \beta_0 | S_0 | \alpha_0 \rangle = \int d\alpha \langle \beta_0 | S_0 | \alpha_0 \rangle \langle \alpha_0 | \Psi_{\text{in}} \rangle.
$$

One is thus led to consider the matrix elements

$$
S_{\beta\alpha} \equiv \langle \beta_0 | S_0 | \alpha_0 \rangle.
$$

From (7.21) it follows that

$$
0 = \langle \beta_0 | [H_0, S_0] | \alpha_0 \rangle = \langle \beta_0 | S_0 | \alpha_0 \rangle \delta(E_\beta - E_\alpha),
$$

which shows that $\langle \beta_0 | S_0 | \alpha_0 \rangle \propto \delta(E_\beta - E_\alpha)$ (because $x\delta(x) = 0$). Furthermore, because for $V_{\text{int}} = 0$ the $S_0$ operator reduces to the unit operator, it is convenient to write

$$
S_0 = \hat{1} - iT_0,
$$

thereby introducing the reaction operator $T_0$. Thus

$$
S_{\beta\alpha} \equiv \langle \beta_0 | S_0 | \alpha_0 \rangle = \delta_{\alpha\beta} - 2\pi i \delta(E_\beta - E_\alpha) t_{\beta\alpha}(E_\alpha),
$$

(7.23)

where $2\pi i \delta(E_\beta - E_\alpha) t_{\beta\alpha}(E_\alpha) = \langle \beta_0 | T_0 | \alpha_0 \rangle$. As will be shown in Section 10, it is precisely the quantity $t_{\beta\alpha}(E_\alpha)$ which is needed to compute the rate of the process $\alpha \rightarrow \beta$. In the case of the nonrelativistic potential scattering the quantity $t_{\beta\alpha}(E_\alpha)$ is directly related to the standard scattering amplitude $f(\theta)$ - see Appendix C. All measurable characteristics of scattering processes predicted by a given theory defined by the Hamiltonian $H = H_0 + V_{\text{int}}$ can be extracted from the matrix elements $S_{\beta\alpha}$ of the corresponding $S_0$ (or $T_0$) operator of this theory. One useful representation for this operator will be derived directly from the differential equation satisfied by the (interaction picture) evolution operator introduced in section 1.1:

$$
U_I(t_2, t_1) = e^{iH_0 t_2} e^{-iH t_2} e^{iH t_1} e^{-iH_0 t_1} = e^{iH_0 t_2} e^{-iH (t_2 - t_1)} e^{-iH_0 t_1} = \Omega_{\pm}^\dagger(t_2) \Omega_{\pm}(t_1),
$$

(7.24)

of which $S_0$ is the double limit:

$$
S_0 = \Omega_{\pm}^\dagger \Omega_{\pm} \lim_{t_2 \rightarrow +\infty} \lim_{t_1 \rightarrow -\infty} U_I(t_2, t_1).
$$

(7.25)

Notice also that $\Omega_{\pm} = U_I(0, \mp \infty)$. Before deriving it one has, however, to introduce the non-normalizable (generalized) $H$ eigenvectors and the rezolvent operators which allow to relate these to the $H_0$ eigenvectors $|\alpha_0\rangle$. 237
7.2 Rezolvents and the $T$ operator representation of the $S$-matrix

A very important role in the formal scattering theory is played by the rezolvent operators

$$ G(z) \equiv (z - H)^{-1}, \quad \text{and} \quad G_0(z) \equiv (z - H_0)^{-1}. \quad (7.26) $$

Their matrix elements between normalizable states are analytic functions of the complex $z$ plane except for isolated poles corresponding to normalizable $H$ ($H_0$) eigenstates and branch cut along the continuous part of the $H$ ($H_0$) spectrum.

Substituting for $A$ and $B$ in the obvious operator identity

$$ \frac{1}{A} - \frac{1}{B} = \frac{1}{B} (B - A) \frac{1}{A}, $$

the operators $z - H$ and $z - H_0$ (the operators $z - H_0$ and $z - H$) one obtains two relations

$$ G(z) = G_0(z) + G_0(z) V_{\text{int}} G(z), $$

$$ G(z) = G_0(z) + G(z) V_{\text{int}} G_0(z), \quad (7.27) $$

It is also easy to see that because $H = H^\dagger$ ($H_0 = H_0^\dagger$),

$$ G(z^*) = [G(z)]^\dagger, \quad G_0(z^*) = [G_0(z)]^\dagger. \quad (7.28) $$

Matrix elements of the resolvent operator $G_0(z)$ between the non-normalizable $H_0$ eigenstates are explicitly given by

$$ \langle \beta_0 | G_0(z) | \alpha_0 \rangle = \delta_{\beta \alpha} \frac{1}{z - E_{\alpha}}. \quad (7.29) $$

Another very important operator is the $T(z)$ operator defined as

$$ T(z) \equiv V_{\text{int}} + V_{\text{int}} G(z) V_{\text{int}}. \quad (7.30) $$

It has the same analytic properties as $G(z)$ and satisfies the following relations

$$ G_0(z) T(z) = G(z) V_{\text{int}}, $$

$$ T(z) G_0(z) = V_{\text{int}} G(z), \quad (7.31) $$

which readily follow from the relations (7.27). They allow to express $G(z)$ through $T(z)$: replacing in (7.27) $V_{\text{int}} G(z)$ (or $G(z) V_{\text{int}}$) using (7.31) one gets

$$ G(z) = G_0(z) + G_0(z) T(z) G_0(z). \quad (7.32) $$
Using (7.31) in (7.30) leads instead to
\[ T(z) = V_{\text{int}} + V_{\text{int}} G_0(z) T(z), \] (7.33)
which is known as the Lippman-Schwinger equation for \( T(z) \). Iterating it yields the series
\[ T(z) = V_{\text{int}} + V_{\text{int}} G_0(z) V_{\text{int}} + V_{\text{int}} G_0(z) V_{\text{int}} G_0(z) V_{\text{int}} + \ldots \] (7.34)

Matrix elements of the \( S_0 \) operator between generalized \( H_0 \) eigenstates can be expressed through the operator \( T(z) \). To this end, instead of representing \( S_0 \) as in (7.25) as the double limit of the \( U_I(t_2, t_1) \) operator, it is written as the single limit
\[ S_0 = \Omega^+ \Omega_+ = \lim_{\tau \to \infty} U_I(\tau, -\tau) = \lim_{\tau \to \infty} e^{iH_0 \tau} e^{-2iH_\tau} e^{iH_0 \tau}. \]
Differentiating with respect to \( \tau \) the operator \( U_I(\tau, -\tau) \) one obtains the differential equation satisfied by it, which, together with the obvious boundary condition at \( \tau = 0 \), allows to write for this operator an integral expression. Using it, matrix element of the \( S_0 \) operator between normalizable states can be written as
\[ \langle \Phi | S_0 | \Psi \rangle = \langle \Phi | \Psi \rangle - i \int_0^\infty dt e^{-\varepsilon t} \langle \Phi | e^{iH_0 t} V_{\text{int}} e^{-2iH_\tau} e^{iH_0 t} + e^{iH_0 t} e^{-2iH_\tau} V_{\text{int}} e^{iH_0 t} | \Psi \rangle. \]

The factor \( e^{-\varepsilon t} \) is not necessary when the matrix element is taken between two normalizable states, but when introduced,\(^{20}\) it allows to replace \( | \Phi \rangle \) and \( | \Psi \rangle \) by the generalized \( H_0 \) eigenstates \( | \alpha_0 \rangle \) and \( | \beta_0 \rangle \):
\[ \langle \beta_0 | S_0 | \alpha_0 \rangle = \delta_{\beta \alpha} - i \int_0^\infty dt \langle \beta_0 | V_{\text{int}} e^{i(E_\beta + E_\alpha - 2H + i0)t} + e^{i(E_\beta + E_\alpha - 2H + i0)t} V_{\text{int}} | \alpha_0 \rangle \\
= \delta_{\beta \alpha} + \frac{1}{2} \langle \beta_0 | V_{\text{int}} G \left( \frac{E_\beta + E_\alpha}{2} + i0 \right) + G \left( \frac{E_\beta + E_\alpha}{2} + i0 \right) V_{\text{int}} | \alpha_0 \rangle. \]

Using the operator identities (7.31) one can replace here the operators \( G(z) \) by \( G_0(z) \) ones which can act directly on the states \( | \alpha_0 \rangle \) and \( | \beta_0 \rangle \). The second term can be then cast in the form
\[ \left\{ \frac{1}{E_\beta - E_\alpha + i0} + \frac{1}{E_\alpha - E_\beta + i0} \right\} \langle \beta_0 | T \left( \frac{E_\beta + E_\alpha}{2} + i0 \right) | \alpha_0 \rangle, \]
\(^{20}\)It is usually (incorrectly) introduced from the beginning as a factor ensuring “adiabatic” switching on and off the interaction - something which certainly does not happen in Nature! Notice also that if \( V_{\text{int}} \) was defined with such a factor, the evolution operator \( U(t, t_0) \) corresponding to the Hamiltonian \( H \) (explicitly time dependent then) would have to have the form (1.9) instead of \( e^{-iH(t-t_0)} \).
which, upon using the Sochocki formula, leads to
\[ \langle \beta_0 | S_0 | \alpha_0 \rangle = \delta_{\beta \alpha} - 2\pi i \delta (E_\beta - E_\alpha) \langle \beta_0 | T(E_\alpha + i0) | \alpha_0 \rangle. \] (7.35)

This shows that the matrix element \( t_{\beta \alpha} \) of the \( T_0 \) operator introduced in (7.23) is given by the special limit \( z \rightarrow E_\alpha + i0 \) of the general matrix element of the \( T(z) \) operator. Combining this with the truncated to its first term iterative solution (7.34) of the Lippmann-Schwinger equation (7.33) for \( T(z) \), one immediately obtains the formula known as the Born approximation
\[ t_{\beta \alpha} (E_\alpha) \approx \langle \beta_0 | V_{\text{int}} | \alpha_0 \rangle. \] (7.36)

If it is known how to compute the action of \( V_{\text{int}} \) on free particle states (which is precisely the case, when \( V_{\text{int}} \) is expressed in terms of the creation and annihilation operators of free particles) this formula provides the simplest working approximation to amplitudes of particle reactions.\(^{21}\)

### 7.3 In and out state-vectors

One can now define the in and out generalized state-vectors \( | \alpha_\pm \rangle \) by the formula
\[ | \alpha_\pm \rangle \equiv \Omega_\pm | \alpha_0 \rangle \equiv \lim_{t \rightarrow \pm \infty} e^{iHt} e^{-iH_0 t} | \alpha_0 \rangle. \] (7.37)

Owing to the intertwining relations (7.19), \( | \alpha_\pm \rangle \) turn out to be just the generalized (non-normalizable) eigenvectors of the Hamiltonian \( H = H_0 + V_{\text{int}} \):
\[ H | \alpha_\pm \rangle = E_\alpha | \alpha_\pm \rangle, \] (7.38)

with the eigenvalue \( E_\alpha \) equal to the energy (w.r.t. \( H_0 \)) of the corresponding \( | \alpha_0 \rangle \) states. With the assumption that the spectra of \( H \) and \( H_0 \) are identical, the vectors \( | \alpha_0 \rangle, | \alpha_+ \rangle \) and \( | \alpha_- \rangle \) related to each other in the same way as are the vectors \( | \Psi \rangle, | \Psi_+ \rangle \) and \( | \Psi_- \rangle \) in (7.15) form three equivalent bases (of generalized vectors) of the theory Hilbert space \( \mathcal{H} \). From this point of view the \( S \)-matrix elements
\[ S_{\beta \alpha} = \langle \beta_- | \alpha_+ \rangle = \langle \beta_0 | S_0 | \alpha_0 \rangle, \] (7.39)
form a collection of numbers, such that
\[ | \alpha_+ \rangle = \int d\beta | \beta_- \rangle S_{\beta \alpha}, \quad \langle \beta_- | = \int d\alpha \langle \alpha_+ | S_{\beta \alpha}. \] (7.40)

\(^{21}\)However, frequently in relativistic theories of interacting particles \( t_{\beta \alpha} (E_\alpha) = 0 \) in this approximation. In Quantum Field Theory the name “Born approximation” is sometimes also used to denote what otherwise is called the tree-level approximation (see section 9); it coincides with (7.36) only for very special interactions \( V_{\text{int}} \).
As a matrix connecting two complete sets of orthonormal states (it is just the matrix of the change of bases) $S_{\beta\alpha}$ must be unitary:

$$\int d\beta S^{*}_{\beta\gamma} S_{\beta\alpha} = \int d\beta \langle \gamma_+ | \beta_- \rangle \langle \beta_- | \alpha_+ \rangle = \langle \gamma_+ | \alpha_+ \rangle = \delta_{\gamma\alpha}. \quad (7.41)$$

This reflects also the unitarity of the $S_0$ operator: $S_0^{-1} = S_0^\dagger$. The states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are, in turn, connected by the $S$ operator defined in (7.18):

$$S|\alpha_-\rangle = |\alpha_+\rangle, \quad \text{or} \quad \langle \beta_+ | S = \langle \beta_- |, \quad (7.42)$$

so that, in analogy to (7.17)

$$S_{\beta\alpha} = \langle \beta_+ | S |\alpha_+\rangle = \langle \beta_- | S |\alpha_-\rangle. \quad (7.43)$$

From the practical point of view (7.37) establishes the strict one-to-one correspondence between the in and out eigenstates of $H$ and the eigenstates of $H_0$ on which the formulation of the perturbative calculation of the $S$-matrix elements will be based. (This strict correspondence will be relaxed only in Section 13 where a more flexible, nonperturbative in essence, way of accessing $S$-matrix elements will be formulated).

Since any normalizable state $|\Psi\rangle$ can be decomposed into the generalized $H_0$ eigenvectors $|\Psi\rangle = \int d\alpha |\alpha_0\rangle \psi(\alpha)$, from the relation $|\Psi_\pm\rangle = \Omega_\pm |\Psi\rangle$ one gets

$$|\Psi_\pm\rangle = \Omega_\pm \int d\alpha |\alpha_0\rangle \psi(\alpha) = \int d\alpha |\alpha_\pm\rangle \psi(\alpha), \quad (7.44)$$

That is, normalizable state-vectors $|\Psi_\pm\rangle$ decompose onto the generalized $H$ eigenvectors $|\alpha_\pm\rangle$ with the same profile $\psi(\alpha)$ as do their $\Omega_\pm$ images onto the generalized $H_0$ eigenvectors $|\alpha_0\rangle$. Moreover, from the fact that the $\Omega_\pm$ operators preserve the scalar product of normalizable states (cf. (7.11)) it follows that

$$\langle \beta_\pm | \alpha_\pm \rangle = \langle \beta_0 | \alpha_0 \rangle = \delta_{\beta\alpha}. \quad (7.45)$$

Since the in and out state-vectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are in the one-to-one correspondence with the free particle vectors $|\alpha_0\rangle$, in addition to the operators $a(k, \sigma)$, $a^\dagger(k, \sigma)$ (which build the states $|\alpha_0\rangle$ out of $|\Omega_0\rangle$) one can define also the in and out creation and annihilation operators $a_{in}(k, \sigma)$, $a_{in}^\dagger(k, \sigma)$ and $a_{out}(k, \sigma)$, $a_{out}^\dagger(k, \sigma)$ which acting on the corresponding vacua$^{22}$ $|\Omega_\pm\rangle = |\Omega_{\pm}\rangle$ vacua with the Møller operators $\Omega_\pm$. The vacua $|\Omega_+\rangle$ and $|\Omega_-\rangle$ of closed systems, i.e. system whose Hamiltonians $H$ do not depend on time, differ only by a phase factor.
Ω±|Ω0⟩ build the in and out state-vectors and satisfy the same commutation relations as do the original operators a(k, σ), a†(k, σ) and have the same transformation properties (in the relativistic case with respect to the full Poincaré symmetry group generated by H = H0 + Vint, P, J and K = K0 + W - see section 7.4) as do the operators creating and annihilating the free-particle states α0⟩. From (7.42) it then follows (cf. (1.57)) that

\[ S†a†_{\text{in}}(k, \sigma)S = a_{\text{out}}(k, \sigma), \quad S†a_{\text{in}}(k, \sigma)S = a_{\text{out}}(k, \sigma). \]  

(7.46)

Finally, it should be stressed that by themselves the vectors U(t, 0)|α±⟩ do not converge to U0(t, 0)|α0⟩ = e−iEαt|α0⟩ in the limits t → ±∞. The convergence holds only for normalizable states built as smooth superpositions of such states. Nevertheless, (7.37) stay true in the literal sense.

The operator identities established above allow to derive useful representations for the in and out states |α±⟩ either in terms of the rezolvent G(z) or in terms of the T(z) operator. To this end we consider first the action of Ω± on a normalizable state-vector |Ψ⟩. One gets then the scattering states |Ψ±⟩ which, using the formula (7.10) can be written as

\[ |Ψ±⟩ = Ω±|Ψ⟩ = |Ψ⟩ + i \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} U†(t', 0)V_{\text{int}} U_0(t', 0)|Ψ⟩. \]  

(7.47)

Again the factor e−ε|t'| is not necessary for convergence when |Ψ⟩ is a normalizable state, but is necessary when |Ψ⟩ is decomposed into generalized H0 eigenstates |α0⟩:

\[ |Ψ±⟩ = |Ψ⟩ + \int dα \int_{-\infty}^{\infty} dt e^{-i(E_α - H ± i\epsilon)t} V_{\text{int}} |α0⟩⟨α0|Ψ⟩ \]

\[ = |Ψ⟩ + \int dα G(E_α ± i0) V_{\text{int}} |α0⟩⟨α0|Ψ⟩. \]  

(7.48)

To obtain the representations of the in and out states |α±⟩ one rewrites (7.48), decomposing |Ψ⟩ onto the |α0⟩ states, in the form

\[ |Ψ±⟩ = \int dα \left( |α0⟩ + G(E_α ± i0) V_{\text{int}} |α0⟩ \right) ψ(α). \]

Comparing this with (7.44) one gets the representation

\[ |α±⟩ = |α0⟩ + G(E_α ± i0) V_{\text{int}} |α0⟩. \]  

(7.49)

Yet another representation can be obtained using the identity

\[ T(E_α ± i0)|α0⟩ = V_{\text{int}}[\hat{1} + G(E_α ± i0)V_{\text{int}}]|α0⟩ = V_{\text{int}}|α±⟩, \]  

(7.50)
which follows from the definition (7.30) of the $T(z)$ operator and (7.49). This relation combined with the result (7.35) immediately allows to write the element $t_{\beta\alpha}(E_\alpha)$ in the $S_0$ matrix element (7.23) as

$$t_{\beta\alpha}(E_\alpha) = \langle \beta_0 \mid T(E_\alpha + i0) \mid \alpha_0 \rangle = \langle \beta_0 \mid V_{\text{int}} \mid \alpha_{\pm} \rangle.$$  

(7.51)

The identity (7.50) applied to (7.49) after trading in this formula the product $G(E_\alpha \pm i0) V_{\text{int}}$ for $G_0(E_\alpha \pm i0) T(E_\alpha \pm i0)$ in agreement with (7.31), leads to the Lippmann-Schwinger equation for $|\alpha_{\pm}\rangle$:

$$|\alpha_{\pm}\rangle = |\alpha_0\rangle + G_0(E_\alpha \pm i0) V_{\text{int}} |\alpha_{\pm}\rangle \equiv |\alpha_0\rangle + \frac{1}{E_\alpha - H_0 \pm i0} V_{\text{int}} |\alpha_{\pm}\rangle,$$  

(7.52)

or

$$|\alpha_{\pm}\rangle = |\alpha_0\rangle + \int d\beta |\beta_0\rangle \frac{t_{\beta\alpha}(E_\alpha)}{E_\alpha - E_\beta \pm i0}.$$  

(7.53)

Notice that the formula (7.52) agrees with the identification of the $|\alpha_{\pm}\rangle$ as the eigenstates of $H$, if the relation (7.38) is rewritten in the form

$$(E_\alpha - H_0) |\alpha_{\pm}\rangle = V_{\text{int}} |\alpha_{\pm}\rangle.$$  

The $\pm i0$ prescription specifies the way of inverting the operator $(E_\alpha - H_0)$ which has $|\alpha_0\rangle$ as its zero eigenvector.\(^{24}\) Iterating the Lippmann-Schwinger equation (7.52) eg. for $|\alpha_+\rangle$ gives the series

$$|\alpha_+\rangle = |\alpha_0\rangle + G_0(E_\alpha + i0) V_{\text{int}} |\alpha_0\rangle + G_0(E_\alpha + i0) V_{\text{int}} G_0(E_\alpha + i0) V_{\text{int}} |\alpha_0\rangle + \ldots,$$  

(7.54)

When closed from the left with $\langle \beta_0 | V_{\text{int}}$, it reproduces the Born series for $t_{\beta\alpha}(E_\alpha) = \langle \beta_0 \mid T(E_\alpha + i0) \mid \beta_0 \rangle$ which can be obtained from (7.34); the latter series, truncated to the first term, gives the Born approximation (7.36).

Another useful approximation can be obtained if the interaction $V_{\text{int}}$ consists of two parts: $V_{\text{int}} = V_{\text{strong}} + V_{\text{weak}}$ of which one is “strong” and the other one “weak”. One is then interested in accounting for the strong interactions exactly, while the effects of the weak ones can be treated in the simplest approximation. To this end, in addition to the in and out eigenstates $|\alpha_{\pm}\rangle$

\(^{23}\)Similar representation of $t_{\beta\alpha}(E_\alpha)$ in terms of the out state is obtained by taking the Hermitian conjugation of $T(E_\alpha - i0) |\beta_0\rangle = V_{\text{int}} |\beta_-\rangle$ and using the property $T^\dagger(z) = T(z^*)$. This leads to $t_{\beta\alpha}(E_\alpha) = \langle \beta_- | V_{\text{int}} |\alpha_0\rangle$.

\(^{24}\)Weinberg in his book derives the formula (7.53) directly from this equality. His derivation (quicker than the one given here) suffers, however, from the not fully convincing application of the residue method to the integral over $E_\beta$ implicit in (7.53); it does not extend to the whole real axis as requires this method, but is restricted to $E_\beta > M_{\text{min}} \geq 0$ (energy of the states $|\alpha_0\rangle$ representing particles is never negative).
Figure 7.1: Strong interaction induced rescattering of pions produced in the decay of Kaon.

of the full Hamiltonian \( H = H_0 + V_{\text{strong}} + V_{\text{weak}} \) one defines also the \textit{in} and \textit{out} states with respect to the strong interaction

\[
|\beta_{\pm}\rangle = |\beta_0\rangle + \frac{1}{E_\beta - H_0 \mp i0} V_{\text{strong}} |\beta_{\pm}\rangle,
\]

so that

\[
\langle \beta_0 \rangle = \langle \beta_{\pm}\rangle - \langle \beta_{\pm}\rangle V_{\text{strong}} \frac{1}{E_\beta - H_0 \mp i0}.
\]

The full matrix \( t_{\beta\alpha}(E_\alpha) \) (7.51) can be then written in the form

\[
t_{\beta\alpha} = \left( \langle \beta_{-}\rangle - \langle \beta_{+}\rangle |V_{\text{strong}} \frac{1}{E_\beta - H_0 + i0} \right) (V_{\text{strong}} + V_{\text{weak}}) |\alpha_+\rangle
\]

\[
\approx \langle \beta_{-}\rangle |V_{\text{weak}} |\alpha_+\rangle + \langle \beta_{+}\rangle |V_{\text{strong}} |\alpha_0\rangle,
\]

where the formula (7.52) with \( V_{\text{int}} \) replaced by \( V_{\text{strong}} + V_{\text{weak}} \) has been used (in the denominator \( E_\beta \) can be replaced by \( E_\alpha \) because we need \( t_{\beta\alpha} \) for \( E_\beta = E_\alpha \)) to replace the product \( |E_\beta - H_0 + i0|^{-1} (V_{\text{strong}} + V_{\text{weak}}) |\alpha_+\rangle \) by \( |\alpha_+\rangle - |\alpha_0\rangle \).

This (exact) formula is most useful if the strong interaction cannot induce the \( \alpha \rightarrow \beta \) transition. The second term, which is just \( t_{\beta\alpha} \) in the absence of weak interactions (just set \( V_{\text{weak}} \) to zero in the formula given in the footnote related to the formula (7.51) to see it!) which describes all possible transitions \( \alpha \rightarrow \beta \) induced by \( V_{\text{strong}} \) alone, is then zero and, moreover, since the effects of \( V_{\text{weak}} \) are small, one can approximate the full Hamiltonian \textit{in} state \( |\alpha_+\rangle \) in the first term by \( |\alpha_{+}\rangle \). The resulting formula \( t_{\beta\alpha} \approx \langle \beta_{-}\rangle |V_{\text{weak}} |\alpha_{+}\rangle \) is used e.g. in nuclear physics to compute rates of nuclear weak beta decays \( |\alpha_{+}\rangle \) and \( |\alpha_{-}\rangle \) are then the initial and final nucleon states). Furthermore, using the property (7.40) of the \( S \) matrix, this formula can be rewritten as

\[
t_{\beta\alpha} = \int d\gamma S_{\beta\gamma} \langle \gamma_{+}\rangle |V_{\text{weak}} |\alpha_{+}\rangle.
\]

In this form it is used to account for the strong interaction re-scattering effects (shown graphically in figure 7.1) in hadronic weak decays; such effects are crucial for CP violation effects in the Kaon system.
The Born formula (7.36) is the first term of the entire perturbative series which is obtained either by sandwiching the series (7.34) between the states $\langle \beta_0 |$ and $| \alpha_0 \rangle$ and evaluating it for $z = E_\alpha + i0$, or by using the Lippmann-Schwinger formula (7.53) for $| \alpha_+ \rangle$ in the exact expression (7.51) for $t_{\beta\alpha}$:

$$
t_{\beta\alpha} \equiv \langle \beta_0 | V_{\text{int}} | \alpha_+ \rangle = V_{\beta\alpha} + \int d\gamma \frac{V_{\beta\gamma} t_{\gamma\alpha}(E_\alpha)}{E_\alpha - E_\gamma + i0},
$$

(7.59)

where $V_{\beta\alpha} \equiv \langle \beta_0 | V_{\text{int}} | \alpha_0 \rangle$. Iterating this equation yields the series:

$$
t_{\beta\alpha} = V_{\beta\alpha} + \int d\gamma \frac{V_{\beta\gamma} V_{\gamma\alpha}}{E_\alpha - E_\gamma + i0}
+ \int d\gamma \int d\gamma' \frac{V_{\beta\gamma} V_{\gamma\gamma'} V_{\gamma'\alpha}}{(E_\alpha - E_\gamma + i0)(E_\alpha - E_{\gamma'} + i0)} + \ldots,
$$

(7.60)

This is the so-called “old-fashioned” perturbation calculus. While in some situations it is convenient to investigate some specific issues, its main drawback in relativistic theories is the lack of manifest Lorentz covariance.

An alternative, more satisfactory in this respect, approach to perturbative calculation of the $S$-matrix elements is developed by starting directly from the formulae (7.25) and (7.24). Differentiating the latter with respect to $t_2$ we get as in section 1.1 the formula (1.23) and, hence,

$$
S_0 = U_I(+\infty, -\infty) = T \exp \left( -i \int_{-\infty}^{+\infty} dt V_{\text{int}}^I(t) \right),
$$

(7.61)

where the interaction operator in the Dirac picture reads

$$
V_{\text{int}}^I(t) \equiv e^{iH_0 t} V_{\text{int}} e^{-iH_0 t}.
$$

(7.62)

The formula (7.61) is the basis of the commonly used time-dependent perturbation calculus.

The equivalence of the formula (7.61) and the formulae (7.23) and (7.60) should be clear from the construction (at least at the formal level), but can also be seen directly by making use of the identity

$$
\frac{1}{E_\alpha - E_\beta + i0} = \frac{1}{i} \int_0^{+\infty} d\tau e^{i(E_\alpha - E_\beta + i0)\tau},
$$

(7.63)

to represent the energy denominators in (7.60). For example, the first terms in the expansion of (7.61) give

$$
S_{\beta\alpha} = \langle \beta_0 | S_0 | \alpha_0 \rangle = \langle \beta_0 | 1 - i \int_{-\infty}^{+\infty} dt V_{\text{int}}^I(t) + \ldots | \alpha_0 \rangle
$$

245
\[ = \delta_{\beta\alpha} - i \int_{-\infty}^{+\infty} dt e^{-i(E_{\alpha} - E_{\beta})t} V_{\beta\alpha} + \ldots \]
\[ = \delta_{\beta\alpha} - 2\pi i \delta(E_{\alpha} - E_{\beta}) V_{\beta\alpha} + \ldots \]
\[ + \frac{(-i)^2}{2!} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \langle \beta_0 | T(V_{\text{int}}(t_1) V_{\text{int}}(t_2)) | \alpha_0 \rangle + \ldots \]

and so on. The advantage of the time-dependent perturbative expansion based on the formula (7.61) lies in the fact that in relativistic theories it allows to keep Lorentz invariance manifest at each stage of the calculations.

### 7.4 S matrix in relativistic quantum mechanics

We now ask the question, what are the conditions the interaction \( V_{\text{int}} \) should satisfy in order to lead to a relativistically covariant S matrix. Obviously, the first (but by no means sufficient) requirement is that the free particle states, which describe particles long before and long after the reaction, transform as described in Section 6 when the reference frame is changed. Thus, in the Hilbert space there should act a representation of the Poincaré group (or of its universal covering) by unitary operators \( U_0(\Lambda, a) \) whose action on the multi-particle \( H_0 \) generalized eigenstates constructed as tensor products of the 1-particle states follows from the rules (6.111) with \( U_0(\Lambda, a) \) and reads

\[ U_0(\Lambda, a)|(|p_1\sigma_1, \ldots, p_N\sigma_N)_0 \rangle = e^{-ia(\Lambda p_1 + \ldots + \Lambda p_N)} \times \sum_{\sigma_1, \ldots, \sigma_N} |(\Lambda p_1\sigma_1, \ldots, \Lambda p_N\sigma_N)_0 \rangle D^{(s_1)}_{\sigma_1\sigma_1}(W(\Lambda, p_1)) \ldots D^{(s_N)}_{\sigma_N\sigma_N}(W(\Lambda, p_N)). \]

The subscript “0” denotes the free particle states \( |\alpha_0 \rangle \) and \( \Lambda p \) stand for the spatial components of the four-vectors \( \Lambda_{\mu} p_{\mu} \). As explained at the end of Section 6.5, the operators \( P_0^\mu \) and \( J_0^\mu \), that is, \( H_0, P_0, J_0 \) and \( K_0 \) generating \( U_0(\Lambda, a) \) satisfying the commutation rules (6.21) can be constructed given the creation and annihilation operators of free particles (provided the energy operator \( H_0 \) has the appropriate relativistic form).

Furthermore, in a relativistic theory of interacting particles there should also exist a set of unitary operators \( U(\Lambda, a) \), also furnishing a representation of the Poincaré group, and acting on the in and out state-vectors, whose labels are in one-to-one correspondence with the momentum and spin labels of the free multi-particle vectors, according to the rule:

\[ U(\Lambda, a)|(|p_1\sigma_1, \ldots, p_N\sigma_N)_{\pm} \rangle = e^{-ia(\Lambda p_1 + \ldots + \Lambda p_N)} \times \sum_{\sigma_1, \ldots, \sigma_N} |(\Lambda p_1\sigma_1, \ldots, \Lambda p_N\sigma_N)_{\pm} \rangle D^{(s_1)}_{\sigma_1\sigma_1}(W(\Lambda, p_1)) \ldots D^{(s_N)}_{\sigma_N\sigma_N}(W(\Lambda, p_N)). \]
Because the in and out state-vectors are the two sets of eigenvectors of the same Hamiltonian $H$, the operators $U(\Lambda, a)$ must act on them in exactly the same way. This means that the in and out creation and annihilation operators $a_{\text{in}}(\mathbf{p}, \sigma), a_{\text{in}}^\dagger(\mathbf{p}, \sigma)$ and $a_{\text{out}}(\mathbf{p}, \sigma), a_{\text{out}}^\dagger(\mathbf{p}, \sigma)$ should satisfy the rules (6.111) with $U(\Lambda, a)$ (instead of $U_0(\Lambda, a)$). Of course, the Hamiltonian $H$ is itself one of the operators generating $U(\Lambda, a)$.

The problem we want to investigate in this subsection can be reformulated as follows: for what interactions $V_{\text{int}}$ it is possible to construct the generators $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{K}$ which satisfy the commutation rules (6.21) and which act the same way on the in and out states (thereby ensuring that also the operators $U(\Lambda, a)$ will have the required properties)? Since $H = H_0 + V_{\text{int}} \neq H_0$, it follows from the commutation rules

\[
\begin{align*}
[K^i, P^j] &= iH\delta^{ij}, & [K^i, H] &= iP^i, \\
[K^i_0, P^j_0] &= iH_0\delta^{ij}, & [K^i_0, H_0] &= iP^i_0,
\end{align*}
\]

that either $\mathbf{P} \neq \mathbf{P}_0$ or $\mathbf{K} \neq \mathbf{K}_0$ (or both).

Assuming for the moment that such operators $U(\Lambda, a)$ do exist, we can write the condition for the Lorentz covariance of the $S$-matrix. Let $|\alpha_+\rangle = |(\mathbf{p}_1\sigma_1, \ldots, \mathbf{p}_N\sigma_N)_+\rangle$ and $|\beta_-\rangle = |(\mathbf{p}_1'\sigma_1', \ldots, \mathbf{p}_M'\sigma_M')_-\rangle$. Then

\[
S_{\beta\alpha} = \langle \beta_-|\alpha_+\rangle = \langle \beta_-|U^\dagger(\Lambda, a)U(\Lambda, a)|\alpha_+\rangle = e^{-i\alpha \cdot (\mathbf{p}_1 + \ldots + \mathbf{p}_N - \mathbf{p}_1' + \ldots + \mathbf{p}_M')} \sum_{\sigma_1', \ldots, \sigma_M', \sigma_1, \ldots, \sigma_N} D^{(s_1')}_{\sigma_1'\sigma_1} (W(\Lambda, \mathbf{p}_1)) \ldots D^{(s_M')}_{\sigma_M'\sigma_M} (W(\Lambda, \mathbf{p}_N)) \ldots D^{(s_{\beta})}_{\sigma_{\beta}} (W(\Lambda, \mathbf{p}_{\beta})) D^{(s_{\alpha})}_{\sigma_{\alpha}} (W(\Lambda, \mathbf{p}_{\alpha})) \langle \Lambda^\dagger(\mathbf{p}_1'\sigma'_1, \ldots, \mathbf{p}_M'\sigma'_M)_-|\Lambda(\mathbf{p}_1, \ldots, \mathbf{p}_N\sigma_N)_+\rangle.
\]

(7.67)

(The last line in the above formula is just $S_{\beta\alpha \Lambda}$). Applying this formula for $\Lambda = 1$, we infer that the total four-momentum has to be conserved because otherwise the relation

\[
S_{\beta\alpha} = e^{-i\alpha \cdot (\mathbf{p}_1 + \ldots + \mathbf{p}_N - \mathbf{p}_1' + \ldots + \mathbf{p}_M')} S_{\beta\alpha} \equiv e^{-i\alpha \cdot (P_\alpha - P_{\beta})} S_{\beta\alpha},
\]

would imply $S_{\beta\alpha} = 0$. It follows, that the formula (7.23) can be written in the form

\[
S_{\beta\alpha} = \delta_{\beta\alpha} + (2\pi)^4 \delta^{(4)}(P_{\beta} - P_{\alpha}) (-iM_{\beta\alpha}),
\]

(7.68)

that is, the delta function expressing the overall three-momentum conservation can be factorized from the reaction matrix $t_{\beta\alpha}$ introduced in the preceding subsection:

\[
t_{\beta\alpha} = (2\pi)^3 \delta^{(3)}(\mathbf{P}_{\beta} - \mathbf{P}_{\alpha})M_{\beta\alpha}.
\]

(7.69)
The factor $M_{\beta\alpha}$ is frequently called *invariant amplitude* (despite being rather covariant than invariant...)

The condition equivalent to (7.67) can also be derived for the $S_0$ operator defined by (7.14). Writing

$$S_{\beta\alpha} = \langle \beta_0 | S_0 | \alpha_0 \rangle = \langle \beta_0 | U_0^\dagger(\Lambda, a) U_0(\Lambda, a) S_0 | \alpha_0 \rangle ,$$

we see that with (7.65) the formula (7.67) will be recovered provided

$$[S_0, U_0(\Lambda, a)] = 0 ,$$

or, equivalently, provided the operator $S_0$ given by (7.61) commutes with all generators $H_0, P_0, J_0$ and $K_0$. This can hold only for special forms of $V_{\text{int}}$ and we will see, that the existence of the generators $P, J$ and $K$ which satisfy the commutation rules (6.21) and act the same way on *in* and *out* state-vectors and the condition that $S_0$ commutes with $H_0, P_0, J_0$ and $K_0$ are equivalent.

Let us now construct the generators $P, J$ and $K$. Most of the interactions $V_{\text{int}}$ are such that they commute with the $P_0$ and $J_0$ generators. In such cases we can identify

$$P = P_0 , \quad \text{and} \quad J = J_0 .$$

Indeed, since $P_0$ and $J_0$ all commute with $H_0$, their commutation with the interaction operator $V_{\text{int}}$

$$[P_0, V_{\text{int}}] = [J_0, V_{\text{int}}] = 0 ,$$

implies that they commute also with $H = H_0 + V_{\text{int}}$ and, hence, also with the Møller operator $\Omega(t) \equiv e^{iHt} e^{-iH_0t}$ (also for finite $t$). Trivially then

$$P |\alpha_\pm\rangle = P \Omega_\pm |\alpha_0\rangle = P_0 \Omega_\pm |\alpha_0\rangle = \Omega_\pm P_0 |\alpha_0\rangle ,$$

(and the same for the operators $J$). This shows that the operators $P$ and $J$ act the same way on the *in* and *out* state-vectors. It is also clear that if (7.73) holds, the operators $P_0$ and $J_0$ commute with the evolution operator,

$$[P_0, U_I(\tau_2, \tau_1)] = [J_0, U_I(\tau_2, \tau_1)] = 0 ,$$

for arbitrary finite $\tau_1$ and $\tau_2$ and, therefore, also with the operator $S_0 = U_I(-\infty, +\infty)$. Since the Poincaré group generator $H$ acts the same way on *in* and *out* state-vectors owing to the relations (7.19) and, as has been shown, $[H_0, S_0] = 0$, we conclude that those of the commutation rules (6.21) which involve only $H, P$ and $J$ are (in most cases) easy to satisfy.
It remains to construct the operator $K$ satisfying the commutation rules (6.21) and to prove that $[K_0, S_0] = 0$. This is the most tricky part of the construction. If $P = P_0$ but $H \neq H_0$, then from the commutation rule $[K, H] = iP = iP_0$ it follows that $K \neq K_0$. Therefore we write

$$K = K_0 + W.$$ (7.76)

The operator $W$ has to be such that

$$[K_0, V_{\text{int}}] = -[W, H_0 + V_{\text{int}}] \equiv -[W, H].$$ (7.77)

The operator $K_0$ is known (its action on the free multi-particle vectors $|\alpha_0\rangle$ is known) and one has to construct $W$ satisfying the condition (7.77) and such, that $K$ act the same way on $\text{in}$ and $\text{out}$ state-vectors. The last requirement is the most important. One could try to define $W$ (without imposing any conditions on the interaction $V_{\text{int}}$ itself) by simply giving its matrix elements between a complete set of state vectors, say the $\text{in}$ (or $\text{out}$ or free multi-particle) state-vectors, e.g. by

$$\langle \beta_+ | W | \alpha_+ \rangle := \frac{\langle \beta_+ | [K_0, V_{\text{int}}] | \alpha_+ \rangle}{E_\beta - E_\alpha},$$ (7.78)

(so that it would automatically fulfill the condition (7.77)), but the matrix elements defined in this way would not be smooth functions of energy and $K = K_0 + W$ would not (as will become clear shortly) then act the same way on $\text{in}$ and $\text{out}$ state-vectors.

We will first show that if there exist an operator $W$ satisfying the condition (7.77) and whose matrix elements are smooth functions of energy, then $K$ acts the same way on $\text{in}$ and $\text{out}$ vectors and, at the same time, the operator $K_0$ commutes with the $S_0$ operator. To this end we consider the commutator $[K_0, U_I(\tau_2, \tau_1)]$ for finite $\tau_1$ and $\tau_2$. From the commutation rule $[K_0, H_0] = iP_0$ and the fact that $P_0$ and the fact that $P_0$ commute with $H_0$ one gets the relation

$$[K_0, e^{iH_0\tau}] = -\tau P_0 e^{iH_0\tau}.$$ (7.79)

Similarly, from $[K, H] = iP = iP_0$ and the fact that $P_0$ commutes also with $V_{\text{int}}$ (which means that $P$ commutes with $H$) one finds

$$[K, e^{iH\tau}] = -\tau P e^{iH\tau} = -\tau P_0 e^{iH\tau}.$$ (7.80)

Therefore,

$$[K_0, U_I(\tau_2, \tau_1)] = [K_0, e^{iH_0\tau_2}e^{-iH(\tau_2-\tau_1)}e^{-iH_0\tau_1}]$$

$$= -\tau_2 P_0 U_I(\tau_2, \tau_1) + e^{iH_0\tau_2} [K_0, e^{-iH(\tau_2-\tau_1)}] e^{-iH_0\tau_1}$$

$$+ \tau_1 U_I(\tau_2, \tau_1) P_0.$$
where (7.79) has been used. In the second term we set then $K_0 = K - W$ which enables us to make use of (7.80), after which we find that the term obtained from the commutator of $K$ precisely cancels the two terms containing $P_0$ (recall that $[P_0, U_I(\tau_2, \tau_1)] = 0$). One is therefore left with

$$[K_0, U_I(\tau_2, \tau_1)] = -W_I(\tau_2)U_I(\tau_2, \tau_1) + U_I(\tau_2, \tau_1)W_I(\tau_1), \quad (7.81)$$

where the operator $W_I(\tau)$ in the Dirac picture has been defined

$$W_I(\tau) = e^{iH_0\tau}W e^{-iH_0\tau}. \quad (7.82)$$

From the result (7.81) it is clear that $K_0$ does not commute with the evolution operator for finite times $\tau_1$ and $\tau_2$. However, Lorentz-covariance of the $S$-matrix requires only vanishing of the right hand side of (7.81) in the limits $\tau_1 \to -\infty$ and $\tau_2 \to +\infty$. This is ensured if the matrix elements of $W$ are nonsingular functions of energy, for then, for any two smooth normalizable profiles $\psi(\alpha)$ and $\phi(\beta)$ the Fourier transforms

$$\int d\alpha \int d\beta \phi^*(\beta) \psi(\alpha) \langle \beta_0 | W_I(\tau) | \alpha_0 \rangle = \int d\alpha \int d\beta \phi^*(\beta) \psi(\alpha) e^{i(E_{\beta} - E_{\alpha})\tau} \langle \beta_0 | W | \alpha_0 \rangle, \quad (7.83)$$

are well defined and, by the Riemann-Lebesgue theorem vanish as $\tau \to \pm \infty$, provided the integrand is smooth enough, in particular provided it does not have poles such as the right hand side of (7.78). Thus, if the matrix elements of $W$ between the complete set of generalized free multi-particle eigenvectors $|\alpha_0\rangle$ of $H_0$ are nonsingular functions of energy, all matrix elements of $W_I(\tau)$ between smooth superpositions of such generalized vectors\footnote{Such superpositions form a dense set in the Hilbert space.} vanish for $\tau \to \pm \infty$, that is, in these limits the operator $W_I(\tau)$ vanishes (in the weak sense). The right hand side of (7.81) then vanishes in the double limit $\tau_2 \to \infty$, $\tau_1 \to -\infty$ too, and $K_0$ commutes with the $S_0$ operator.

In this case also the operator $K = K_0 + W$ acts the same way on in and out states. To see this, consider the commutator

$$[K_0, \Omega_\pm] \equiv [K_0, U_I(0, \mp \infty)] = -W_I(0)\Omega_\pm + \Omega_\pm W_I(\mp \infty). \quad (7.84)$$

The result (7.81) with $\tau_2 = 0$ and $\tau_1 = \mp \infty$ has been used here. If matrix elements of $W$ are nonsingular functions of energy, the last term in (7.84) vanishes and one obtains the intertwining relations

$$K\Omega_\mp = K_0\Omega_\mp + W\Omega_\mp = \Omega_\mp K_0 - W\Omega_\mp + W\Omega_\mp = \Omega_\mp K_0, \quad (7.85)$$

where we have used the commutator (7.84) and the fact that $W_I(0) = W$ analogous to (7.74) satisfied by the generators $P$ and $J$ and (7.19) relating.
the Hamiltonians $H$ and $H_0$. Note that if the matrix elements of $W$ were singular functions of energy (as in (7.78), the last term in the commutator (7.84) would be nonzero and the specification of the way of going around the singularities would introduce the difference in $W_I(-\infty)$ and $W_I(+\infty)$; $K$ would then act differently on in and out state-vectors.

Thus, if the operator $W$ with the required properties can be constructed, all the generators $G_0$ of the Poincaré group acting on smooth superpositions of the generalized state-vectors $|\alpha\rangle_0$ commute with the $S_0$ operator and, simultaneously, hold the related intertwining relation:

\[ i) \quad [G_0, S_0] = 0 \quad \Rightarrow \quad [U_0(\Lambda, a), S_0] = 0, \]

\[ ii) \quad G \Omega_\pm = \Omega_\pm G_0 \quad \Rightarrow \quad U(\Lambda, a) \text{ act the same way on the } in \text{ and out states.} \tag{7.86} \]

Analogous intertwining relations will be also crucial for operator quantization of theories of non-Abelian gauge fields (Section 20.3) based on the BRST symmetry. The same scheme (7.86) works also for parity and charge conjugation operators $P$ and $C$, as well as for generators of various possible internal symmetries like isospin or the “eightfold way” $SU(3)$ (see Section 12) etc. Usually all generators $Q^a$ of internal symmetries are such that $Q^a = Q_0^a$ (as is the case with $P$ and $J$), because their free particle counterparts $Q_0^a$ commute with the interaction operator $V_{\text{int}}$. Slightly more tricky is the action of the time reversal operator $T$ because it is antiunitary and interchanges the in and out vectors. We will explore consequences of these symmetries for the $S$ matrix in due course.

The question now is, for which interactions $V_{\text{int}}$ can the operators $W$ with the required properties be constructed? It turns out that a rather broad class of such interactions can be easily identified. Belong to it all interactions $V_{\text{int}}$ whose interaction picture counterparts

\[ V_{\text{int}}^I(t) = e^{iH_0 t} V_{\text{int}} e^{-iH_0 t}, \tag{7.87} \]

appearing in (7.61), can be obtained as the space integral of a local interaction Hamiltonian density $\mathcal{H}_{\text{int}}(t, x)$:

\[ V_{\text{int}}^I(t) = \int d^3x \mathcal{H}_{\text{int}}(t, x), \tag{7.88} \]

which is such that (using now the four-dimensional notation $x^\mu = (t, x)$)

\[ U_0(\Lambda, a) \mathcal{H}_{\text{int}}(x) U_0^{-1}(\Lambda, a) = \mathcal{H}_{\text{int}}(\Lambda \cdot x - a), \tag{7.89} \]

and

\[ [\mathcal{H}_{\text{int}}(x), \mathcal{H}_{\text{int}}(y)] = 0 \quad \text{if} \quad (x - y)^2 < 0. \tag{7.90} \]
Note that for $\Lambda = I$ and $a^\mu = (t, 0)$ (7.89) is consistent with (7.87). It should also be noted that with the condition (7.90) the formula (7.61), which for $V_{\text{int}}^I(t)$ given by (7.88), reads

$$S_0 = \sum_{N=0}^\infty \frac{(-i)^N}{N!} \int d^4x_1 \ldots \int d^4x_N T \{ \mathcal{H}_{\text{int}}(x_1) \ldots \mathcal{H}_{\text{int}}(x_N) \},$$

becomes fully covariant: the time ordering of two spacetime points $x_1$ and $x_2$ is Lorentz invariant only when $(x_1 - x_2)^2 > 0$; but the condition (7.90) makes the time ordering irrelevant for $(x_1 - x_2)^2 \leq 0$.

To these requirements one must also add the one spelled out at the beginning of the subsection 7.3, namely that that structure of $\mathcal{H}_{\text{int}}(x)$ (i.e. $V_{\text{int}}$) must be such that there is a one-to-one correspondence between the eigenstates of $H_0$ and the in and out eigenstates of $H$. In section 9 we will construct interactions $V_{\text{int}}$ satisfying the conditions (7.88)-(7.90) using field operators introduced in section 8). The additional condition will be investigated in section 13.

The property (7.90) of $\mathcal{H}_{\text{int}}(x)$, called the local causality condition, has no counterpart in nonrelativistic quantum mechanics. As it will turn out, it is precisely this property of $\mathcal{H}_{\text{int}}(x)$ that is responsible for the fundamental spin-statistics connection (i.e. it enforces that the creation/annihilation operators of integer and half-integer spin particles satisfy respectively the commutation and the anticommutation rules, thereby implying that these two classes of particles obey respectively the Bose-Einstein and the Fermi-Dirac statistics), as well as for the non-conservation of the number of particles in relativistic theories and, finally, for the existence of antiparticles.

To check that when $\mathcal{H}_{\text{int}}(x)$ satisfies (7.89) the operator $W$ with the required properties can indeed be constructed, we set in (7.89) $a^\mu = 0$ and write

$$U_0(\Lambda) = \exp \left( -\frac{i}{2} \omega_{\mu\nu} J_{0}^{\mu\nu} \right). \quad (7.91)$$

Taking the parameters $\omega_{\mu\nu}$ to be infinitesimal, we get from (7.89) the relation

$$-\frac{i}{2} \omega_{\mu\nu} [J_{0}^{\mu\nu}, \mathcal{H}_{\text{int}}(x)] \approx \mathcal{H}_{\text{int}}(x + \omega \cdot x) - \mathcal{H}_{\text{int}}(x)$$

$$\approx \frac{\partial \mathcal{H}_{\text{int}}(x)}{\partial x^\lambda} \omega_{\mu, \nu} x^\nu = \frac{1}{2} \omega_{\mu\nu} \left( g^{\lambda\mu} x^\nu - g^{\lambda\nu} x^\mu \right) \frac{\partial \mathcal{H}_{\text{int}}(x)}{\partial x^\lambda}, \quad (7.92)$$

which, in view of the arbitrariness of $\omega_{\mu\nu}$, implies

$$-i [J_{0}^{\mu\nu}, \mathcal{H}_{\text{int}}(x)] = \left( g^{\lambda\mu} x^\nu - g^{\lambda\nu} x^\mu \right) \frac{\partial \mathcal{H}_{\text{int}}(x)}{\partial x^\lambda}. \quad (7.93)$$
Taking now $J^0_i \equiv K^i_0$ (i.e. $\mu = 0$ and $\nu = i$ in the above equality) we get:

$$-i \left[ K^i_0 , \mathcal{H}_{\text{int}}(x) \right] = t \nabla_i \mathcal{H}_{\text{int}}(x) + x^i \frac{\partial \mathcal{H}_{\text{int}}(x)}{\partial t}. \quad (7.94)$$

Finally, setting $t = 0$, and integrating both sides over $d^3x$ we find

$$\left[ K^i_0 , \int d^3x \mathcal{H}_{\text{int}}(0, x) \right] \equiv \left[ K^i_0 , V_{\text{int}} \right] = i \int d^3x x^i \left( \frac{\partial \mathcal{H}_{\text{int}}(x)}{\partial t} \right)_{t=0}
= i \int d^3x x^i \left. \left( e^{iH_0t} \mathcal{H}_{\text{int}}(0, x) e^{-iH_0t} \right) \right|_{t=0}
= \left[ H_0, - \int d^3x x^i \mathcal{H}_{\text{int}}(0, x) \right]. \quad (7.95)$$

This suggests that one can take

$$W = - \int d^3x x \mathcal{H}_{\text{int}}(0, x). \quad (7.96)$$

Owing to the condition (7.90) then

$$\left[ V_{\text{int}}, W \right] = - \int d^3y \int d^3x \left[ \mathcal{H}_{\text{int}}(0, y), \mathcal{H}_{\text{int}}(0, x) \right] = 0, \quad (7.97)$$

and the condition (7.77), $[K_0, V_{\text{int}}] = -[W, H_0 + V_{\text{int}}]$ is satisfied because (7.95) ensures that $[K_0, V_{\text{int}}] = -[W, H_0]$. Thus, if the matrix elements of the operator $\mathcal{H}_{\text{int}}(0, x)$ between generalized $H_0$ eigenvectors $|\alpha \rangle$ are nonsingular functions of energy, the operators $K = K_0 + W$ act the same way on in and out generalized eigenvectors of $H$.

### 7.5 Unitarity of the $S$ matrix

As has been argued in Section 7.1, the $S_0$ operator, which is a (double) limit of the unitary evolution operator $U_I(\tau_2, \tau_1)$ (7.61), is itself unitary, i.e. satisfies $S^*_0 S_0 = 1$. This relation written in terms of the $S$-matrix elements $S_{\beta \alpha}$ and the amplitudes $M_{\beta \alpha}$ takes the form

$$S_{\beta \alpha} = \delta_{\beta \alpha} + (2\pi)^4 \delta^{(4)}(P_\beta - P_\alpha) (-iM_{\beta \alpha}),$$

one can therefore write

$$\delta_{\beta \alpha} = \int d\gamma S^*_{\gamma \beta} S_{\gamma \alpha}
= \int d\gamma \left[ \delta_{\gamma \beta} + (2\pi)^4 \delta^{(4)}(P_\gamma - P_\beta) (+iM^*_{\gamma \beta}) \right]
= \left[ \delta_{\gamma \alpha} + (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha) (-iM_{\gamma \alpha}) \right]
= \delta_{\beta \alpha} + (2\pi)^4 \delta^{(4)}(P_\beta - P_\alpha) + iM^*_{\alpha \beta} - iM_{\beta \alpha}
= \int d\gamma (2\pi)^4 \delta^{(4)}(P_\gamma - P_\beta) \delta^{(4)}(P_\gamma - P_\alpha) M^*_{\gamma \beta} M_{\gamma \alpha}.$$
Hence the matrices $M_{\beta\alpha}$ satisfy the following important unitarity condition

$$-i \left( M_{\alpha\beta}^* - M_{\beta\alpha} \right) = \int d\gamma (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha) M_{\gamma\beta}^* M_{\gamma\alpha},$$

(7.98)

in which both sides are to be taken for $P_\beta = P_\alpha$. Recall that in the adopted notation the integral over $d\gamma$ involves also summation over different numbers of particles in the state $|\gamma_0\rangle$ and includes appropriate factors $1/n_i!$ for each set of $n_i$ identical particles of type $i$ in this state.

In the perturbative expansion of the $S$-matrix, when the amplitudes $M_{\alpha\beta}$ are computed as power series in some (small) coupling constant(s), the importance of the condition (7.98) stems from the fact that it relates contributions to $M_{\alpha\beta}$ which are of different orders in the couplings. One is therefore able to say something about higher order contributions to $M_{\alpha\beta}$ knowing it in lower orders. This will be exploited in Section 7.5. Here we discuss some consequences of the condition (7.98) which do not rely on any perturbative expansion and have therefore a general character.

The first useful relation is obtained by setting in (7.98) $\beta = \alpha$. One then gets

$$-2 \text{Im} M_{\alpha\alpha} = \int d\gamma (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha) |M_{\gamma\alpha}|^2.$$  

(7.99)

This can for example be used to argue that in the framework of relativistic quantum field theory the total decay widths of a particle and of its antiparticle are equal. To this end we write the condition (7.99) for the CPT transformed state $|\alpha\rangle$ of a single unstable particle at rest

$$-2 \text{Im} M_{(\text{CPT}\alpha)(\text{CPT}\alpha)} = \int d\gamma (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha) |M_{\gamma(\text{CPT}\alpha)}|^2,$$

(7.100)

($P_{\text{CPT}\alpha}^\mu = P_\alpha^\mu$ because if the $\text{CPT}$ operator commutes with the Hamiltonian, the states $|\alpha\rangle$ and $\text{CPT}|\alpha\rangle$ have the same energy and the action of the $\text{CPT}$ operator does not reverse the three-momenta), and use the fact that because the CPT operation is a valid symmetry of any relativistic, unitary quantum field theory, one always has

$$S_{(\text{CPT}\alpha)(\text{CPT}\beta)} = S_{\beta\alpha},$$

(7.101)

---

26 One can arrive at the same result writing $S_0 = \hat{1} - iT_0$ and taking the matrix elements between the $|\alpha_0\rangle$ and $|\beta_0\rangle$ state-vectors of the operator equality

$$-i(T_0^\dagger - T_0) = T_0^\dagger T_0,$$

using the relation $\langle \beta_0|T_0^\dagger|\alpha_0\rangle = (\langle \alpha_0|T_0|\beta_0\rangle)^*$.  

27 Since the $S$-matrix elements can, strictly speaking, be defined only for absolutely stable particles, the “proof” presented here cannot be considered truly rigorous.
which in turn implies that
\[ M_{(CPT \alpha)(CPT \beta)} = M_{\beta \alpha} \] (7.102)
Thus, the left hand sides of (7.99) and (7.100) are equal (for \( \alpha = \beta \) all phase factors resulting from the CPT action mutually cancel out) and so are the RHSs. Up to a multiplicative constant factor (see Section 10.2) the expressions on the right hand sides of (7.99) and (7.100) represent the total decay widths of a particle and its antiparticle with reversed spin, respectively. However, by rotational invariance the full decay rate (integrated over all possible directions of the final state particles and summed over possible projections of their spins or over their helicities) cannot depend on the spin projection of the decaying particle represented by \(|\alpha\rangle\). This proves the proposition.

Another general consequence of the unitarity of the S-matrix can be obtained by taking for \(|\alpha_0\rangle\) in (7.99) some particular two-particle state. Dividing both sides by the initial state flux factor \( F \) (to be defined in section 10.2) we obtain the optical theorem in the form
\[ \sigma_{\text{tot}}(\alpha \to \text{anything}) = -\frac{2}{F} \text{Im} M_{\alpha \alpha}. \] (7.103)

Further consequences of (7.99) can be explored by going over to the helicity basis of multiparticle state-vectors introduced (on the example of two-particle state-vectors) in section 6.4. Consider a collision process of two particles which are represented by the state \(|\alpha_0\rangle\) and a two-particle final state represented by \(|\beta_0\rangle\) in the center of mass (CM) frame. Inserting in the formula
\[ \langle \beta_0|T_0|\alpha_0\rangle = (2\pi)^4 \delta^{(4)}(P_\alpha - P_\beta)M_{\beta \alpha}. \] (7.104)
in which \( T_0 = i(S_0 - 1) \) (cf. (7.22)), the vectors \(|\alpha_0\rangle\) and \(|\beta_0\rangle\) decomposed as in (6.110) into the states with definite total angular momentum, and denoting the labels of the state \(|\beta_0\rangle\) with primes, we get
\[ \langle P', p', \lambda'_1, \lambda'_2|T_0|0, p, \lambda_1, \lambda_2\rangle = \sum_{j'} \sum_{m'_j} \sum_{j} \sum_{m_j} D_{j'm'_j}^{(j)*} \langle 0, s, \lambda_1, \lambda_2, j, m_j| T_0 |0, s, \lambda_1, \lambda_2, j, m_j\rangle. \] (7.105)
Since the total angular momentum as well as its z-axis projection are conserved by the interaction, the matrix element in (7.105) must take the form
\[ \langle P', \sqrt{s}', \lambda'_1, \lambda'_2, j', m'_j | T_0 |0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle = (2\pi)^4 \delta^{(3)}(P'_{00} - \sqrt{s}) 64\pi^2 T_{\lambda'_1, \lambda'_2; \lambda_1, \lambda_2}^{(j)}(s) \delta_{j', j} \delta_{m'_j, m_j}. \] (7.106)
\[^{28}\text{Recall (Section 6.4) that the vectors }|P, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle \text{ are } H_0 \text{ eigenvectors with the eigenvalue } \sqrt{P^2 + s}.\]
with the factor $64\pi^2$ introduced for further convenience. $T_{\lambda_1,\lambda_2;\lambda_1,\lambda_2}(s)$, defined by (7.106), are called the partial wave amplitudes. We recall (see the formula (6.107)) that for identical particles the states $|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ vanish for $j$ odd if $\lambda_1 = \lambda_2$ and so must do the corresponding partial wave amplitudes (for $j$ odd and $\lambda_1 = \lambda_2$ or $\lambda_1' = \lambda_2'$). Comparing with (7.104) we get

$$
\mathcal{M}_{\beta\alpha} = 16\pi \sum_{j} \sum_{m_j} (2j+1) T_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s) D_{m_j,\lambda_1' - \lambda_2'}^{(j)\ast} (\Omega_{\mathbf{p}'}^\prime) D_{m_j,\lambda_1 - \lambda_2}(\Omega_{\mathbf{p}}). \tag{7.107}
$$

The choice of the angular momentum quantization axis in the direction of the momentum $\mathbf{p}$ reduces $D_{m_j,\lambda_1 - \lambda_2}(\Omega_{\mathbf{p}})$ to $\delta_{m_j,\lambda_1 - \lambda_2}$, so that:

$$
\mathcal{M}_{\beta\alpha} = 16\pi \sum_{j} (2j+1) T_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s) D_{\lambda_1 - \lambda_2,\lambda_1',\lambda_2'}(\Omega_{\mathbf{p}}). \tag{7.108}
$$

In the following we will take the state $|\beta_0\rangle$ to have the same two-particle content as $|\alpha_0\rangle$; the factors $T_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(s)$ will therefore be the elastic (with possible spin flip) scattering partial wave amplitudes. We will also need the formula

$$
\mathcal{M}_{\gamma\beta} = 16\pi \sum_{j} \sum_{m_j} (2j+1) \tilde{T}_{\lambda_a,\lambda_b;\lambda_1',\lambda_2'}(s) D_{m_j,\lambda_a - \lambda_b}(\Omega_{\mathbf{p}}) D_{m_j,\lambda_1' - \lambda_2'}(\Omega_{\mathbf{p}'}),
$$

with the state $|\gamma_0\rangle = |0,\mathbf{p},\lambda_a,\lambda_b\rangle \equiv |\mathbf{P} = 0,\mathbf{p},\lambda_1\lambda_2\rangle$ of two particles $a$ and $b$ (not necessarily the same as in $|\alpha_0\rangle$ and $|\beta_0\rangle$) characterized by the helicities $\lambda_a$, $\lambda_b$ and the momentum $\mathbf{p}$ (in their CM frame); we denote the corresponding partial wave amplitudes by $\tilde{T}_{\lambda_a,\lambda_b;\lambda_1',\lambda_2'}(s)$. For $|\gamma_0\rangle = |\alpha_0\rangle$, that is, for $\lambda_a = \lambda_1$, $\lambda_b = \lambda_2$ and $\mathbf{p} = \mathbf{p}$, we get from this formula

$$
\mathcal{M}_{\alpha\beta} = 16\pi \sum_{j} (2j+1) T_{\lambda_1,\lambda_2;\lambda_1',\lambda_2'}(s) D_{\lambda_1 - \lambda_2,\lambda_1',\lambda_2'}(\Omega_{\mathbf{p}}), \tag{7.109}
$$

with the same partial wave amplitudes $T_{\lambda_1,\lambda_2;\lambda_1',\lambda_2'}(s)$ as in (7.108) but with the helicity labels interchanged.

These formulae allow to single out the contribution of two-particle states $|\gamma_0\rangle = |\mathbf{p}_a,\lambda_a,\mathbf{p}_b,\lambda_b\rangle$ to the unitarity condition (7.98) specified to the elastic scattering amplitude, i.e. for $|\alpha_0\rangle$ and $|\beta_0\rangle$ with the same particle content.

The integral $\int d\gamma$ in (7.98) involves the following contribution of a two-particle state $|\gamma_0\rangle$:

$$
\sum_{\lambda_a,\lambda_b} N_{\lambda_a,\lambda_b} \int d\mathbf{p}_a \int d\mathbf{p}_b (2\pi)^4 \delta(E_a + E_b - \sqrt{s}) \delta^{(3)}(\mathbf{p}_a + \mathbf{p}_b) \mathcal{M}_{\gamma\beta}^\ast \mathcal{M}_{\gamma\alpha}
\quad = \frac{1}{32\pi^2 s} \lambda^{1/2}(s,m_a^2,m_b^2) \sum_{\lambda_a,\lambda_b} N_{\lambda_a,\lambda_b} \int d\Omega_{\mathbf{p}} (\mathcal{M}_{\gamma\beta}^\ast \mathcal{M}_{\gamma\alpha}) \text{on shell}, \tag{7.110}
$$

256
where \(m_a\) and \(m_b\) are the masses of the particles \(a\) and \(b\) in the state \(|\gamma_0\rangle\), the subscript “on shell” means \(\mathbf{p}_a = -\mathbf{p}_b \equiv \mathbf{p}\) with \(|\mathbf{p}|\) determined by \(\sqrt{s}\) (and \(m_a\) and \(m_b\)), and the function \(\lambda(s, m_a^2, m_b^2)\) (do not confuse it with the helicity labels!) reads

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{7.111}
\]

The factor \(N_{\lambda_a, \lambda_b} = 1/2\) for \(a = b\) (i.e. for identical particles) with \(\lambda_a = \lambda_b\), and \(N_{\lambda_a, \lambda_b} = 1\) otherwise, follows from the explicit form of the completeness relation (7.5) (explained in detail in Section 5.1). Using now the property (6.109) of the \(D\)-functions we arrive at

\[
-i \sum_j (2j + 1) D^{(j)*}_{\lambda_1 - \lambda_2, \lambda'_1 - \lambda'_2} (\Omega_p') \left[ T^{(j)*}_{\lambda_1; \lambda'_1; \lambda_2; \lambda'_2} (s) - T^{(j)}_{\lambda'_1; \lambda'_2; \lambda_1; \lambda_2} (s) \right]
= \sum_{(ab)} \sum_j (2j + 1) D^{(j)*}_{\lambda_1 - \lambda_2, \lambda'_1 - \lambda'_2} (\Omega_p') \times \sum_{\lambda_a, \lambda_b} \frac{2}{s} \lambda^{1/2}(s, m_a^2, m_b^2) N_{\lambda_a, \lambda_b} \tilde{T}^{(j)*}_{\lambda_a, \lambda_b; \lambda'_1; \lambda'_2} (s) \tilde{T}^{(j)}_{\lambda_a, \lambda_b; \lambda_1; \lambda_2} (s)
+ \frac{1}{16\pi} \int d\gamma (2\pi)^4 \delta^{(4)} (P_\gamma - P_a) \mathcal{M}^*_\gamma \mathcal{M}_\gamma . \tag{7.112}
\]

where the sum in the second line is over all kinematically allowed final states with two\(^{29}\) particles \((ab)\) and the integral in the last line includes all kinematically allowed three- and more particle final states.

If the three- and more-particle channels are kinematically inaccessible (or forbidden by some conservation laws) the last line is absent and integrating both sides of (7.112) over \(d\Omega_p'\) with \(D^{(j)}_{\lambda_1 - \lambda_2, \lambda'_1 - \lambda'_2} (\Omega_p')\) we obtain the unitarity condition in the form

\[
-i \left[ T^{(j)*}_{\lambda_1; \lambda'_1; \lambda_2; \lambda'_2} (s) - T^{(j)}_{\lambda'_1; \lambda'_2; \lambda_1; \lambda_2} (s) \right]
= \frac{2}{s} \lambda^{1/2}(s, m_a^2, m_b^2) \sum_{\lambda_1, \lambda_2} N_{\lambda_1, \lambda_2} T^{(j)*}_{\lambda_1; \lambda'_1; \lambda_2; \lambda'_2} (s) T^{(j)}_{\lambda_1; \lambda'_1; \lambda_2; \lambda'_2} (s) \tag{7.113}
+ \sum_{(ab)\neq (12)} \frac{2}{s} \lambda^{1/2}(s, m_a^2, m_b^2) \sum_{\lambda_a, \lambda_b} N_{\lambda_a, \lambda_b} \tilde{T}^{(j)*}_{\lambda_a, \lambda_b; \lambda'_1; \lambda'_2} (s) \tilde{T}^{(j)}_{\lambda_a, \lambda_b; \lambda_1; \lambda_2} (s) ,
\]

where in the second line we have explicitly singled out the contribution of the elastic channel. In the fully scattering elastic scattering channel with no

\(^{29}\)Pairs \((ab)\) and \((ba)\) must be treated here as one and the same state i.e. only one of them should be included in the sum. Alternatively, one can sum over \(a\) and \(b\) independently, including in the sum both states, \((ab)\) and \((ba)\), and setting \(N_{\lambda_a, \lambda_b} = \frac{1}{2}\).
change of helicities, i.e. for \( \lambda' = \lambda_1, \lambda'_2 = \lambda_2 \), this can be rewritten in the form

\[
\left[ \text{Re} \, T_{\lambda_1\lambda_2;\lambda_1\lambda_2}^{(j)}(s) \right]^2 + \left[ \text{Im} \, T_{\lambda_1\lambda_2;\lambda_1\lambda_2}^{(j)}(s) + \frac{s \lambda_{12}^{-1/2}}{2N_{\lambda_1\lambda_2}} \right]^2 = \frac{s^2 \lambda_{12}^{-1}}{4N_{\lambda_1\lambda_2}^2} - R_j^2(s),
\]

in which

\[
R_j^2(s) = \sum_{(\lambda'_1\lambda'_2)\neq(\lambda_1\lambda_2)} \frac{N_{\lambda'_1\lambda'_2}}{N_{\lambda_1\lambda_2}} \left| T_{\lambda'_1\lambda'_2;\lambda_1\lambda_2}^{(j)}(s) \right|^2
\]

\[+ \sum_{(ab)\neq(12)} \sum_{\lambda_u,\lambda_b} \frac{N_{\lambda_u\lambda_b}}{N_{\lambda_1\lambda_2}} \frac{\lambda_{ab}^{1/2}}{\lambda_{12}^{1/2}} \left| \tilde{T}_{\lambda_u\lambda_b;\lambda_1\lambda_2}^{(j)}(s) \right|^2,
\]

and we have used the notation \( \lambda_{12} \equiv \lambda(s, m_1^2, m_2^2) \) and \( \lambda_{ab} \equiv \lambda(s, m_a^2, m_b^2) \).

It is easy to see that inelastic processes leading to multi-particle final states can also be included in \( R_j^2(s) \). To show this without introducing an explicit kinematical characterization of the multi-particle states, we write, using the formula (6.110), the amplitudes of transitions from the \( |\alpha_0\rangle \) and \( |\beta_0\rangle \) two-particle states with the same particle content into any three- or more-particle state \( |\gamma_0\rangle \) as

\[
\mathcal{M}_{\gamma_\alpha} = \sum_{j} \sum_{m_j'} \sqrt{\frac{2j'+1}{4\pi}} T_{\gamma;\lambda_1,\lambda_2,m_j'}(\gamma; s') \, D_{m_j'\lambda_1\lambda_2}^{(j')} \lambda \rho \, \Omega \rho \rho,
\]

\[
\mathcal{M}_{\gamma_\beta} = \sum_{j''} \sqrt{\frac{2j''+1}{4\pi}} T_{\gamma;\lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s),
\]

where the amplitudes \( T_{\gamma;\lambda_1,\lambda_2,m_j}(\gamma; s) \) are defined by the equality

\[
(2\pi)^4 \delta^{(4)}(P_\gamma) \delta(P_\gamma - \sqrt{s}) \, T_{\gamma;\lambda_1,\lambda_2,m_j}(\gamma; s) \equiv \langle \gamma_0 | T_0 | 0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j \rangle.
\]

The symbol \( \gamma \) used as the argument of \( T^{(j)} \) is to remind that this amplitude depends, apart from \( s \), also on the variables (relative momenta and helicities) needed to specify the multiparticle state \( |\gamma_0\rangle \). In \( \mathcal{M}_{\gamma_\alpha} \), similarly as in (7.108), the equality \( D_{m_j\lambda_1\lambda_2}^{(j)}(\Omega \rho \rho) = \delta_{m_j,\lambda_1-\lambda_2} \) has been used. With these formulae, and setting \( \lambda'_1 = \lambda_1, \lambda'_2 = \lambda_2 \), the last line of (7.112) takes the form

\[
\frac{1}{16\pi} \sum_{j''} \sum_{m_j'} \sum_{j'} \sqrt{(2j''+1)(2j'+1)} D_{m_j'\lambda_1-\lambda_2}^{(j')} \lambda \rho \, \Omega \rho \rho
\]

\[\times \int d\gamma (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha) \, T_{\gamma;\lambda_1,\lambda_2,m_j'}(\gamma; s) \, T_{\gamma;\lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s).
\]

258
Integrating now as previously both sides of (7.112) with $D_{(j)}^{(j')}_{\lambda_1-\lambda_2,\lambda'_1-\lambda'_2}(\Omega_{p'})$ over $d\Omega_{p'}$ we get in (7.113) an extra term

$$+ \frac{1}{16\pi} \sum_{j''} \sqrt{2j'' + 1} \int d\gamma (2\pi)^4 \delta^{(4)}(P_{\gamma} - P_{\alpha}) \times T_{\gamma; \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s) T_{\gamma; \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s).$$

However, angular momentum conservation implies that only $j'' = j$ can contribute to the sum and therefore the last term in (7.112) adds to $R_j^2$ given by (7.115) a strictly nonnegative contribution proportional to

$$\frac{1}{16\pi} \int d\gamma (2\pi)^4 \delta^{(4)}(P_{\gamma} - P_{\alpha}) \left| T_{\gamma; \lambda_1,\lambda_2,\lambda_1-\lambda_2}(\gamma; s) \right|^2. \quad (7.118)$$

The relation (7.114) demonstrates that the amplitude $T_{\lambda_1,\lambda_2;\lambda_1,\lambda_2}(s)$ of the elastic scattering with no change of helicities must lie on a circle, called the Argand circle, whose radius in not greater than $s\lambda^{-1/2}/2N_{\lambda_1,\lambda_2}$ and the center is at the point $(0, -s\lambda^{-1/2}/2N_{\lambda_1,\lambda_2})$ in the complex plane, as shown graphically in Figure 7.2. This shows, that the elastic scattering amplitude must have a nonzero imaginary part which grows as more and more inelastic channels open up with increasing $\sqrt{s}$ (at high energies elastic scattering amplitudes are therefore predominantly imaginary). From (7.114) it also follows, that the fully elastic (with no change of helicities) scattering partial wave amplitude

![Argand circles: if inelastic channels are closed, i.e. if $R_j^2(s) = 0$, (left) the radius is $s\lambda^{-1/2}/2N_{\lambda_1,\lambda_2}$; if inelastic channels are open (right) the radius is smaller. Partial amplitudes of the elastic scattering must lie on the Argand circle. Short-dashed lines show possible partial elastic scattering amplitudes in a weakly coupled theory (small corrections in the perturbative expansion) whereas the long-dashed ones illustrate elastic scattering amplitudes typical for a strongly coupled (nonperturbative) theory.](image)
\[ T^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2}(s) \] can be represented in terms of the (complex) phase shift \( \delta_j(s) \) and \( \beta_j(s) \) as

\[ T^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2}(s) = i \frac{s \lambda_1^{-1/2}}{2 N_{\lambda_1 \lambda_2}} \left( e^{2i \delta_j(s)-2\beta_j(s)} - 1 \right), \quad (7.119) \]

where \( 0 < \delta_j(s) < \pi \) and

\[ \beta_j(s) = -\frac{1}{4} \ln \left( 1 - \frac{4 N_{\lambda_1 \lambda_2}^2}{s^2 \lambda^{-1}} R_j^2 \right). \quad (7.120) \]

Of course, if only the elastic channel with no helicity change is open \( \beta_j(s) = 0 \) and the phase shifts are real numbers.

These results, used in (7.106) and combined with the formula (6.108), allows to write down the \( S \)-matrix element corresponding to elastic (with no change of helicities) scattering in the basis of states with definite angular momentum in the form

\[ \langle P', \sqrt{s}, \lambda_1, \lambda_2, j, m_j | 1 - iT_0 | P, \sqrt{s}, \lambda_1, \lambda_2, j, m_j \rangle = (2\pi)^4 \delta^{(4)}(P' - P) 64\pi^2 \delta_{jj'} \delta_{mjm_j} \frac{s \lambda_1^{-1/2}}{2 N_{\lambda_1 \lambda_2}} e^{2i \delta_j(s)-2\beta_j(s)}, \quad (7.121) \]

where for identical particles in the initial state the factor \( N_{\lambda_1 \lambda_2}^{-1} \) should be written as \( 1 + (-1)^j \delta_{\lambda_1 \lambda_2} \) to account for the fact that for identical particles the states \( |P, \sqrt{s}, \lambda_1, \lambda_2, j, m_j \rangle \) vanish for \( j \) odd and \( \lambda_1 = \lambda_2 \). We have also used the fact that \( |p| \) (the length of the momentum of the first particle in the CM frame) in the formula (6.108) is given by

\[ |p| = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_1^2, m_2^2). \quad (7.122) \]

The restriction to the elastic scattering amplitude with no helicity flip i.e. to \( \lambda'_1 = \lambda_1 \), \( \lambda'_2 = \lambda_2 \) can be removed by diagonalizing the elastic scattering amplitude in the spin space

\[ \delta_{kl} T^{(j,k)}(s) = (U^\dagger)_{k,\lambda_1 \lambda_2} T^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2}(s) U_{l,\lambda_1 \lambda_2}, \quad (7.123) \]

with the help of a \((2s_1 + 1) \times (2s_2 + 1)\) unitary \((s \text{ dependent})\) matrix \( U \) and writing the unitarity condition (7.98) in the basis in which \( T^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} \) is diagonal (it suffices to sandwich the relation (7.113) between \((U^\dagger)_{k,\lambda_1 \lambda_2}\) and \(U_{l,\lambda_1 \lambda_2}\)). One then gets the representations (7.114), and consequently, also (7.119) with appropriate phase shifts \( \delta^{(j)}_k(s) + i\beta^{(j)}_k(s) \), for each of the elastic scattering amplitudes \( T^{(j,k)}(s) \) with \( R^2_{jk}(s) \) which now does not include the contribution from the elastic channel.
In principle, by appropriately choosing the basis of states, i.e. the unitary transformations $U$, the whole scattering matrix could be diagonalized leading to only purely elastic diagonal amplitudes $\mathcal{T}^{(jk)}$ with purely real phase shifts $\delta_{jk}(s)$. In such a basis, the $S$ matrix can, extending the formula (7.121), succinctly be written as

$$S_{(j'k')(jk)} = \delta_{j'j} \delta_{k'k} e^{2i\delta_{jk}(s)}. \quad (7.124)$$

This makes clear that the phase shift factors are just the $S$ matrix eigenvalues on such states (since $S$ matrix is unitary its eigenvalues must be numbers of unit modulus).

Since not all elements of the scattering matrix are known, symmetries of the interactions can be exploited in practice to diagonalize the scattering amplitudes at least for low energies (small $\sqrt{s}$), when only a limited number of channels can be reached from a given initial state due to kinematical restrictions. A canonical example is provided by low energy strong interactions of pions (the lightest strongly interacting spinless particles), if the electromagnetic and weak interactions are neglected: due to the isospin invariance of the strong interactions (see Section 12) their $S$ matrix is diagonal in the isospin basis

$$\langle I', I_3' | S_0 | I, I_3, k, -k \rangle = \delta_{I'I} \delta_{I'_3 I_3} S^{(I)}(s), \quad (7.125)$$

so that in the isospin basis $|I, I_3, j, m\rangle$ the formula (7.124) takes the form

$$S_{(j'k')(jI_3)} = \delta_{j'j} \delta_{k'k} e^{2i\delta_I(s)}. \quad (7.126)$$

where the (purely real) isospin-angular momentum phase shifts $\delta_I(s)$ parametrize the corresponding isospin partial wave amplitudes $\mathcal{T}^{(jI)}$. The combined formulae (7.126) and (7.58) are the basis for accounting for the final state re-scattering effects in decay processes induced by the weak interactions. They are particularly important in the analysis of the CP violation in the kaon system (see section 12.4).

From the relation (7.114) or directly from Figure 7.2 one gets two unitarity bounds pertaining to elastic scattering (with no change of helicities) partial wave amplitudes:

$$N_{\lambda_1 \lambda_2} \left| \mathcal{T}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \right| \leq \frac{s}{\lambda^{1/2}(s, m_1^2, m_2^2)}, \quad (7.127)$$

$$N_{\lambda_1 \lambda_2} \left| \text{Re} \mathcal{T}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \right| \leq \frac{s}{2\lambda^{1/2}(s, m_1^2, m_2^2)}.$$
Moreover, since \( R_j^2 \) cannot exceed \( s^2 \lambda_1^{-1}/4N_{\lambda_1 \lambda_2}^2 \) (the right hand side of (7.114) must be positive), one gets also the bounds on partial wave amplitudes of any two body (not necessarily elastic) scattering:\[^{31}\]

\[
\sqrt{N_{\lambda_1 \lambda_2} N_{\lambda_1 \lambda_2}} \left| \mathcal{T}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \right| \leq \frac{s}{2 \lambda^{1/4} (s, \tilde{m}_a^2, \tilde{m}_b^2) \lambda^{1/4} (s, m_1^2, m_2^2)}. \tag{7.128}
\]

Notice, that at the reaction threshold, where \( \lambda^{1/2} (s, m_2^2, m_2^2) = 0 \), the bounds (7.128) and (7.127) disappear. If \( \sqrt{s} \) is much greater than any of the masses involved, the unitarity bounds become:\[^{32}\]

\[
N_{\lambda_1 \lambda_2} \left| \mathcal{T}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \right| \leq 1,
\]

\[
N_{\lambda_1 \lambda_2} \left| \text{Re} \mathcal{T}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \right| \leq \frac{1}{2}, \tag{7.129}
\]

\[
\sqrt{N_{\lambda_1 \lambda_2} N_{\lambda_1 \lambda_2}} \left| \tilde{\mathcal{T}}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \right| \leq \frac{1}{2}.
\]

The bounds (7.127) and (7.128) have been derived assuming only that the evolution of the quantum system is unitary.\[^{33}\] In particular they do not rely on any perturbative expansion. Scattering amplitudes derived from local quantum field theory models which (are believed to) give rise to unitary \( S \)-matrices should in principle, respect these bounds. Since elastic scattering partial wave amplitudes computed in the lowest order of the perturbative expansion in quantum field theories are (usually) real (i.e. lie on the horizontal axis in Figure 7.2), they cannot satisfy the unitarity relation (7.114). Higher order contributions must therefore bring elastic amplitudes back on the Argand circle. Two distinct situations can be then encountered. If the (absolute value of the) real part of the lowest order amplitude is bounded by \( 1/2N_{\lambda_1 \lambda_2} \), higher order contributions required to restore unitarity can be relatively small (short dashed lines in Figure 7.2) and the perturbative expansion is likely to be reliable. In contrast, if the real part of the lowest order amplitude greatly exceeds \( 1/2N_{\lambda_1 \lambda_2} \), the necessary higher order contributions must be comparable or even larger than the lowest order term and the perturbative expansion evidently fails. In specific quantum field theory models

\[^{31}\]It is clear that the quantities (7.118) related to the contribution of multi-particle production to the total cross section are also bounded by this requirement.

\[^{32}\]In the literature it is customary to include the factor \( \sqrt{N_{\lambda_1 \lambda_2} N_{\lambda_1 \lambda_2}} \) in \( \tilde{\mathcal{T}}^{(j)}_{\lambda_1 \lambda_2; \lambda_1 \lambda_2} (s) \). In this way the \( N_{\lambda_1 \lambda_2} \) factors disappear altogether from the formulae (7.112)-(7.119) and the unitarity bounds (7.129) for distinct and identical particles look the same. We preferred not to do so, in order to keep control over such factors and, moreover, because the factors \( \sqrt{N_{\lambda_1 \lambda_2}} \) should \emph{not} be included (for the cross section calculation) in the amplitudes of scatterings of identical particles in the initial state.

\[^{33}\]Another assumption is that the partial wave expansion of amplitudes makes sense. This may not be true in the presence of massless particles which produce long range interactions.
the magnitude of the lowest order amplitudes depends usually on the energy $\sqrt{s}$. In renormalizable theories (see Section 14) the lowest order amplitudes are bounded for $\sqrt{s} \to \infty$ by some constants and reliability of the perturbation expansion depends on the magnitude of such a limiting value (whether such a constant is smaller or bigger than $1/2N_{\lambda_1\lambda_2}$). In nonrenormalizable theories the lowest order amplitudes usually grow with $\sqrt{s}$ and above some critical energy the perturbation expansion unavoidably breaks down.

Example of the latter situation is provided by the phenomenological Fermi theory of weak interactions (introduced in Section 12). Amplitudes computed in this model in the lowest order grow linearly with $s$ and violate the unitarity bounds (7.129) at $\sqrt{s} \gtrsim 600$ GeV, indicating that at such energies either the theory of weak interaction becomes strongly coupled (and the perturbative expansion cannot be applied to it) or that the Fermi theory is only an effective model which should be replaced by a more fundamental theory in which exchanges of new particles restore unitarity of amplitudes computed in the lowest order. It is the second option that is realized in the Nature - the Fermi theory turned out to provide only an effective, low energy approximation to the results obtained in the Standard Theory of electroweak interactions.

Unitarity bound derived in this section were also important in discussing (before 2012) possible versions of the extension (ultraviolet completion in the modern parlance) of the Fermi theory. Scattering amplitudes of longitudinally polarized massive spin 1 (vector) bosons - particles which are experimentally now well known to mediate weak interactions - computed in the lowest order grow linearly with $s$ and violate these bounds if the contributions of the sector of the theory responsible for electroweak symmetry breaking is not taken into account. This sector was, before 2012, experimentally unexplored (and remains largely such even now). The Standard Model - the concrete renormalizable realization of such an extension, in which which the electroweak gauge symmetry is broken by a single doublet of scalar fields - predicted the existence, without fixing its mass, of a single neutral spinless particle, $h^0$, whose contribution to the discussed scattering amplitudes cuts their rising with $s$ - they reach a constant value proportional to the mass squared of $h^0$. It could be then concluded that the bounds would be violated if the mass of $h^0$ were greater than $\sim 1$ TeV. The discovery of $h^0$ with mass equal 125 GeV, giving strong support for the mechanism of electroweak symmetry breaking realized in the Standard Model, cuted (forever?) the speculations concerning more exotic possibilities of saving unitarity in the scattering of longitudinally polarized electroweak vector bosons.
7.6 Other symmetries of the $S$ matrix

Discrete symmetries: parity, time reversal and charge conjugation may or may not be exact symmetries of a given model of relativistic quantum mechanics (a quantum field theory model). Even if they are not exact symmetries of the real world (we know they are not), it is still interesting to consider them because in physics, in contrast to pure mathematics in which the statement that some operation is not a symmetry closes the issue, we are interested not only in whether they are symmetries but also, how they are violated, i.e. by which type of interactions and in which processes.

The action of the parity operator on one-particle states has been discussed in Section 6.3. Multi-particle states transform of course as tensor products:

$$P_0 \left( |p_1 \sigma_1, \ldots, p_N \sigma_N \rangle \right) = \eta_1 \ldots \eta_N | -p_1 \sigma_1, \ldots, -p_N \sigma_N \rangle \tag{7.130}$$

(we have assumed that all particles are massive; modifications for massless particles are obvious). The same formula, with $P$ instead of $P_0$ is also true for the in and out states.

If $P_0$ commutes with the interaction operator $V_{\text{int}}$ (and with $H_0$, so in all with $H$), one can set $P = P_0$. The $S$-matrix satisfies then the following identity

$$S_{p_1' \sigma_1', p_2' \sigma_2' \ldots : p_1 \sigma_1, p_2 \sigma_2 \ldots} = \eta_1^* \eta_2^* \ldots \eta_1 \eta_2 \ldots S_{-p_1' \sigma_1', -p_2' \sigma_2' \ldots : -p_1 \sigma_1, -p_2 \sigma_2 \ldots} \tag{7.131}$$

This shows that if parity is conserved in elementary processes, then for $\eta_1^* \eta_2^* \ldots \eta_1 \eta_2 \ldots = +1$ the $S$ matrix is an even function of the particle momenta, whereas for $\eta_1^* \eta_2^* \ldots \eta_1 \eta_2 \ldots = -1$ it must be an odd function.

What are the internal parities $\eta$ of various known elementary particles? For a small set particles $\eta$ can be fixed by a convention. This is because in the real world the parity operator $P = P_0$ can be redefined

$$P' = P e^{-ic \hat{Q} - ic_2 \hat{B} - ic_3 \hat{L} - \ldots} \tag{7.132}$$

where $\hat{Q}$, $\hat{B}$ and $\hat{L}$ are the operators of the electric charge, baryon and lepton numbers and other quantum numbers conserved in parity-conserving interactions.\textsuperscript{34} If $P$ commutes with the Hamiltonian so does $P'$. This freedom in the definition of the parity operator allows to assign (by convention) $\eta = +1$ to $p$, $n$ and $e^-$ (these particles have all different combinations of $B$, $L$ and $Q$.

\textsuperscript{34}Even if the lepton and baryon numbers are not strictly conserved - at least as far as the lepton number is concerned, there are strong indications that it is indeed violated by interactions responsible for small, but experimentally well established, neutrino masses - this does not hurt, because they are conserved in the same interactions which preserve parity.
and one can choose \(c_1, c_2\) and \(c_3\) so that the action of \(P'\) is consistent with this assignment. In the same way using the factor with the strangeness or charm operators \(S, C\) one can by convention assign \(\eta = 1\) to one strange baryon and one charmed baryon.

Internal parities of the remaining particles should be assigned in such a way that parity is conserved in as broad class of processes as possible. Consider for example the process of radiative capturing of \(\pi^-\) in the “\(\pi\) meson deuteron” i.e. in the \(d\pi^-\) bound state: \(d + \pi^- \to n + n\). The capturing occurs from the orbital \(l = 0\) ground state of such an “atom”. The initial state has \(j = 1\) (deuteron is predominantly an \(l = 0\) proton-neutron bound state with the total spin \(s = 1\)) and, therefore, the final state should also have \(j = 1\). Since the final state, being the state of the two identical fermions, must be antisymmetric in their labels, it must have \(s = 1, l = 1\) (other possibilities allowing for \(j = 1\): \(s = 0, l = 1\) or \(s = 1, l = 0\) or \(s = 1, l = 2\) all lead to symmetric final states). Thus, if parity is conserved, we must have

\[
\eta_d \eta_{\pi^-} = -\eta_n^2. \tag{7.133}
\]

Since \(\eta_d = +1\) (\(\eta_p = \eta_n = +1\) by convention and \(l = 0\)), it follows that

\[
\eta_{\pi^-} = -1. \tag{7.134}
\]

From the isospin symmetry (see section 12) it then follows that also

\[
\eta_{\pi^+} = \eta_{\pi^0} = -1. \tag{7.135}
\]

Negative parity of the \(\pi\) mesons has important consequences. One is that if parity is conserved, a particle decaying into three pions must necessarily have negative parity. Indeed, in the rest frame of the decaying particle the \(S\)-matrix element corresponding to its decay into three pions can depend only on \(p_1 \cdot p_2, p_1 \cdot p_3\) or \(p_2 \cdot p_3\) (as pions are spinless, no other vectors are available; moreover, since in this frame \(p_1 + p_2 + p_3 = 0\) the triple product \((p_1 \times p_2) \cdot p_3 = 0\)). Hence, parity of the final state is negative and so must be parity of the initial state. Similarly, a particle decaying into two pions must necessarily have positive parity. Non-conservation of parity in weak interactions indicated first by the observation of \(K^+\) decays into two and three pions became evident around 1950 (see Section 12). It is by now firmly established and is one of the cornerstones of the theory of electroweak interactions (Section ??).

The action of the time reversal operator on one-particle states has been given in Section 6.3. Its action on free multi-particle states therefore reads

\[
\mathcal{T}_0|\{p_1 \sigma_1, \ldots, p_N \sigma_N\}_0\rangle = \zeta_1 (-1)^{s_1 - \sigma_1} \cdots \zeta_N (-1)^{s_N - \sigma_N} |(-p_1 - \sigma_1, \ldots, -p_N - \sigma_N)_0\rangle. \tag{7.136}
\]
(as for parity we have assumed that all particles are massive; modifications for massless particles are obvious). If \( T_0 \) commutes with the interaction operator \( V \), we can set \( T = T_0 \). It is also easy to see that the action of \( T \) changes the in states into the out states and vice-versa:

\[
T |(p_1 \sigma_1, \ldots, p_N \sigma_N \rangle)_{\pm} = \zeta_1(-1)^{s_1-s_2} \ldots \zeta_N(-1)^{s_{N-1}-s_N} |(-p_1 - \sigma_1, \ldots, -p_N - \sigma_N)_{\mp}\rangle. \tag{7.137}
\]

This can be seen from the formal expression (7.37):

\[
T |\alpha_{\pm}\rangle = \lim_{\tau \to \mp \infty} T_0 e^{iH_0 \tau} e^{-iH_0 \tau} |\alpha_0\rangle = \lim_{\tau \to \mp \infty} e^{-iH_\tau} e^{iH_0 \tau} |\alpha_0\rangle = \lim_{\tau \to \mp \infty} e^{-iH_\tau} e^{iH_0 \tau} |(T \alpha)_0\rangle \equiv |(T \alpha)_{\mp}\rangle, \tag{7.138}
\]

(recall that \( T \) is antilinear!). We have introduced the compact notation \(|(T \alpha)\rangle\) for the state (including its phase factors) appearing on the right hand side of (7.137); thus if \(|\alpha_0\rangle = |(p, \sigma)\rangle_0\), then \(|(T \alpha)_0\rangle \equiv \zeta(-1)^{s-\sigma}|(-p, -\sigma)_0\rangle\). The same conclusion follows also from the expression (7.52).

If the time reversal is a symmetry operation, from the properties of antiunitary operators (see Section 4) it follows that\(^{35}\)

\[
S_{\beta \alpha} = (\beta_- |\alpha_+\rangle = (\beta_- |T^-1 T \alpha_+\rangle = (\beta_- |T^\dagger T \alpha_+\rangle = (T \beta_- |T \alpha_+\rangle) = (T \alpha_+ |T \beta_-\rangle) = (|(T \alpha)_-\rangle |(T \beta)_+\rangle \equiv S_{(T \alpha)(T \beta)}, \tag{7.139}
\]

where if \( S_{\beta \alpha} \equiv S_{(p'_1 \sigma'_1, p'_2 \sigma'_2, \ldots, p_1 \sigma_1, p_2 \sigma_2, \ldots)} \), the symbol on the right hand side should be understood as

\[
S_{(T \alpha)(T \beta)} \equiv \zeta_{1v}(-1)^{s_1-s_1'} \zeta_2(-1)^{s_2-s_2'} \ldots \zeta_1(-1)^{s_1-s_1'} \zeta_2(-1)^{s_2-s_2} \ldots \times S_{-p_1 - \sigma_1 - p_2 - \sigma_2 \ldots, -p'_1 - \sigma'_1, -p'_2 - \sigma'_2 \ldots}. \tag{7.140}
\]

Notice, that in general the time reversal does not imply that the rate of the reaction \( \alpha \rightarrow \beta \) is the same as of the reaction \( T \alpha \rightarrow T \beta \). There are however some special situations in which it does imply this. This is when the \( S \)-matrix can be split into two parts (\( S_{\beta \alpha}^{\text{strong}} \) can be viewed as the zeroth order term in the expansion of \( S_{\beta \alpha} \) in some small parameter)

\[
S_{\beta \alpha} = S_{\beta \alpha}^{\text{strong}} + S_{\beta \alpha}^{\text{weak}}, \tag{7.141}
\]

\(^{35}\)We have to abandon for a while the Dirac bra-ket notation. The alternative derivation of this result is \((T = T_0)\)

\[
S_{\beta \alpha} = (\beta_0 |S_0 \alpha_0\rangle) = (\beta_0 |T^\dagger T S_0 \alpha_0\rangle) = (T \beta_0 |S_0^\dagger T \alpha_0\rangle)^* = ((T \alpha)_0 |S_0(T \beta)_0\rangle) = S_{(T \alpha)(T \beta)},
\]

because \( TS_0 = S_0^\dagger T \), owing to the antiunitarity of \( T \).
where $|S^{\text{weak}}_{\beta\alpha}| \ll |S^{\text{strong}}_{\beta\alpha}|$ because $S^{\text{strong}}$ is due to the strong interactions, whereas $S^{\text{weak}}$ is due to the weak ones. In the first order in $S^{\text{weak}}_{\beta\alpha}$ the unitarity condition for $S$ then reads

$$\hat{1} = S^\dagger S = S^{\text{strong}}\dagger S^{\text{strong}} + S^{\text{strong}}\dagger S^{\text{weak}} + S^{\text{weak}}\dagger S^{\text{strong}}.$$

(7.142)

So, approximately,

$$S^{\text{weak}} \approx -S^{\text{strong}} S^{\text{weak}}\dagger S^{\text{strong}},$$

(7.143)

because $S^{\text{strong}}\dagger S^{\text{strong}} = \hat{1}$ (the zeroth order terms must match). More concretely,

$$S^{\text{weak}}_{\beta\alpha} \approx -\int d\gamma \int d\gamma' S^{\text{strong}}_{\beta\gamma} [S^{\text{weak}}\dagger]_{\gamma'\gamma} S^{\text{strong}}_{\gamma'\alpha}$$

$$= -\int d\gamma \int d\gamma' S^{\text{strong}}_{\beta\gamma} [S^{\text{weak}}]_{\gamma'\gamma} S^{\text{strong}}_{\gamma'\alpha}$$

$$= -\int d\gamma \int d\gamma' S^{\text{strong}}_{\beta\gamma} [S^{\text{weak}}(T\gamma)(T\gamma')]^* S^{\text{strong}}_{\gamma'\alpha}.$$  

(7.144)

This relation is particularly useful if, as in the case of the nuclear $\beta$ decays, $S^{\text{strong}}_{\beta\alpha} = 0$ for the relevant states $\alpha$ and $\beta$ (the process cannot occur through the strong interactions). In the basis of states with fixed total angular momentum using the formula (7.124) $S^{\text{strong}}_{\gamma\alpha} = \delta_{\gamma\alpha} \exp(2i\delta_\alpha)$ we get from (7.144)

$$S^{\text{weak}}_{\beta\alpha} \approx -e^{2i(\delta_\beta + \delta_\alpha)} [S^{\text{weak}}(T\beta)(T\alpha)]^*.$$  

(7.145)

which does imply that the rate of the process $\alpha \rightarrow \beta$ is approximately\textsuperscript{36} the same as the rate of the reaction $(T\alpha) \rightarrow (T\beta)$.

Action of the charge conjugation operator $C_0$ on free one-particle states was defined in Section 6.3. If the interaction operator $V_{\text{int}}$ commutes with $C_0$, one can take $C = C_0$ and the charge conjugation symmetry implies that

$$S_{p_1^'\sigma_1^',p_2^'\sigma_2',...;p_1^\sigma_1,p_2^\sigma_2,...} = \xi_1^{*}\xi_2^{*} \cdots \xi_1\xi_2 \cdots S_{p_1^'\sigma_1^',p_2^'\sigma_2',...;p_1^\sigma_1,p_2^\sigma_2,...},$$

(7.146)

where we have denoted antiparticle momentum and spin labels by bars.

Charge conjugation parities of different particles are assigned in a similar way as the intrinsic parities. First of all, $C$ can always be redefined

$$C' = C e^{-ia_1Q-ia_2B-ia_3L},$$

(7.147)

\textsuperscript{36}Since the weak interaction is much much weaker than the strong one, this is in fact an almost perfect approximation.
so that for three particles that have different $Q$, $B$ and $L$ charge conjugation parities $C$ can be fixed by a convention. Furthermore, charge conjugation parities of neutral particles, which like photon or neutral pion do not carry any conserved quantum numbers, are uniquely determined. As will become evident in Section 8, a state of a (massive) fermion-antifermion pair has negative charge conjugation parity (more precisely, the electromagnetic current operator $J_{\text{EM}}^\mu C^{-1} = -J_{\text{EM}}^\mu$). Therefore, the photon which couples to such pairs must also have negative charge conjugation parity (i.e. the photon field operator must have the property $CA^\mu C^{-1} = -A^\mu$). Since $\pi^0$ decays into two photons, $\xi_{\pi^0} = +1$ and, by isospin symmetry, the same must be also true for $\pi^\pm$. It then follows that the process $\pi^0 \to 3\gamma$ is forbidden (experimentally $\text{Br}(\pi^0 \to 3\gamma) < 3.1 \times 10^{-8}$).

### 7.7 The cluster decomposition principle

In this subsection we shall briefly describe, without entering into details, the so-called cluster decomposition principle, which any physically sensible $S$ matrix should satisfy in order the theory predictions for measurements made in spatially remote laboratories be uncorrelated. The cluster decomposition principle imposes a simple but nontrivial constraint on the general structure of quantum field theory Hamiltonians expressed in terms of the creation and annihilation operators of free particles in the momentum representation.

We begin with the almost obvious statement that any operator $O$ acting in the multi-particle Hilbert space spanned by the states (7.1) can be represented in the form\(^{37}\)

\[
O = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int d\Gamma_{p_1} \cdots \int d\Gamma_{p_M} \int d\Gamma_{q_1} \cdots \int d\Gamma_{q_N} C_{MN}(p_1, \ldots, p_M; q_1, \ldots, q_N) a^\dagger(p_1) \cdots a^\dagger(p_M) a(q_1) \cdots a(q_N),
\]

where for simplicity of the notation we consider only one type of spinless particles. This means that given an operator $O$, or equivalently, given all possible matrix elements of $O$ between multi-particle states, we can always choose the functions $C_{MN}(p_1, \ldots, p_M; q_1, \ldots, q_N)$ in (7.148) to reproduce the matrix elements defining $O$. The proof is inductive: one adjusts first the value of $C_{00}$ so as to reproduce $\langle \Omega | O | \Omega \rangle$, then one considers matrix elements of $O$ between the vacuum and the one-particle states to fix $C_{01}$ and $C_{10}$, then one adjusts the functions $C_{11}$ to reproduce matrix elements of $O$ between two one-particle states and so on.

\(^{37}\)The multi-particle states considered here can be the free multi-particle states but can also be the in or the out states, for which the corresponding creation and annihilation operators $a_{\text{in}}(p)$, $a_{\text{in}}(p)$ and $a_{\text{out}}^\dagger(p)$, $a_{\text{out}}(p)$ can also be introduced (see section 8.7).
Consider now the $S$ matrix in the position representation

$$S_{y_1, \ldots, y_M; x_1, \ldots, x_N} = \int d\Gamma_{p_1} \cdots \int d\Gamma_{p_M} \int d\Gamma_{q_1} \cdots \int d\Gamma_{q_N}$$

and imagine a process $\alpha \rightarrow \beta$ (where $\alpha$ and $\beta$ stand for collections of positions and spins of the initial and final state particles, respectively), in which particles in the initial and final states exhibit some clustering: $\alpha = (\alpha_1)(\alpha_2)(\alpha_3) \ldots (\alpha_n)$, $\beta = (\beta_1)(\beta_2)(\beta_3) \ldots (\beta_n)$. By clustering we mean that the subprocesses $\alpha_1 \rightarrow \beta_1$, $\alpha_2 \rightarrow \beta_2$, $\ldots$, $\alpha_n \rightarrow \beta_n$ are measured separately in remote laboratories (like e.g. CERN, SLAC and FNAL). The $S$ matrix satisfies the cluster decomposition principle if it factorizes in such a case, that is, if

$$S_{\beta \alpha} = S_{\beta_1 \alpha_1} S_{\beta_2 \alpha_2} S_{\beta_3 \alpha_3} \cdots S_{\beta_n \alpha_n},$$

when the distances $|x_i - x_j|$ and $|y_i - y_j|$ are large for $x_i$, $y_i$ and $x_j$, $y_j$ belonging to different clusters\(^{38}\) (large compared to the typical distances in the same cluster). This in turn ensures independence of the theory predictions for experiments performed in different laboratories: if we are interested in the probability of a concrete final state $|\beta_1\rangle$ in an experiment performed e.g. at CERN, we sum over all possible final states which can be found in far-away laboratories:

$$P(\alpha_1 \rightarrow \beta_1) = \sum_{\beta_2, \beta_3, \ldots} S_{\beta \alpha} S_{\beta \alpha}^*$$

$$= S_{\beta_1 \alpha_1} S_{\beta_1 \alpha_1}^* \sum_{\beta_2} S_{\beta_2 \alpha_2} S_{\beta_2 \alpha_2}^* \sum_{\beta_3} S_{\beta_3 \alpha_3} S_{\beta_3 \alpha_3}^* \cdots = S_{\beta_1 \alpha_1} S_{\beta_1 \alpha_1}^*,$$

where in the last step unitarity of the $S$ matrix has been used.

To implement the factorization of the $S$ matrix let us first define its connected part $S_{\beta \alpha}^C$ by using a combinatoric trick\(^{39}\). We write

$$S_{\beta \alpha} = \sum_{\text{partitions}} (\pm) S_{\beta_1 \alpha_1}^C S_{\beta_2 \alpha_2}^C \cdots,$$
as different those partitions which differ only by a permutation of labels within the same group, or differ by a permutation of the clusters as wholes). The sign \( \pm \) depends on whether the number of interchanges of fermionic labels is even or odd. The definition (7.152) is recursive:

\[
S_{\beta \alpha} = S_{\beta \alpha}^C + \sum_{\text{partitions}'} (\pm) S_{\beta_1 \alpha_1}^C S_{\beta_2 \alpha_2}^C \ldots ,
\]

(7.153)

where now the sum goes over the partitions in which all clusters \((\alpha_i), (\beta_i)\) contain less particles than \(\alpha\) and \(\beta\). (We assume here that no one of the clusters \((\alpha_i)\) and/or \((\beta_i)\) is empty; this requires that in (7.152) \(S_{\beta \alpha}^C\) is set to zero). It is easier to understand this on the example. We have by definition\(^{40}\)

\[
S_{y,x} \equiv S_{y,x}^C \propto \int d\Gamma_p \int d\Gamma_q e^{-ipy} e^{iqx} \delta^{(3)}(p - q) \equiv \delta_{y,x}.
\]

(7.154)

Then, for the 2 \(\rightarrow\) 2 element of the \(S\) matrix we have

\[
S_{y_1 y_2; x_1 x_2} = S_{y_1 y_2; x_1 x_2}^C + \delta_{y_1, x_1} \delta_{y_2, x_2} \pm \delta_{y_1, x_2} \delta_{y_2, x_1}.
\]

Similarly,

\[
S_{y_1 y_2 y_3; x_1 x_2 x_3} = S_{y_1 y_2 y_3; x_1 x_2 x_3}^C + S_{y_2 y_3; x_1 x_2}^C \delta_{y_1, x_1} \pm S_{y_1 y_3; x_1 x_2}^C \delta_{y_2, x_2} + S_{y_1 y_2; x_1 x_3}^C \delta_{y_3, x_3}
\]

for the 2 \(\rightarrow\) 3 transitions,

\[
S_{y_1 y_2 y_3; x_1 x_2 x_3} = S_{y_1 y_2 y_3; x_1 x_2 x_3}^C + \delta_{y_1, x_1} S_{y_2 y_3; x_2 x_3}^C \pm \text{permutations}
\]

\[
+ \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y_3, x_3} \pm \text{permutations},
\]

for the 3 \(\rightarrow\) 3 transitions, and

\[
S_{y_1 y_2 y_3 y_4; x_1 x_2 x_3 x_4} = S_{y_1 y_2 y_3 y_4; x_1 x_2 x_3 x_4}^C + S_{y_1 y_2; x_1 x_2}^C S_{y_3 y_4; x_3 x_4}^C \pm \text{permutations}
\]

\[
+ \delta_{y_1, x_1} S_{y_2 y_3 y_4; x_2 x_3 x_4}^C \pm \text{permutations}
\]

\[
+ \delta_{y_1, x_1} \delta_{y_2, x_2} S_{y_3 y_4; x_3 x_4}^C \pm \text{permutations}
\]

\[
+ \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y_3, x_3} \delta_{y_4, x_4} \pm \text{permutations},
\]

(7.155)

for the 4 \(\rightarrow\) 4 transitions etc.

The main point is that \(S_{\beta \alpha}\) (7.153) satisfies the cluster decomposition principle (7.150) if the connected matrices \(S_{\beta \alpha}^C\) vanish when at least one

\(^{40}\)In the presence of interactions \(S_{p,q}\) differs from \(\delta^{(3)}(p - q)\) by a phase factor which is not relevant for what follows.
of the particles in clusters \((\alpha_i), (\beta_i)\) is spatially separated from the other particles in the same cluster. To see it on an example, consider the process \(4 \rightarrow 4\) and the corresponding \(S\) matrix element \((7.155)\). Let us assume that the initial and final particles \(1 \) and \(2\) are far away from the initial and final particles \(3 \) and \(4\). Then we get from \((7.155)\) throwing out all vanishing terms

\[
S_{y_1y_2y_3y_4;x_1x_2x_3x_4} = S^C_{y_1y_2;x_1x_2}S^C_{y_3y_4;x_3x_4} \\
+ (\delta_{y_1,x_1} \delta_{y_2,x_2} \pm \delta_{y_1,x_2} \delta_{y_1,x_1})S^C_{y_1y_4;x_1x_4} \\
+ (\delta_{y_3,x_3} \delta_{y_4,x_4} \pm \delta_{y_3,x_4} \delta_{y_3,x_3})S^C_{y_1y_2;x_1x_2} \\
+ (\delta_{y_1,x_1} \delta_{y_2,x_2} \pm \delta_{y_1,x_2} \delta_{y_1,x_1})(\delta_{y_3,x_3} \delta_{y_4,x_4} \pm \delta_{y_3,x_4} \delta_{y_3,x_3}),
\]

which is precisely \(S_{y_1y_2;x_1x_2}S_{y_3y_4;x_3x_4}\).

To see what form of \(S^C_{p_1p_2...q_1q_2...}\) should take in order to have this property, let us note that for

\[
S^C_{y_1y_2...x_1x_2...} = \int d\Gamma_{p_1} \int d\Gamma_{p_2} \cdots \int d\Gamma_{q_1} \int d\Gamma_{q_2} \cdots e^{-ip_1y_1}e^{-ip_2y_2} \cdots e^{iq_1x_1}e^{iq_2x_2} \cdots S^C_{p_1p_2...q_1q_2...}, \quad (7.156)
\]

the Riemann-Lebesgue theorem says that if \(|S^C_{p_1p_2...q_1q_2...}|/\sqrt{E_{p_1} \cdots E_{q_1} \cdots}\) is a Lebesgue integrable function then \(S^C_{y_1y_2...x_1x_2...}\) vanishes if one of the \(|x_i|\) and/or \(|y_i|\) is large. This is, however, too strong a condition because \(S^C_{y_1y_2...x_1x_2...}\) should not vanish if all \(x_i\) and \(y_i\) are simultaneously shifted by the same vector \(a\) (no matter how large \(|a|\) is). By translational invariance \(S^C_{y_1y_2...x_1x_2...}\) should depend only on differences of the positions of initial and final state particles. This means that \(S^C_{p_1p_2...q_1q_2...}\) has to be proportional to the single delta function (therefore it cannot be Lebesgue integrable) expressing the conservation of the total 3-momentum (and, by Lorentz covariance, also to the delta function expressing the conservation of the total energy):

\[
S^C_{p_1p_2...q_1q_2...} = \delta^{(3)}(p_1 + p_2 + \cdots - q_1 - q_2 - \cdots) \delta(E_{p_1} + E_{p_2} + \cdots - E_{q_1} - E_{q_2} - \cdots)\tilde{S}^C_{p_1p_2...q_1q_2...}, \quad (7.157)
\]

The function \(\tilde{S}^C_{p_1p_2...q_1q_2...}\) cannot then contain any additional delta functions. If \(\tilde{S}^C_{p_1p_2...q_1q_2...}\) was an analytic function of the momenta, \(S^C_{y_1y_2...x_1x_2...}\) would vanish exponentially fast with growing differences of the particle positions. We can however, allow also for poles (they appear in theories with massless particles) in \(\tilde{S}^C_{p_1p_2...q_1q_2...}\) which leads to power-like fall-off of \(S^C_{y_1y_2...x_1x_2...}\) with growing differences of particle positions (such a fall-off signals the presence of long-range forces).

The final question is what interactions \(V_{\text{int}}\) lead to \(S\) matrices satisfying the cluster decomposition principle. The answer is simple if \(V_{\text{int}}\) is built
out of the creation and annihilation operators of the eigenstates $|\alpha_0\rangle$ of a free-particle Hamiltonian $H_0$. The cluster decomposition principle is then satisfied if the coefficient functions $h_{MN}(p_1, \ldots, p_M; q_1, \ldots, q_N)$ of various terms of the interaction

$$V_{\text{int}} = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \int d\Gamma_{p_1} \ldots \int d\Gamma_{p_M} \int d\Gamma_{q_1} \ldots \int d\Gamma_{q_N}$$

$$h_{MN}(p_1, p_2, \ldots; q_1, q_2, \ldots) a^\dagger(p_1) \ldots a^\dagger(p_M)a(q_1) \ldots a(q_N),$$

contain only a single overall delta function

$$h_{MN}(p_1, \ldots, p_M; q_1, \ldots, q_N) = \delta^{(3)}(p_1 + \ldots + p_M - q_1 - \ldots - q_N)$$

$$\times \bar{h}_{MN}(p_1, \ldots, p_M; q_1, \ldots, q_N).$$

This is automatically ensured when the quantum field theory Hamiltonians are built from field operators (to be introduced in the next Section).
Appendix C  Potential scattering

In this Appendix we briefly recall standard scattering theory based on non-relativistic Quantum Mechanics of a single (for simplicity spinless) particle moving in an external spherically symmetric potential \( V(r) \) (playing the role of the interaction operator \( V_{\text{int}} \)) on which some restrictions (pertaining to its behaviour as \( r \to 0 \) and \( r \to \infty \)) are usually imposed.

The Hilbert space \( \mathcal{H} \) is in this case \( L^2(\mathbb{R}^3) \) - the space of all Lebesque square-integrable functions. \( H_0 = \frac{\mathbf{p}^2}{2m} \) has in \( \mathcal{H} \) no normalizable eigenvectors. In contrast, \( H = H_0 + V(r) \) can have discrete normalizable eigenstates \( |\phi_n\rangle \). Thus the spectrum of \( H \) typically consists of isolated discrete values \( E_n < 0 \) (we assume \( V(\infty) = 0 \)) and a continuous part starting from \( E = 0 \).

In this case the convergence of the operators \( \Omega(t) = e^{iHt}e^{-iH_0t} \) can be proved rigorously (for a class of potentials \( V(r) \)). This reduces to showing that the integral term in the formula (7.10) has a finite limit for \( t \to \pm \infty \) on any normalizable vector \( |\psi\rangle \). This is so, if the sequence of Hilbert space vectors

\[
|\psi_t\rangle = \int_0^t d\tau U^\dagger(\tau,0)V_{\text{int}}U_0(\tau,0)|\psi\rangle,
\]

is a Cauchy sequence for \( t \to \pm \infty \) which in turns is ensured if\(^1\)

\[
\| \int_0^{\pm \infty} d\tau U^\dagger(\tau,0)V_{\text{int}}U_0(\tau,0)|\psi\rangle \| < 0.
\]

Because \( U^\dagger(t,0) \) is unitary, i.e. it preserves the norm, this holds if

\[
\int_0^{\pm \infty} d\tau \| V_{\text{int}}U_0(\tau,0)|\psi\rangle \| < 0.
\]

Since the time evolution governed by \( H_0 \) ultimately drives any localized (normalizable) wave packet (the vector \( |\psi\rangle \)) outside the domain in which \( V(r) \) acts, the condition is (for an appropriate class of potentials) satisfied.

A priori vectors of the form \( \Omega_{\pm}|\psi\rangle \) span two subspaces \( \mathcal{H}_{\pm} \) of the Hilbert space \( \mathcal{H} \). One can easily show that normalizable eigenstates of \( H \) are orthogonal to all vectors of this form. Indeed, let \( |\phi_n\rangle \) be such a vector corresponding to \( E_n \) and \( |\psi_\pm\rangle = \Omega_{\pm}|\psi\rangle \). Then

\[
\langle \phi_n|\psi_\pm\rangle = \langle \phi_n|U^\dagger(t)U(t)|\psi_\pm\rangle = e^{iE_nt}\langle \phi_n|U(t)|\psi_\pm\rangle.
\]

\(^1\)This is because the usual rule “tails contribute nothing to convergent integrals whose domains extend to \( \infty \) (or to \( -\infty \))” applied to this integral turns out to be just the condition for the sequence of vectors being a Cauchy sequence.
As this holds for any time, \( t \) can be pushed to \( \pm \infty \) in which limit \( U(t, 0)|\psi_{\pm}\rangle \) can be replaced by \( U_0(t, 0)|\psi\rangle \). This vector represents a wave packet moving freely, which ultimately, for \( t \to \pm \infty \) leaves the region in which \( |\phi_n\rangle \) is localized and the scalar product must be zero. Furthermore, using a reasoning similar to the one applied above, one shows that the operator \( \Omega(t) = e^{iH_0t}e^{-iHt} \) does not have the \( t \to \pm \infty \) limits when acting on normalizable vectors \( |\phi_n\rangle \). Thus, while the operator limits \( \lim_{t \to \pm \infty} \Omega(t) \) do exist, if \( H \) has normalizable eigenstates (bound states) \( |\phi_n\rangle \) the limits \( \lim_{t \to \pm \infty} \Omega^{-1}(t) \) do not. Notice that this is not in conflict with the fact that \( \Omega^{-1}(t) \) satisfy \( \Omega(t)\Omega^{-1}(t) = 1 \) for any finite \( t \), because the existence of the operator limits of \( \Omega(t) \) does not imply the same for \( \Omega^{-1}(t) \); the operators \( \Omega^{-1}_{\pm} \) (defined below) are then not limits of \( \Omega(t) \).

An important assumption (whose validity can be rigorously established for some classes of potentials \( V(r) \)) is the one about the asymptotic completeness. It states that \( H_{+} = H_{-} = H_{\text{scatt}} \) and that \( H = H_{\text{scatt}} \oplus H_{\text{bound}} \). Defining the Hermitian conjugations \( \Omega_{\pm} \) of \( \Omega_{\pm} \) one has to specify their action on the vectors \( |\phi_n\rangle \) belonging to \( H_{\text{bound}} \) by applying the rule (section 4.1)

\[
(\psi|\Omega_{\pm}^{-1}|\phi_n\rangle = (\Omega_{\pm}\psi|\phi_n\rangle = (\psi_{\pm}|\phi_n\rangle = 0.
\]

As \( \psi \) is an arbitrary vector of \( H \), this implies that the operators \( \Omega_{\pm}^{-1} \) must necessarily annihilate the whole subspace \( H_{\text{bound}} \) (on \( H_{\text{scatt}} \) they act as operators inverse to \( \Omega_{\pm} \: \Omega_{\pm}^{-1}|\psi_{\pm}\rangle = |\psi\rangle \)). Thus while \( \Omega_{\pm}^{-1}\Omega_{\pm} = 1 \),

\[
\Omega_{\pm}^{-1}\Omega_{\pm} = \hat{1} - \Pi_{\text{bound}}, \tag{C.1}
\]

where \( \Pi_{\text{bound}} \) is the projector onto \( H_{\text{bound}} \). The proof (7.20) that \( S_0H_0 = H_0S_0 \) goes however unmodified:

\[
\Omega_{-}^{-1}H_{+} = \Omega_{+}^{-1}H (\Omega_{-}\Omega_{-}^{-1} + \Pi_{\text{bound}})_{+} = \Omega_{-}^{-1}H_{-}\Omega_{-}\Omega_{-}^{-1}_{+}, \tag{C.2}
\]

because \( \Pi_{\text{bound}}\Omega_{+} = 0 \).

The state-vectors \( |\alpha_0\rangle \) are in the case considered here simply the state-vectors \( |p\rangle \), that is the plane waves \( \psi_{p}(x) = e^{i\mathbf{p}\cdot\mathbf{x}} = \langle x|\mathbf{p}\rangle \), and the formula (7.23) takes the form

\[
S_0^{\prime}\mathbf{p} = \langle p'|S_0|\mathbf{p}\rangle = (2\pi)^3\delta^{(3)}(p' - p) - 2\pi i\delta(E_{p'} - E_p) t(p', \mathbf{p}). \tag{C.3}
\]

The usual scattering amplitude \( f(p', p) \equiv f(\theta) \) \( (\theta \) is the angle between \( p' \) and \( p \) is then related to \( t(p', p) \) by

\[
f(p', p) = -\frac{m}{2\pi} t(p', p). \tag{C.4}
\]

952
This can be justified by analysing scattering of localized wave packets peaked around a well defined momentum \( p \) - one finds that the differential elastic scattering cross section is just given by \( \sigma(\theta) = |f(\theta)|^2 \).

The formula (7.36) with the definition (C.4) gives the standard Born approximation

\[
    f(p', p) = -\frac{m}{2\pi} \langle p'|V_{\text{int}}|p \rangle = -\frac{m}{2\pi} \int d^3x \text{e}^{i\mathbf{q} \cdot \mathbf{x}} V(|x|),
\]

with \( q \equiv p' - p \).

The \textit{in} and \textit{out} state-vectors \( |\alpha_\pm\rangle \), denoted \( |p_\pm\rangle \) in the case considered here, satisfy the equation (7.49) which, when written in the standard position representation, reads

\[
    \langle x|p_\pm(x) = \text{e}^{ip \cdot x} + \int d^3y \langle x|G_0(E_p \pm i0)|y \rangle V(|y|) \langle y|p_\pm \rangle.
\]

The matrix element \( \langle x|G_0(E_p + i0)|y \rangle \) can be given explicitly:

\[
    \langle x|G_0(z)|y \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\text{e}^{ip \cdot (x-y)}}{z - E_p} = \frac{im}{2\pi^2 |x-y|} \int_{-\infty}^{\infty} dp \frac{pe^{i|y-x|}}{p^2 - 2mw}.
\]

The remaining integral can be computed by the residue method. It has two simple poles at \( p = \sqrt{2mw_{\pm}} \), where \( w_{\pm}^2 = z \) and \( \text{Im}(w_+) > 0, \text{Im}(w_-) < 0 \). According to the Jordan lemma, the integration contour must be closed with a large semicircle in the upper half plane. This gives

\[
    \langle x|G_0(z)|y \rangle = -\frac{m}{2\pi} \frac{\text{exp}(i\sqrt{2mw_+}|x-y|)}{|x-y|}.
\]

To find the asymptotic form of \( \psi_{p_\pm}(x) \) for \( |x| \to \infty \), when \( V(|y|) \) vanishes sufficiently fast for \( |y| \to \infty \), it is sufficient to approximate

\[
    |x-y| = r (1 - n \cdot y) + \ldots,
\]

where \( r \equiv |x| \) and \( n \equiv x/|x| = x/r \). For \( z = E_p \pm i0 \), so that \( \sqrt{2mw_\pm} = \pm |p| + i0 \) The formula (C.6) takes then the form

\[
    \psi_{p_\pm}(x) \approx \text{e}^{ip \cdot x} + \frac{\text{e}^{\pm ip \cdot r}}{r} \left( -\frac{m}{2\pi} \int d^3y \text{e}^{\mp i|y|} V(|y|) \psi_{p_\pm}(y) \right).
\]

The factor in the bracket is just \(-m/2\pi\) times the matrix element

\[
    \langle \pm p'|V_{\text{int}}|p_\pm \rangle \equiv t(\pm p', p),
\]

\footnote{Setting \( w_{\pm} = \xi + i\eta \) with \( \eta > 0 \), one has solve for \( \xi \) the equation \( 2mw_{\pm}^2 = 2m(\xi^2 - \eta^2 + 2i\eta) = p^2 \pm i0 \); since \( \eta > 0 \), it is \( \xi \) which must keep the sign of the small imaginary part \( \pm i0 \).}

953
where $p' \equiv n|p|$ which, upon using the definition (C.4) allows to identify $\psi_{p\pm}(x)$ (playing here the role of the the $|\alpha_{\pm}\rangle$ states) as the (generalized) eigenfunctions of $H$ commonly used in the ordinary stationary scattering theory

$$
\psi_{p+}(x) \approx e^{i p \cdot x} + \frac{f(p', p)}{r} e^{+i|p|r},
$$

$$
\psi_{p-}(x) \approx e^{i p \cdot x} + \frac{f(-p', p)}{r} e^{-i|p|r},
$$

and representing asymptotically $\psi_{p+}(\psi_{p-})$ the incoming plane wave and the outgoing (incoming) spherical wave. The convergence (7.15) of wave packets (well localized in space) built on the $in$ or $out$ states to the wave packets built with the same profile $g(k)$ on the plane waves takes here the form

$$
e^{-iHt} \int \frac{d^3p}{(2\pi)^3} g(p) \psi_{p\pm}(x) \rightarrow e^{-iH_0t} \int \frac{d^3p}{(2\pi)^3} g(p) \psi_p(x).$$

Heuristically, it can be justified by appealing to the fact that for $t \rightarrow \mp \infty$ any localized wave packet formed from $\psi_{p\pm}(x)$ is driven far from $x = 0$, into the region in which $\psi_{p\pm}(x)$ effectively do not differ much from the plane waves $\psi_p(x) \equiv e^{i p \cdot x}$. 