14 Renormalization

Przesuwa sie akcent: nie chodzi o to, czy liczmy f. Greena zrenormalizowanych operatorów, tylko o to, jak rozbijamy $H$ na $H_0$ i $V_{\text{int}}$.

As we have already seen in the preceding Section, integrations corresponding to closed loops in Feynman diagrams are in many cases ultraviolet (UV) divergent and at first sight seem to render the whole quantum field theory nonsensical. Historically, they were first noticed in quantum electrodynamics (QED) in the thirties of XXth century. The problem in QED was solved in lowest orders around 1948 by R.P. Feynman, J. Schwinger and S.-I. Tomonaga. Their approach was generalized by F. Dyson who sketched a proof that the procedure of Feynman, Schwinger and Tomonaga works to all orders, thereby laying foundations for the complete renormalization procedure elaborated subsequently by N.N. Bogoliubov, O.N. Parasiuk, K. Hepp, W. Zimmermann, V. Glaser and H. Epstein. In this Section we describe renormalization putting it in the framework developed in the preceding sections.

As explained in sections 9.7, 11.10, the postulates formulated in section 7.3, on which direct calculation of $S$-matrix elements is based, force us to work with the Heisenberg picture field operators $\phi_{\text{ph}}$ normalized in the particular way and to identify mass parameters in the $H_0$ part of the full Hamiltonian with physical masses of particles described by the in and out states. In the perturbative expansion satisfying these requirements, which translate into conditions on the position and residues of simple poles of the appropriate two-point Green’s functions, is achieved by adding to $V_{\text{int}}$ well defined interaction terms which are called counterterms. In the approach based on quantization of fields these extra terms are interpreted as the particular choice of the canonical field variables and a particular splitting of the original Lagrangian mass parameters into the part which enters $H_0$ and the counterterm factors. Although a priori not related to infinities other than the ones identified in section 9.7 (associated with poles of propagators corresponding to external lines), working with physically normalized field operators and physical masses automatically removes some of the ultraviolet (UV) infinities associated with integrations over infinitely large momenta of virtual particles circulating in loops in Feynman diagrams (but not all of them).

In general, UV infinities are encountered also when the perturbative method is applied to compute vacuum Green’s functions (13.9). In this case organization of the expansion is not restricted by the requirements of section 7.3. In particular, one can work with arbitrarily rescaled (renormalized) canonical field operators (i.e. in the approach based on field quantization one can use also the canonical, “bare” field operators) and mass parameters in $H_0$ need
not be identical with masses of physical particles. Some of the UV infinities present in the vacuum Green’s functions do not affect $S$-matrix elements extracted in this approach with the help of the LSZ procedure (see section 13.4) because they are (after regularization) removed by the LSZ prescription. Still, for many purposes it is better to work with finite vacuum Green’s functions. Elimination of the UV infinities that do not affect $S$-matrix elements can be achieved by working not with the canonical (bare) field operators, but with the renormalized operators $\phi_R = Z^{-1/2}\phi_B \equiv Z^{-1/2}\phi_H$ and adjusting appropriately order by order the renormalization constant $Z$. Removing infinities that do affect $S$-matrix elements requires realizing that also the coupling constants in the interaction terms must be given a physical meaning.

Thus, the key idea of the renormalization is that the parameters of the Lagrangian, like mass parameters and coupling constants are not in fact true masses of particles and their measured couplings. They are only Lagrangian parameters (called “bare” parameters) which, even independently of the question of infinities, have to be related to measurable quantities. In turns out that when $S$ matrix elements computed using the perturbative expansion are expressed, order by order, in terms of physical, measurable quantities, the infinities disappear. Renormalization formulated in such a simple way works only in a class of quantum field theory models, called renormalizable. $\lambda\varphi^4$ theory, QED, QCD, and the Standard Model are examples of renormalizable models. There are however infinitely many theories, an example of which is the Fermi theory of weak interactions, in which this simple recipe is not sufficient. Such quantum field theory models are traditionally called nonrenormalizable. As we will see, they too can be dealt with by generalizing the renormalization procedure (one can say that nonrenormalizable models are not much worse than renormalizable ones).

Physically ultraviolet divergences appearing in the perturbation expansion signal that quantities computed in a given quantum field theory model depend on physics at some very large momentum scale, i.e. on physics on very short distances, at which these theories are no longer applicable. Classical theories which treat matter as continuous necessarily break down at length scales comparable to atomic sizes. Thus, existence in these theories of implicit small distance cutoffs (in the field theory language) of this order is quite natural. The cutoff scale does not appear explicitly in the quantities computed using these theories but is hidden in their parameters. For instance, in fluid dynamics quantities like the viscosity coefficient, the sound speed etc. are phenomenological parameters determined by direct measurements. Their magnitudes agree, however, with the appropriate combinations of atomic radii and velocities whose magnitudes set the cutoff scale for the classical fluid dynamics. Similarly, in a classical description of a magnet the magnitude of the magnetic susceptibility $\chi_T$ is of the order of the energy
needed to flip a single electron spin. In many cases phenomenological pa-
rameters used in classical theories can be computed theoretically because the
underlying theory at the atomic level is known. In quantum field theory we
still do not know physics at the shortest length scales (perhaps it is described
by the string theory, or the M-theory, but this is still not certain, and in,
particular, we still do not know how to link these theories to the known
physics). For this reason parameters like the fine structure constant \( \alpha_{\text{EM}} \) or
the electron mass \( m_e \) cannot at present be computed from first principles.
Therefore we are forced to use experimentally measured values \( \alpha_{\text{EM}}, m_e \) etc.
to parametrize our ignorance and accept the fact that all quantum field theo-
ries are only effective theories (valid only at sufficiently large distance scales,
that is, only at sufficiently low energies). As we will see, \( S \)-matrix elements
indeed become finite, once they are expressed in terms of measured masses
and couplings.

\[ \text{ale tu tež dowolność } m_R, \lambda_R \leftrightarrow m_{\text{ph}}, \lambda_{\text{ph}}. \]

In order to construct such effective quantum field theory models describ-
ing physical phenomena at low energies (low compared to some chosen en-
ergy scale like the Planck scale \( M_{\text{pl}} \), Grand Unification scale or even the
electroweak scale \( G_F^{-1/2} \) one needs to know two things: i) what are the rel-
levant low energy degrees of freedom and ii) how many parameters (masses
and couplings) are relevant. In fluid dynamics the degrees of freedom are
waves and the cutoff means that the classical theory can describe only suf-
ciently long waves (long compared to the atomic scales) which play the role
of low energy degrees of freedom). The quantum field theory analog of long
waves in fluid dynamics are light particles (with sufficiently small momenta).
The question is therefore, what light particles (light compared to the char-
acteristic mass scale \( M \) of the more fundamental theory) there are, and ii) how many terms in the effective Lagrangian are necessary to describe the ob-
served low energy phenomena. (The latter question is also related to another
one: what are the conserved - at least approximately - quantum numbers).
Naively, infinitely many interaction terms can be written down but the renor-
malization group point of view, developed in the next Section, will teach us
that in most cases, the interaction terms which are called non-renormalizable
(whose number is in principle infinite) become irrelevant for description of
processes whose characteristic energies \( E \) are small compared to the mass
scale \( M \) characterizing the underlying a more fundamental theory. Corre-
cctions to amplitudes due to nonrenormalizable interactions represented in \( \mathcal{H}_{\text{int}}^{\text{eff}} \)
by operators of dimension \( 4 - \Delta \) (where \( -\Delta = 1, 2, \ldots \) ) are suppressed com-
pared to contributions of renormalizable operators by powers of the factor
\( (E/M) \). This is why with the present experimental accuracy, the renormal-
izable theory known as the Standard Model, is sufficient\(^1\) to describe known

\[^1\text{Perhaps with the exception of neutrino oscillations, which most probably are due to}\]
particle interactions up to energies of order a few hundred GeV.

In this section we first show how renormalization works in the simple case of the $\phi^4$ theory and then discuss it in more general terms.

### 14.1 One loop renormalization in the $\phi^4$ theory

We return to the problem of removing infinities from the scattering amplitude (13.57). As we have noticed, the coupling $\lambda$ is not yet given any physically meaningful definition. To fill this gap we define the physical coupling $\lambda_{ph}$ of the $\phi^4$ theory using the value of the measured differential scattering cross section at some particular (but chosen arbitrarily) kinematical point $s_0$, $t_0$:

$$\lambda^2_{ph} = 64\pi^2 s_0 \left. \frac{d\sigma}{d\Omega} \right|_{s_0,t_0}. \quad (14.1)$$

Since the theoretical formula for the differential cross section for elastic scattering is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}(s,t)|^2, \quad (14.2)$$

this means that for $s = s_0$, $t = t_0$ the (modulus of the) amplitude $\mathcal{M}(s_0,t_0)$ is just equal $\lambda_{ph}$. The value of $\mathcal{M}(s_0,t_0)$ is measurable experimentally and the coupling $\lambda_{ph}$ is therefore well defined. On the theoretical side, in the one-loop approximation, the amplitude $\mathcal{M}(s_0,t_0)$ is given by (13.57), from which it follows that

$$\lambda_{ph} = \lambda + \frac{\lambda^2}{2(4\pi)^2} [3I_{\text{div}} + f(s_0) + f(u_0) + f(t_0)] + \mathcal{O}(\lambda^3). \quad (14.3)$$

Inverting this relation to one loop accuracy we get

$$\lambda = \lambda_{ph} - \frac{\lambda^2_{ph}}{2(4\pi)^2} [3I_{\text{div}} + f(s_0) + f(u_0) + f(t_0)] + \mathcal{O}(\lambda^3_{ph}). \quad (14.4)$$

If the amplitude of the elastic scattering is expressed in terms of $\lambda_{ph}$ instead of $\lambda$ the infinities cancel out:

$$-i\mathcal{M}(p_1,p_2 \rightarrow p_3p_4) = -i\lambda_{ph} - i\frac{\lambda^2_{ph}}{2(4\pi)^2} [f(s) + f(u) + f(t)$$

$$- f(s_0) - f(u_0) - f(t_0)] + \mathcal{O}(\lambda^3_{ph}), \quad (14.5)$$

a nonrenormalizable interaction represented by a dimension 5 operator. The search for proton decay conducted since many years in various laboratories (such as Kamiokande and Superkamiokande in Japan) is in fact a search for a manifestation of another nonrenormalizable interaction.

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2To the order we are working the mass parameter $M^2$ on which the amplitude (13.57) depends can be replaced by $M^2_{ph}$ because as follows from (13.54) $M^2_{ph} = M^2 + \mathcal{O}(\lambda)$. 599
and we obtain finite predictions\(^3\) for the cross section for elastic scattering in order \(\lambda_{\text{ph}}^2\) for arbitrary values of the kinematical variables \(s, t\) and \(u\).

In the above example there were originally two free parameters: \(M^2\) and \(\lambda\). After renormalization predictions of the theory still depend on two free parameters: \(M_{\text{ph}}^2\) and \(\lambda_{\text{ph}}\), i.e. the number of free parameters did not change. This is not always the case. Had we started with

\[
\mathcal{H}_{\text{int}} = \frac{\lambda}{4!} \varphi^4 + \frac{\hbar}{6!} \varphi^6 ,
\]

we could compute at one-loop the amplitude for the production of six scalars in the collision of two particles through the diagrams like the one shown in figure 14.1. This diagram is divergent, so, in addition to defining the physical coupling \(\lambda_{\text{ph}}\), we would have to introduce yet another renormalization condition to fix the finite part of the amplitude of figure 14.1 (e.g. by requiring that the amplitude equals \(g_{\text{ph}}\) for some particular values of its external momenta). In this way we would introduce a new free parameter \(g_{\text{ph}}\). Moreover, in higher orders we would need more and more conditions, introducing thereby more and more new free parameters, to make all amplitudes finite. Such a situation is typical for theories called nonrenormalizable. We will discuss this in more detail in what follows.

### 14.2 Renormalized parameters and field operators

The renormalization procedure described in the preceding subsection can be formulated in a more flexible way. (This flexibility will be exploited to partially resumm the perturbation series by using the renormalization group). We will discuss this on the example of the \(\varphi^4\) theory. to do zmiany: oprzeć na LSZ

\(^3\)In the \(\varphi^4\) theory, to the order we are working, \(Z = 1\). As follows from the LSZ procedure described in section 13.4, in more complicated QFT models, in which \(Z \neq 1\) at one loop, inclusion of the appropriate \(Z^{1/2}\) factor per each external leg (see section 13.4) is necessary to render the physical amplitudes finite.
We begin by saying that the original Lagrangian
\[
L = \frac{1}{2} \partial_\mu \varphi_B \partial^\mu \varphi_B - \frac{1}{2} M_B^2 \varphi_B^2 - \frac{\lambda_B}{4!} \varphi_B^4 ,
\]
(14.6)
is written in terms of the “bare”, i.e. canonical, field operators \( \varphi_B \equiv \varphi_H \) and bare parameters (as indicated by the superscripts “B”).

The bare parameters are unphysical in the sense that they are not measurable quantities. They may well be infinite. We relate the “bare” field operators and parameters to the renormalized ones by the formulae
\[
\varphi_B = Z^{1/2} \varphi_R , \quad M_B^2 = M_R^2 + \delta M^2 , \quad \lambda_B = \lambda_R + \delta \lambda .
\]
(14.7)

In terms of the renormalized field operators and parameters the Lagrangian (14.6) takes the form
\[
L = \frac{1}{2} \partial_\mu \varphi_R \partial^\mu \varphi_R - \frac{1}{2} M_R^2 \varphi_R^2 - \frac{\lambda_R}{4!} \varphi_R^4 + \frac{1}{2} (Z - 1) \partial_\mu \varphi_R \partial^\mu \varphi_R
\]
\[
- \frac{1}{2} [(Z-1)M_R^2 + Z \delta M^2] \varphi_R^2 - \frac{1}{4!} [(Z^2 - 1) \lambda_R + Z^2 \delta \lambda] \varphi_R^4 .
\]
(14.8)
The last three terms are called counterterms. It should be stressed that the renormalized parameters like \( M_R^2 \) and \( \lambda_R \) need not be “physical” (i.e. may not coincide with measurable physical masses and values of the scattering amplitudes at some kinematical accessible points). They are just finite parameters which will parametrize all measurable quantities predicted by the theory. Similarly the renormalized fields need not be physical (i.e. normalized according to the condition (9.99). We also stress, that the field renormalization constant \( Z \) is not the same quantity as the \( Z \) factor defined in section 13.4.

Upon going to the interaction picture, i.e. by equating \( \varphi_R \) taken at \( t = 0 \) with the free field operators \( \varphi_I \) the Lagrangian density (14.8) generates the Feynman rules.

The last three terms in (14.8) must be treated as additional interactions. The new Feynman rules are given in figure 14.2. The rules on the right hand side are the rules for the counterterms. In the perturbative expansion the “wave function” renormalization constant \( Z \), and the factors \( \delta m^2 \) and \( \delta \lambda \) will be given by power series in \( \lambda_R \):
\[
Z = 1 + \sum_{k=1}^{\infty} \delta Z_{(k)} , \quad \delta M^2 = \sum_{k=1}^{\infty} \delta M_{(k)}^2 , \quad \delta \lambda = \sum_{k=1}^{\infty} \delta \lambda_{(k)} ,
\]
(14.9)
where
\[
\delta Z_{(k)} \propto \lambda_R^k , \quad \delta M_{(k)}^2 \propto \lambda_R^k , \quad \delta \lambda_{(k)} \propto \lambda_R^{k+1} .
\]
(14.10)

nawiazać do LSZ i przejścià przez Hamiltonian

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Therefore, the counterterm vertices involving the factors \( \delta Z^{(1)} \), \( \delta M^{(1)} \) and \( \delta \lambda^{(1)} \) inserted in tree graphs should be combined with one-loop graphs with “normal” vertices, then \( \delta Z^{(2)} \), \( \delta M^{(2)} \) and \( \delta \lambda^{(2)} \) inserted in tree graphs should be combined with two-loop graphs with “normal” vertices and one-loop graphs with a single insertion of \( \delta Z^{(1)} \), \( \delta M^{(1)} \) or \( \delta \lambda^{(1)} \) and the other vertices “normal”, and so on. The counterterm diagrams allow then to make finite all off-shell Green’s functions of the theory (recall, that no counterterms are needed to make finite the \( S \) matrix elements provided one includes the \( Z \) factors appropriately and expresses amplitudes in terms of measurable masses and couplings).

To discuss renormalization we will use the 1PI Green’s functions. As we will show, in the \( \varphi^4 \) theory only \( \Gamma^{(2)} \) and \( \Gamma^{(4)} \) 1PI Green’s functions are superficially divergent.\(^4\) By definition \( i\tilde{\Gamma}^{(2)} = -[\tilde{G}_c^{(2)}]^{-1} \) where \( \tilde{G}_c^{(2)} \) is the full propagator (the two-point connected Green’s function). In one loop order we have for these 1PI Green’s functions:

\[
i\tilde{\Gamma}_R^{(2)}(p^2) = i \left( p^2 - M_R^2 \right) - i\Sigma(p^2) + i\delta Z^{(1)}p^2 - i \left( M_R^2\delta Z^{(1)} + \delta M^{(1)} \right), \tag{14.11}
\]

\[
i\tilde{\Gamma}_R^{(4)}(s, t, u) = -i\lambda_R - \frac{\lambda_R^2}{2(4\pi)^2} \left[ 3I_{\text{div}} + f(s) + f(t) + f(u) \right]
- i \left( 2\lambda_R\delta Z^{(1)} + \delta \lambda^{(1)} \right), \tag{14.12}
\]

where we have expanded the counterterms of figure 14.2 to the appropriate order. The counterterms \( \delta Z^{(1)} \), \( \delta M^{(1)} \) and \( \delta \lambda^{(1)} \) must be fixed by some arbitrary renormalization conditions imposed on \( \tilde{\Gamma}^{(2)}(p^2) \) and \( \tilde{\Gamma}^{(4)}(s, t, u) \) 1PI Green’s functions. There are two popular choices called (renormalization schemes) which we briefly characterize below. Other renormalization schemes can be

\(^4\)The meaning of the word “superficially” will become clear soon.
also used and are sometimes more convenient for some specific purposes.

1) On-Shell renormalization scheme. In this scheme one imposes on \( \tilde{\Gamma}^{(2)}(p^2) \) the conditions

\[
\tilde{\Gamma}_R^{(2)}(p^2 = M_R^2) = 0, \quad \text{and} \quad \frac{d}{dp^2} \tilde{\Gamma}_R^{(2)}(p^2) \bigg|_{p^2 = M_R^2} = 1. \tag{14.13}
\]

As a result, in the vicinity of \( p^2 = M_R^2 \) the full propagator behaves as

\[
\frac{i}{p^2 - M_R^2 - \Sigma_R(p^2) + i0} \approx \frac{i}{p^2 - M_R^2 + i0}, \tag{14.14}
\]

i.e. \( M_{\text{ph}}^2 = M_R^2 \) (the renormalized mass parameter is the same as the physical mass) and \( Z = 1 \). At one loop this requires

\[
\delta Z^{(1)} = \Sigma'(p^2 = M_R^2), \quad \delta M^2 = -\Sigma(p^2 = M_R^2) \tag{14.15}
\]

(of course, in the \( \phi^4 \) theory at one loop \( \Sigma' = 0 \) and \( \delta Z^{(1)} = 0 \)). For \( \tilde{\Gamma}^{(4)}(s,t,u) \) no canonical condition exists. We can for example impose the condition

\[
\tilde{\Gamma}_R^{(4)}(s_0, t_0, u_0) \equiv -\lambda_R + \Delta \tilde{\Gamma}_R^{(4)}(s_0, t_0, u_0) = -\lambda_{\text{ph}}, \tag{14.16}
\]

for some physically realizable (i.e. allowed kinematically) \( s_0 \sim t_0 \sim u_0 \sim M_R^2 \). This gives

\[
\delta \lambda = \Delta \tilde{\Gamma}^{(4)}(s_0, t_0, u_0) - 2\lambda_R \Sigma'(p^2 = M_R^2), \tag{14.17}
\]

and \( \lambda_{\text{ph}} = \lambda_R \). It should be however clear that no “canonical” definition of \( \lambda_{\text{ph}} \) exist and one can even define \( \lambda_R \) to be e.g. the value of the amplitude at some kinematically inaccessible point, like \( s = t = 0 \).

Thus in this renormalization scheme the renormalized parameters (like \( m_R^2 \) and \( \lambda_R \)) are just the physical ones. The only difference compared to the approach of section 14.1 is that now the counterterms will make finite also off-shell Green’s functions whereas in the approach of section 14.1 only on-shell S-matrix elements were made finite.

2) (Modified) Minimal Subtraction renormalization scheme. This scheme is related to a particular way of regularizing divergent integrals called Dimensional Regularization (DIMREG). DIMREG consists of continuing the loop integrals in the number of space-time dimensions. For the present purpose
it is sufficient to say that in this method divergent loop integrals, after appropriately removing their subdivergences (see section 14.5), take the form

\[ I = \sum_{k \geq 1} \frac{a_k}{\epsilon^k} + I_{\text{finite}}, \]  

(14.18)

where \( \epsilon \propto (d - 4) \). The pole terms are unambiguously determined by the DIMREG procedure. By definition, in the Minimal Subtraction (MS) scheme the counterterms cancel only such pole terms. Thus, in the MS scheme \( Z, \delta M^2 \) and \( \delta \lambda \) take the forms of power series in \( 1/\epsilon \) (the coefficients \( a_k \) of the powers of \( 1/\epsilon \) are determined in perturbative expansion as power series in coupling constants). Off-shell Green’s functions become then finite, but the relation of \( M^2_R \equiv \hat{M}^2 \) and \( \lambda_R \equiv \hat{\lambda} \) (a hat will be used to denote the quantities renormalized in this scheme) to measurable quantities has to be established order by order in perturbation calculus. Denoting the renormalized in the MS scheme self energy by \( \hat{\Sigma}(p^2) \) we have for the full propagator

\[ \frac{i}{p^2 - \hat{M}^2 - \hat{\Sigma}(p^2)} \approx \frac{iZ}{p^2 - \hat{M}^2_{\text{ph}}} \quad \text{for} \quad p^2 \approx \hat{M}^2_{\text{ph}}, \]  

(14.19)

where the physical mass \( \hat{M}^2_{\text{ph}} \) is the solution of the equation

\[ \hat{M}^2_{\text{ph}} = \hat{M}^2 + \hat{\Sigma}(p^2 = M^2_{\text{ph}}, \hat{M}^2, \hat{\lambda}) \]  

(14.20)

(we have displayed the dependence of \( \hat{\Sigma} \) on the mass parameter and on the coupling constant) and the factor \( Z \) is given by

\[ Z = \left( 1 - \frac{d\hat{\Sigma}(p^2)}{dp^2} \bigg|_{p^2 = \hat{M}^2_{\text{ph}}} \right)^{-1}. \]  

(14.21)

In general \( Z \neq 1 \) in this scheme, as in our calculation in subsection 13.5. However, now it is finite owing to the infinite “wave function” renormalization constant \( Z \).

**że w obu podejśćach f. Greena sa skończone**

The Modified Minimal Subtraction scheme (called the MS scheme) differs from the Minimal Subtraction only in that apart from the pole terms one subtracts also another (finite) piece which is also unambiguously singled out by the DIMREG procedure.

We can now take the point of view that a theory is renormalizable if its Green’s functions can be made finite by renormalizing its original fields and parameters as in (14.7). This is equivalent to adding to the original Lagrangian counterterms of the special form (see (14.8)) whose (infinite when
the regularization is removed) coefficients are given by power series in coupling constant(s); the form of the counterterms is such that they can be interpreted as the renormalization of the original Lagrangian fields and parameters.

ten akapit od czapy!

### 14.3 Regularization: simplest examples

In this subsection we will regularize the integrals encountered in one loop calculations in the $\varphi^4$ theory. We will first apply the simple momentum cut-off and next develop the dimensional regularization method.

Consider the integral

$$I(p^2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - M^2][(k + p)^2 - M^2]} , \quad (14.22)$$

(since the integral is Lorentz invariant it can depend only on $p^2$). It is divergent, but the divergent part is independent on the (external) momentum $p$. To see it we differentiate $I(p^2)$

$$\frac{d}{dp^2} I(p^2) = \frac{p^\mu}{2p^2} \frac{\partial}{\partial p^\mu} I(p^2) = -\frac{1}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{i(k\cdot p + p^2)}{[k^2 - M^2][(k + p)^2 - M^2]} , \quad (14.23)$$

obtaining a finite integral. Thus

$$I(p^2) = I(0) + \int_0^{p^2} dp^2 \frac{d}{dp^2} I(p^2) , \quad (14.24)$$

where only $I(0)$ is divergent. To compute the integral in (14.23) we can use the general formula

$$\frac{1}{a^\alpha b^\beta c^\gamma \cdots} = \frac{\Gamma(\alpha + \beta + \gamma + \cdots)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \cdots} \times \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdots \frac{\delta(1 - x - y - z - \cdots) x^{\alpha-1} y^{\beta-1} z^{\gamma-1} \cdots}{[xa + yb + zc + \cdots]^{\alpha+\beta+\gamma+\cdots}} , \quad (14.25)$$

5In practical calculations it is frequently more convenient to use this formula successively, combining at each step only two denominators.
This gives
\[
\frac{d}{dp^2} I(p^2) = -\frac{2}{p^2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{i(k \cdot p + p^2)}{[(k + xp)^2 - x(x-1)p^2 - M^2 + i0]^3}.
\]

Shifting the integration variable\(^6\) gives
\[
\frac{d}{dp^2} I(p^2) = -\frac{2}{p^2} \int_0^1 dx \ x(1-x) p^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - a^2 + i0]^3},
\]
where \(a^2 = x(x-1)p^2 + M^2\). We have dropped odd powers of \(k^\mu\) in the numerator because they give zero upon symmetric integration over \(d^4k\). To evaluate the integral over \(dk^0\) we note that\(^7\) the poles of the integrand are located as shown by symbols “×” in figure 14.3 and, therefore, the integral over the contour shown by the dashed lines in figure 14.3 is zero (the contour does not enclose any pole). Since the integrals over the arcs at infinities also vanish, we conclude that\(^8\)
\[
\int_{-\infty}^{+\infty} dk^0(\ldots) = \int_{-i\infty}^{+i\infty} dk^0(\ldots) = i \int_{-\infty}^{+\infty} dk^4(\ldots)!,
\]
where we have changed the variable \(k^0 = ik^4\) to get again the integral (now over \(dk^4\)) from \(-\infty\) to \(+\infty\). We then have
\[
\frac{d}{dp^2} I(p^2) = -2 \int_0^1 dx \ x(1-x) \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k^2_E + a^2 - i0]^3},
\]
where the subscript \(E\) means “Euclidean” (i.e. \(k^2_E = k_1^2 + k_2^2 + k_3^2 + k_4^2\)). Performing the integral over the whole (4-dimensional) solid angle (whose volume is \(2\pi^2\)) and using the variable \(\bar{k} = k/a\) we get
\[
\frac{d}{dp^2} I(p^2) = -2 \int_0^1 dx \ x(1-x) \frac{1}{a^2 - i0} \frac{2\pi^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\bar{k}^3 d\bar{k}}{[\bar{k}^2 + 1]^3}
\]
\[
= \frac{1}{16\pi^2} \int_0^1 dx \ \frac{x(x-1)}{x(x-1)p^2 + M^2 - i0}.
\]
Applying now (14.24) we find
\[
I(p^2) = I(0) + \frac{1}{16\pi^2} \int_0^1 dx \ \ln \frac{x(x-1)p^2 + M^2 - i0}{M^2} \equiv I_{\text{div}} + f(p^2),
\]
\(^6\)This is allowed because the integral if convergent.
\(^7\)We compute the integral over \(dk^0\) for \(p^2 < 2M^2\) for which \(a^2 > 0\) for any \(x\), and then continue it analytically into the domain \(p^2 \geq 2M^2\). The factor \(i0\) is then crucial.
\(^8\)This is sometimes called the “Wick rotation”.
as anticipated in (13.49) and (14.3). It remains to compute $I(0)$. Using the same "Wick rotation" we get

$$I(0) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - M^2 + i0]^2} = -\frac{2}{(4\pi)^2} \int_0^\Lambda \frac{k^3dk}{[k^2 + M^2]^2}$$

$$= -\frac{1}{(4\pi)^2} \int_0^{\Lambda^2/M^2} \frac{z dz}{(z + 1)^2} = -\frac{1}{(4\pi)^2} \left(-1 + \ln \frac{\Lambda^2}{M^2}\right), \quad (14.28)$$

where we have retained only terms nonvanishing in the limit $\Lambda \to \infty$. Similarly we can compute the integral needed for the one loop self energy $\Sigma$ in the $\phi^4$ model:

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0} = \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + M^2}$$

$$= \frac{1}{(4\pi)^2} M^2 \int_0^{\Lambda^2/m^2} \frac{z dz}{z + 1} = \frac{1}{(4\pi)^2} \left(\Lambda^2 - M^2 \ln \frac{\Lambda^2}{M^2}\right) + \ldots \quad (14.29)$$

where the ellipses denote terms vanishing as $\Lambda \to \infty$.

### 14.4 Dimensional Regularization

In the previous subsection we have regularized the divergent integrals by imposing the ultraviolet (UV) cutoff on the integrals over virtual momenta. This is not always convenient. For instance, in gauge theories such a regularization violates gauge invariance, i.e. it leads to amplitudes which do not satisfy the relation (9.78). More convenient in this respect is the Dimensional Regularization (DIMREG) introduced by G. ’t Hooft. It consists in formulating the theory in arbitrary number of dimensions. The expressions resulting from loop integrals treated as functions of the (complex) number $d$ of dimensions have then poles at $d = 4$. It is important to stress that while
the MS or \( \overline{\text{MS}} \) schemes are defined only in this regularization, DIMREG itself can be also used to set other renormalization schemes like the on-shell one.

We have to consider first the dimensions of various quantities. For \( h = c = 1 \) the action, i.e. \( \int d^4x \mathcal{L} \) is dimensionless. The measure \( d^4x \) has, in mass units \([M]\), the dimension mass\(^{-d}\):

\[
\left[ \int d^4x \mathcal{L} \right] = [M]^0, \quad [d^4x] = [M]^{-d}.
\]

Hence

\[
[\mathcal{L}] = [M]^d,
\]

and each term of \( \mathcal{L} \) must, of course, have the same dimension. Considering now different terms of \( \mathcal{L} \) we find the dimensions

\[
\begin{align*}
[\partial \phi \partial \phi] &= [M]^2 [\phi]^2 \Rightarrow [\phi] = [M]^{\frac{d}{2} - 1} \\
[\bar{\psi} \gamma \psi] &= [M] [\psi]^2 \Rightarrow [\psi] = [\bar{\psi}] = [M]^{\frac{d+1}{2}} \\
[M^2 \phi^2] &= [M^2] [M]^{d-2} \Rightarrow [M^2] = [M]^2 \\
[m \bar{\psi} \psi] &= [m] [M]^{d-1} \Rightarrow [m] = [M] \\
[\lambda \phi^4] &= [\lambda] [M]^{2d-4} \Rightarrow [\lambda] = [M]^{4-d} \\
[h \phi \bar{\psi} \psi] &= [h] [M]^{\frac{5d-2}{2}} \Rightarrow [h] = [M]^{2 - \frac{d}{2}} \\
g \phi^4 (D - \bar{D})_{\mu} \phi A^\mu &= g [M]^{\frac{5d-3}{2} + 1} \Rightarrow [g] = [M]^{2 - \frac{d}{2}}
\end{align*}
\]

What matters here is the type of the field operators - bosonic or fermionic - and not their Lorentz indices; hence e.g. the next to last result applies equally well to gauge couplings \( g \bar{\psi} \gamma^\mu \psi A_\mu \) etc. For future use it is convenient to define\(^9\) the parameter \( \epsilon \)

\[
\epsilon = \frac{d}{2} - 2. \tag{14.33}
\]

As we see from (14.32) the original ("bare") coupling \( \lambda_B \) in the \( \phi^4 \) theory has dimension \([M]^{-2\epsilon}\). To keep the renormalized coupling dimensionless we write

\[
\lambda_B = \mu^{-2\epsilon} (\lambda_R + \delta \lambda), \tag{14.34}
\]

where \( \mu \) is an arbitrary mass unit called the "renormalization scale". If DIMREG is applied to the on-shell renormalization scheme, the dependence on the arbitrary mass unit \( \mu \) cancels out in physical predictions. On the

\(^9\)Note that \( \epsilon \) is defined here differently than in most textbooks.
the other hand, in the MS or $\overline{\text{MS}}$ schemes, in order to compensate for the possible changes of the arbitrary mass unit $\mu$ one has to assume that the renormalized parameters of the theory $\lambda_R$ and $M_R^2$ depend implicitly on $\mu$; changing $\mu$ one has also to change the numerical values of $\lambda_R$ and $M_R^2$ in such a way that the physical (measurable) quantities do not change. As we will discuss later, this forms the basis of the most convenient formulation of the renormalization group methods (see Section 18).

Let us see how DIMREG works in the $\phi^4$ theory. Consider the one loop corrections to $\tilde{\Gamma}^{(4)}$ given by the upper diagrams in figure 13.5. For the first of them we have the expression

$$\frac{1}{2} (-i\lambda_R \mu^{-2\epsilon})^2 \int \frac{d^d k}{(2\pi)^d} \frac{(i)^2}{[k^2 - M_R^2 + i0][(k + q)^2 - M_R^2 + i0]} ,$$

where $q = p_1 + p_2$. Using the formula (14.25) with $\alpha = \beta = 1$ and completing the perfect square (in the variable $k$) we arrive at

$$-\frac{i}{2} (\lambda_R \mu^{-2\epsilon})^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{i}{[(k + xq)^2 - x(x - 1)q^2 - M_R^2 + i0]^2} .$$

Shifting the variable $k + xq \to k$ and performing the Wick rotation we get

$$-\frac{i}{2} (\lambda_R \mu^{-2\epsilon})^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{-k^{d-1}dk}{[k^2 + x(x - 1)q^2 + M_R^2 - i0]^2} .$$

The integral over the $d$-dimensional solid angle is

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} , \quad (14.35)$$

so that, after changing the variable to $z = k/[x(x - 1)q^2 + M_R^2]^{1/2}$, we have

$$\frac{i}{2} (\lambda_R \mu^{-2\epsilon})^2 \frac{2\pi^{d/2}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \int_0^1 dx \left[ x(x - 1)q^2 + M_R^2 - i0 \right]^{\frac{d}{2} - 2} \int_0^\infty \frac{z^{d-1}dz}{[z^2 + 1]}$$

$$= \frac{i}{2} \lambda_R^2 \mu^{-2\epsilon} \frac{1}{(4\pi)^2 \Gamma\left(\frac{d}{2}\right)} \int_0^1 dx \frac{\left[ x(x - 1)q^2 + M_R^2 - i0 \right]^{\frac{d}{2} - 2}}{4\pi \mu^2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2 - \frac{d}{2}\right) ,$$

where we have used the formula

$$\int_0^\infty \frac{z^{2m-1}dz}{[z^2 + 1]^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m + n)} , \quad (14.36)$$

Using now the expansions

$$\Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma_E + O(\epsilon) ,$$

$$A^\epsilon = \exp(\epsilon \ln A) = 1 + \epsilon \ln A + O(\epsilon^2) , \quad (14.37)$$

\(^{10}\text{Again this is allowed for sufficiently small } d, \text{ so that the integral is convergent.}\)
where $\gamma_E = 0.5772\ldots$ is the Euler constant, we finally get
\[
-i \frac{\lambda_R^2}{2(4\pi)^2} \mu^{-2\epsilon} \left[ \frac{1}{\epsilon} + \gamma_E + \int_0^1 dx \ln \frac{x(x - 1)q^2 + M_R^2 - i0}{4\pi \mu^2} + O(\epsilon) \right]. \tag{14.38}
\]
Similarly, for a one-loop contribution to $i\tilde{\Gamma}^{(2)}$, after similar steps, we obtain
\[
-i \Sigma = -i \frac{\lambda_R^2}{2(4\pi)^2} M_R^2 \left( \frac{1}{\epsilon} + \gamma_E - 1 + \ln \frac{M_R^2}{4\pi \mu^2} \right). \tag{14.39}
\]

Renormalized one loop $i\tilde{\Gamma}^{(2)}_R$ and $i\tilde{\Gamma}^{(4)}_R$ are obtained by adding counterterms generated by the Feynman rules shown in figure 14.2. At one loop we get\footnote{In the DIMREG scheme, the Feynman rules of figure 14.2 should be modified by including the factor $\mu^{-2\epsilon}$ multiplying $\lambda_R$ and $\delta \lambda$ as follows from (14.34).}
\[
\tilde{\Gamma}^{(2)}_R = i \left( p^2 - M_R^2 \right) + i \delta Z^{(1)} p^2 - i \left( \delta M_{(1)}^2 + M_R^2 \delta Z^{(1)} \right) - i \frac{\lambda_R^2}{2(4\pi)^2} M_R^2 \left( \frac{1}{\epsilon} + \gamma_E - 1 + \ln \frac{M_R^2}{4\pi \mu^2} \right), \tag{14.40}
\]
\[
\tilde{\Gamma}^{(4)}_R = -i \lambda_R \mu^{-2\epsilon} - i \left( \delta \lambda^{(1)} + 2 \lambda_R \delta Z^{(1)} \right) \mu^{-2\epsilon} - i \frac{\lambda_R^2}{2(4\pi)^2} \mu^{-2\epsilon} \left\{ \frac{3}{\epsilon} + 3 \gamma_E + \int_0^1 dx \left[ \ln \frac{x(x - 1)q^2 + M_R^2 - i0}{4\pi \mu^2} + (s \to u) + (s \to t) \right] \right\}. \tag{14.41}
\]

The MS scheme is defined by the conditions that the counterterms $\delta Z$, $\delta \lambda$ and $\delta m^2$ remove only the poles in $\epsilon$. Hence, since at one loop $\Sigma$ is independent of $p^2$, $\delta Z^{(1)} = 0$ and\footnote{Recall that we denote quantities in the MS and MS schemes by a hat.}
\[
\begin{align*}
\delta M_{(1)}^{(1)}_{\text{MS}} & = -\frac{\lambda_R}{2(4\pi)^2} \frac{1}{\epsilon} \equiv -\frac{\lambda}{2(4\pi)^2} \hat{M}^2 \frac{1}{\epsilon} \\
\delta \lambda^{(1)}_{\text{MS}} & = -\frac{3\lambda^2_R}{2(4\pi)^2} \frac{1}{\epsilon} \equiv -\frac{3\hat{\lambda}^2}{2(4\pi)^2} \frac{1}{\epsilon}. \tag{14.42}
\end{align*}
\]

The $\overline{\text{MS}}$ scheme differs in that the counterterms remove in addition the factor $\gamma_E - \ln 4\pi$ which is also unambiguously singled out by dimensional regularization
\[
\begin{align*}
\delta M_{(1)}^{(1)}_{\overline{\text{MS}}} & = -\frac{\dot{\lambda}}{2(4\pi)^2} \hat{M}^2 \left( \frac{1}{\epsilon} + \gamma_E - \ln 4\pi \right) \\
\delta \lambda^{(1)}_{\overline{\text{MS}}} & = -\frac{3\dot{\lambda}^2}{2(4\pi)^2} \left( \frac{1}{\epsilon} + \gamma_E - \ln 4\pi \right).
\end{align*}
\]
The physical (measurable) mass $M_{ph}^2$ and coupling $\lambda_{ph}$ (defined in the same way as in (14.1)) are in the MS scheme (for $\epsilon = 0$) given by

$$M_{ph}^2 = \hat{M}^2 + \frac{\hat{\lambda}}{2(4\pi)^2} \hat{M}^2 \left(-1 + \ln \frac{\hat{M}^2}{\mu^2}\right),$$

$$\lambda_{ph} = \hat{\lambda} + \frac{\hat{\lambda}^2}{2(4\pi)^2} \int_0^1 dx \left[ \ln \frac{x(x-1)s_0 + \hat{M}^2 - i0}{\mu^2} + (s_0 \to u_0) + (s_0 \to t_0) \right].$$

It is then clear that $M_{ph}^2$ and $\lambda_{ph}$ can be independent of the arbitrary mass unit $\mu$ only if $\hat{\lambda}$ and $\hat{M}^2$ depend on $\mu$ appropriately. This is the basis of the renormalization group which we discuss in section 18.

### 14.5 Renormalization: general theory

In this subsection we give a general description of the renormalization procedure. We will not go into detailed proofs of the renormalizability of quantum field theory models, because such proofs are tedious and in fact in most cases are not needed in practical calculations (at least in lowest orders).

We define first the degree of superficial divergence $D$ of a 1PI (one particle irreducible, hence connected) Feynman diagram consisting of propagators, vertices and $n$ integrals over independent loop momenta. $D$ is the degree of divergence when all integrated momenta go simultaneously to infinity. In other words, we treat all $n$ four-momenta over which we integrate as a single $4n$ dimensional vector and in this $4n$ dimensional space consider the integration over the length of the $4n$-momentum. If the superficial of degree divergence is $D$, the integral behaves as

$$I \sim \int d^{4n}k \sim \int^\infty |k|^{D-1}d|k|.$$

For example, the integral

$$\int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{1}{[k_1^2 - M^2][(k_1 + p_1)^2 - M^2][k_2^2 - M^2][(k_2 + p_2)^2 - M^2]}$$

$$\sim \int \frac{d^8k}{(2\pi)^8} \frac{1}{[k^2]^4},$$

has $D = 0$. It is clear that if $D \geq 0$, the diagram gives a divergent integral. If instead $D < 0$, the diagram is said to be superficially convergent. The degree of superficial divergence can also be defined for any subdiagram of a given diagram. It is important to realize, that a superficially convergent diagram may contain superficially divergent subdiagrams with $D_{\text{subgraph}} \geq 0$. 
For the degree $D$ of a Feynman diagram having $I_f$ internal lines of type $f$ and $V_i$ vertices of type $i$ a useful formula can be given. For $|k| \to \infty$ the propagator of a particle of spin $s_f$ corresponding to the internal line of type $f$ is related to its spin: it behaves as $|k|^{-2+2s_f}$ where $s_f$ is the spin of the particle. For a spin 1 particle this gives $|k|^0$, but, as we have seen in section 9.6, if the theory is gauge invariant, i.e. spin 1 particles are massless, the propagator effectively behaves as if the spin was 0. This remains true also for massive spin 1 particles provided their masses arise from the Higgs mechanism (which we will discuss in due course). We this caveat the formula for $D$ reads

$$D = \sum_f (2s_f - 2)I_f + \sum_i V_id_i + \sum_f 4I_f - \sum_i 4V_i + 4 , \quad (14.45)$$

where $d_i$ is the number of derivatives in the vertex of type $i$. To justify this formula, we note that each propagator brings in the power $2s_f - 2$ of the momentum (the factor $+ \sum_f (2s_f - 2)I_f$) and each derivative in the vertex brings in one extra power of momentum (the factor $+ \sum_i V_id_i$). On the other hand, for each internal line $f$ there is an integration over its four-momentum $d^4k_f$ which is equivalent to four powers of $|k|$ (the factor $+ \sum_f 4I_f$) but with each vertex there is associated four-momentum conservation delta-function which cancels one integration over $d^4k$ (the factor $- \sum_i 4V_i$). However not all delta functions can be used to eliminate integrations: in connected diagrams, hence, also in 1PI diagrams, there remains one delta-function expressing the overall conservation of the four-momentum, which cannot be used to cancel any of the integrations (the factor $+4$). The formula (14.45) can be rewritten as

$$D = \sum_f 2I_f(s_f + 1) + \sum_i V_i(d_i - 4) + 4 . \quad (14.46)$$

The degree of superficial divergence calculated here depends on the details of the Feynman diagram. It can be, however, simplified by using the obvious topological relation

$$2I_f + E_f = \sum_i V_in_{if} , \quad (14.47)$$

where $E_f$ is the number of external lines of type $f$ and $n_{if}$ is the number of lines of type $f$ attached to the vertex of type $i$. This relation expresses the obvious fact that one end of each of the $E$ external lines and both ends of $I$ internal lines must terminate in a vertex. Taking (14.47) into account we get

$$D = 4 - \sum_f E_f(s_f + 1) - \sum_i V_i\Delta_i , \quad (14.48)$$

13At this point it is sufficient to treat this relation as a “phenomenological” observation.
Figure 14.4: Degrees of superficial divergence of diagrams in a theory with Yukawa and $\phi^4$ interaction.

with

$$\Delta_i = 4 - d_i - \sum_f n_{if}(s_f + 1). \quad (14.49)$$

The factors $\Delta_i$ depend only on the properties of vertices of type $i$ out of which the diagram is built. The important observation is that if in a given quantum field theory model all vertices have $\Delta \geq 0$, then the internal structure of a Feynman diagram is not important for the degree of superficial divergence: $D$ can only be lower or equal to the value determined by the numbers $E_f$ of its external lines. In particular, $D < 0$ if $\sum_f E_f(s_f + 1)$ exceed the critical value 4 (in four space-time dimensions). $\Delta$ factors for typical vertices are as follows:

- $\lambda \phi^4$: $\Delta = 4 - 0 - 4 \times (0 + 1) = 0$,
- $\kappa \phi^3$: $\Delta = 4 - 0 - 3 \times (0 + 1) = 1$,
- $\kappa \phi^5$: $\Delta = 4 - 0 - 5 \times (0 + 1) = -1$,
- $g \phi \bar{\psi}_1 \psi_2$: $\Delta = 4 - 0 - 2 \times (\frac{1}{2} + 1) - 1 \times (0 + 1) = 0$,
- $G[\bar{\psi}_1 \Gamma^A \psi_2] [\bar{\psi}_3 \Gamma^A \psi_4]$: $\Delta = 4 - 0 - 4 \times (\frac{1}{2} + 1) = -2$,
- $g \phi^* (\partial^\mu - \bar{\partial}^\mu) \phi A^\mu$: $\Delta = 4 - 1 - 3 \times (0 + 1) = 0$,
- $\kappa \bar{\psi}_1 \sigma^{\mu
u} \psi_2 F_{\mu\nu}$: $\Delta = 4 - 1 - 2 \times (\frac{1}{2} + 1) - 1 \times (0 + 1) = -1$.

For the $\phi^4$ theory $D = 4 - E_\phi$, whereas in the theory with Yukawa interactions $D = 4 - E_\phi - \frac{3}{2} E_\psi$. Degrees of superficial divergences for some of the types of diagrams in the latter theory are shown in figure 14.4.

Interaction vertices with $\Delta > 0$ are called superrenormalizable, those with $\Delta = 0$ are called renormalizable and those with $\Delta < 0$ - nonrenormalizable.
In the language of statistical physics (in applications of quantum field theory to critical phenomena) these vertices are called \textit{relevant}, \textit{marginal} and \textit{irrelevant}, respectively (for reasons which we will become clear later). If in a theory there is at least one vertex with $\Delta < 0$, the degree of superficial divergence $D$ of any diagram with a fixed number of external lines depends on its internal structure. $D$ can be then made arbitrarily large upon complicating the internal structure of the diagram by inserting in it an appropriate number of vertices with $\Delta < 0$.

The proof of renormalizability of a quantum field theory model consists in showing that all infinities arising in Feynman diagrams (due to integrations over independent four-momenta of internal lines) can be canceled by adding to the initial Lagrangian local counterterms (that is, operators constructed from field operators and their derivatives of finite order). In theories in which all vertices have $\Delta \geq 0$ this is achieved by adding only a finite number of counterterms (that is a finite number of independent operator structures) which can be then interpreted as redefinitions of a finite number of couplings defining the original Lagrangian and field operators\footnote{We assume that the original Lagrangian contains all vertices whose operator composition corresponds to external lines of diagrams with $D \geq 0$.} as in the $\varphi^4$ theory example in (14.7). Such theories are called \textit{renormalizable}. Theories whose Lagrangians contain vertices with $\Delta < 0$ are traditionally called \textit{nonrenormalizable}. To cancel infinities arising in Feynman diagrams in such theories one has to add to their Lagrangians infinitely many independent local operators structures $O_i$. Therefore, Lagrangians of nonrenormalizable theories should from the beginning consist of infinitely many such terms; the counterterms added to remove UV divergences can be then interpreted as redefinitions of the coefficients of successive interaction terms and with bare coefficients split into $c_B^i = c_R^i + \delta c^i$ just as in renormalizable theories. We will not present technical details of the proof of renormalizability of concrete quantum field theory models as it is not necessary in practical calculations. We will only sketch its most important points.

A diagram with $D < 0$ can have infinities arising from subintegrations, that is, from the regions of the loop four-momenta space in which only a subset of (integrated) four-momenta goes to infinity. An example of such a diagram is shown in figure 14.5a. Weinberg showed that a diagram with $D < 0$ is convergent if all subintegrations in it are made convergent by adding to this diagrams other diagrams with counterterms determined in lower orders. One of the diagrams with insertion of the counterterm, necessary to
Figure 14.5: a) Example of a three-loop diagram with $D = -4$ (eight external lines corresponding to spinless particles), containing a subdivergence in the $\varphi^4$ theory. b) Diagram with the counterterm (determined in the two-loop order) removing the overall divergence of the loop integrations enclosed by the solid box in a) (there are also diagrams with counterterms determined at one loop removing subdivergences in the subdiagram enclosed by the solid box).

make finite the diagram 14.5a is shown in figure 14.5b. After all diagrams with counterterms determined in lower orders are added to the considered diagram, the ultraviolet behaviour of their sum is just as dictated by the degree of superficial divergence of the original diagram, hence it is finite if $D < 0$. tu miałem uwagi, że skok myślowy? że z Weinberga diagram o $D > 0$ z kontrczło nami niższego rzędu?

Assume now, that a diagram has $D \geq 0$ but all its subintegrations are already made finite as explained above. By differentiating the expression $I$ corresponding to it with respect to the four-momentum of one of its external lines the degree $D$ of superficial divergence of the integral is lowered by one. For example if the propagator corresponding to an internal line is $i/((k +
\[ p^2 - M^2 + i0 \], where \( p \) is the four-momentum of an external line, then\(^\text{15}\)

\[
\frac{\partial}{\partial p^\mu} \int (\ldots) \frac{i}{(k+p)^2 - M^2 + i0} = - \int (\ldots) \frac{2i(k+p)^\mu}{[(k+p)^2 - M^2 + i0]^2},
\]

Therefore, by differentiating appropriately the expression \( I \) corresponding to the considered Feynman diagram \( D + 1 \) times with respect to four-momenta of its external legs we get a convergent integral. Thus (symbolically),

\[
I = \int I_{\text{finite}}^{(D+1)} + \text{a polynomial of degree} \ D \ \text{in external momenta},
\]

(compare (14.24)) where \( I_{\text{finite}}^{(D+1)} \) is the \( D + 1 \)-th derivative of \( I \), and the coefficients of the polynomial diverge when the cutoff (regularization) is removed (e.g. \( \Lambda \to \infty \) or \( \epsilon \to 0 \)). But a polynomial in external momenta is just what can be canceled by adding to the original Lagrangian \( L \) appropriate local counterterms \( \Delta L \) constructed out of field operators and their derivatives.

If the diagram considered has \( E_f \) external lines of type \( f \) then the needed counterterms in \( \Delta L \) are equivalent to vertices with \( n_{if} = E_f \) and \( d_i \leq D \). If terms of this type were already present in \( L \) then adding such counterterms reduces to redefining original fields and parameters (masses and coupling constants) of \( L \). For the renormalization to work, the original Lagrangian \( L \) must contain, therefore, terms corresponding to all possible superficially divergent Feynman diagrams (i.e. to all graphs having \( D \geq 0 \)).

If the Lagrangian \( L \) has some (global or local) symmetry then of course not all terms in \( \Delta L \) are allowed if the symmetry is not to be spoiled. Fortunately, the symmetry of the original Lagrangian constrains also appropriately the structure of the divergent terms in amplitudes corresponding to Feynman diagrams. (The mathematical expression of the constraints imposed by symmetries are Ward identities - to be discussed in due course - relating various Green’s functions). As a result, the original Lagrangian \( L \) must a priori contain only all terms compatible with the assumed symmetry.\(^\text{16}\)

\(^{15}\)It is easy to see that differentiation with respect to external momenta does not lower the degree of divergence of the diagram 14.5a treated separately. Indeed, the part of the integrand depending on \( p \) - the sum of the momenta entering the diagram in its left lower “corner“ - is

\[
\frac{1}{[(k+p)^2 - M^2]} \frac{1}{[k_1^2 - M^2]} \frac{1}{[(k+k_1+k_2+p)^2 - M^2]} \frac{1}{[k_2^2 - M^2]} \frac{1}{[(k+p)^2 - M^2]},
\]

where \( k \) is the momentum corresponding to the big loop and \( k_1 \) and \( k_2 \) are the independent loop momenta of the enclosed subdiagram. Differentiation with respect to \( p^\mu \) always leaves one term in which the degree of divergence of the enclosed subdiagram is not lowered.

\(^{16}\)The only exception are supersymmetric theories, in which divergences of different diagrams with virtual bosons and fermions “miraculously” cancel each other: in supersymmetric theories one can exclude some (groups of) terms from the Lagrangian and there is no need to include counterterms of the same structure as the excluded terms.
However, if the original Lagrangian $\mathcal{L}$ contains only vertices with $\Delta_i \geq 0$ (i.e. renormalizable and superrenormalizable vertices only) then as we now show, the counterterms needed will also be equivalent to vertices with $\Delta_i \geq 0$ and this “regression ad infinitum” is tamed! Indeed, if all terms in $\mathcal{L}$ have $\Delta_i \geq 0$ then the degree of superficial divergence of any diagram is

$$D \leq 4 - \sum_f E_f (s_f + 1). \quad (14.50)$$

But as we have shown, the counterterms needed are equivalent to vertices with $n_{if} = E_f$ and $d_i \leq D$ so that

$$0 \leq d_i \leq D \leq 4 - \sum_f E_f (s_f + 1) = 4 - \sum_f n_{if} (s_f + 1). \quad (14.51)$$

Hence, $d_i \leq 4 - \sum_f n_{if} (s_f + 1)$ and all counterterm vertices which have to be added as $\Delta \mathcal{L}$ have

$$\Delta_i \equiv 4 - d_i - \sum_f n_{if} (s_f + 1) \geq 0, \quad (14.52)$$

i.e. are also only of renormalizable and superrenormalizable type. The theory is then of the renormalizable type. If at least one vertex with $\Delta_i < 0$ is added to $\mathcal{L}$ then Feynman diagrams having arbitrarily high $D$ can be constructed (for any fixed number of the external legs $E_f$) and, consequently, counterterm vertices with arbitrarily negative $\Delta_i$ have to be included in $\Delta \mathcal{L}$; the theory becomes of the nonrenormalizable type.

To illustrate some of these points let us consider a theory of spinless and spin $1/2$ particles with the interaction Lagrangian given by

$$\mathcal{L}_{\text{int}} = -g \varphi \bar{\psi} \psi. \quad (14.53)$$

With this interaction alone, this is not yet a renormalizable theory because the $D = 0$ diagram shown in figure 14.6a requires a counterterm equivalent to the $\lambda \varphi^4$ vertex. Similarly, a diagram with fermionic loop and only three legs of the scalar field attached$^{17}$ requires a counterterm equivalent to the $\kappa \varphi^3$ vertex. The theory can however be made renormalizable in a trivial way, just by including such interactions in $\mathcal{L}_{\text{int}}$ from the beginning

$$\mathcal{L}_{\text{int}} = -g \varphi \bar{\psi} \psi - \frac{1}{3!} \kappa \varphi^3 - \frac{1}{4!} \lambda \varphi^4. \quad (14.54)$$

The added interactions have also $\Delta_i \geq 0$.

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$^{17}$Such a diagram has $D = 1$ but Lorentz invariance reduces its degree of superficial divergence to $D = 0$. 

617
We have already mentioned that the ultraviolet behaviour of a diagram (with more than one loop) is as dictated by its degree of superficial divergence $D$ provided the diagram is considered together with the set of other diagrams with insertions of counterterms determined in lowest orders. This is one of the most difficult points in the rigorous proof of renormalizability. It is complicated by the occurrence of the so-called overlapping divergences. For example, for the interaction (14.53) the two loop $D = 2$ diagram shown in figure 14.6b has two divergent subgraphs (each with $D = 0$). These two subgraphs have one common line, so the divergent integrals are not quite independent. A full renormalization procedure must specify a recipe for treating such overlapping divergences. There exists such a complete procedure, called the the BPHZ (Bogoliubov, Parasiuk, Hepp, Zimmerman) procedure, allowing to treat renormalization systematically. It is based on the so-called Zimmerman “forest” formula which is rather complicated. In practical calculations performed to a finite order in perturbation calculus, it is not necessary to use the formal BPHZ machinery. It is just sufficient to systematically include counterterms determined in lower orders: the subdivergences are then removed automatically.

Let us see how this works on the example of figure 14.6b. It gives

$$-i \Sigma^{(b)}(p) = (-i)^F (-ig)^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{i}{(k-k')^2 - M^2} \times \text{tr} [iS^F(k+p)iS^F(k)iS^F(k')iS^F(k' + p)] .$$  \hspace{1cm} (14.55)

At one loop the result for the renormalized 1PI three point function (with two fermion and one scalar legs) is given by

$$(-ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2} iS^F(k)iS^F(k) + \text{finite part} - i\Delta g ,$$  \hspace{1cm} (14.56)

where “finite part” is the difference of the loop expression for the vertex with nonvanishing external line momenta and the same expression (which has been
explicitly factorized above) for the momenta of the external lines set to zero.

The contribution \(-i\Delta g\) is generated by the counterterm Lagrangian

\[
\Delta \mathcal{L} \supset - (Z_{\psi} Z^{1/2}_\varphi (g + \delta g) - g) \varphi \bar{\psi} \psi \equiv - \varphi \bar{\psi} (\Delta g) \psi \quad (14.57)
\]

(it is convenient to treat here \(\Delta g\) as a matrix in spinor indices). Thus, to one loop accuracy

\[
-i (\Delta g)^{(1)} = - (ig)^3 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2} i S^F(k) i S^F(k) - (ig)^3 R_{\text{finite}}. \quad (14.58)
\]

The first term in \(-i (\Delta g)^{(1)}\) cancels the divergent part of (14.56) and the second one, \((-ig)^3 R_{\text{finite}}\), is a finite piece which depends on the renormalization condition imposed on the renormalized 1PI three-point function. Since the three-point vertex diagram has \(D = 0\), \(R_{\text{finite}}\) does not depend on the momenta of the external lines. The crossed circles in diagrams 14.6c and 14.6d stand just for vertices \(-i (\Delta g)\) (with \(-i (\Delta g)\) approximated by \(-i (\Delta g)^{(1)}\)), these two diagrams are then formally of the two-loop order just as the diagram 14.6b). Diagrams 14.6c and d are generated automatically in the perturbative expansion. The sum of diagrams 14.6b, c and d therefore reads

\[
-i \Sigma^{(bcd)}(p) = (-)^F (-ig)^4 \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 k'}{(2\pi)^4} \left\{ \frac{i}{(k - k')^2 - M^2} \text{tr} \left[ i S^F(k + p) i S^F(k) i S^F(k') i S^F(k + p) \right] \right. \\
- \frac{i}{k^2 - M^2} \text{tr} \left[ i S^F(k) i S^F(k) i S^F(k') i S^F(k') + p \right] \\
- \frac{i}{k'^2 - M^2} \text{tr} \left[ i S^F(k + p) i S^F(k) i S^F(k') i S^F(k') \right] \left\} - 2 (-)^F (-ig)^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ R_{\text{finite}} i S^F(k + p) i S^F(k) \right].
\]

Consider now the integration over \(d^4 k\): the logarithmic divergence in the first line is canceled by the logarithmic divergence of the second line. The third line is divergent but its divergence will be a polynomial of first order in \(p^2\) (the integration over \(d^4 k'\) will yield an infinite term independent of \(p^2\) multiplying the result of integration of the third term over \(d^4 k\)). Similar argument applies to integration over \(d^4 k'\). The term in the last line is quadratically divergent but, since \(R_{\text{finite}}\) is independent of the momenta, this divergence takes also the form of a polynomial of first order in \(p^2\). Thus, the divergence of the whole integration over \(d^4 k d^4 k'\) will be of the form \(a_{\text{div}} p^2 + b_{\text{div}}\) and can be canceled by appropriately adjusting \(\delta Z_{\varphi}^{(2)}\) and \(\delta M_{(2)}^2\) in \(\Delta \mathcal{L}\).
14.6 Renormalizable and nonrenormalizable theories

The renormalizability condition, $\Delta_i = 4 - d_i - \sum_f n_{if}(s_f + 1) \geq 0$ for all interactions in $\mathcal{H}_{\text{int}}$ is quite restrictive. For a given set of particles (types of field operators) only a limited number of terms in $\mathcal{H}_{\text{int}}$ can satisfy it. In particular, for particles of spin $s_f = 1$ no renormalizable interaction terms could be written down were it not for gauge invariance which effectively reduces $s_f$ for spin 1 gauge bosons from 1 to 0. Additional symmetries enforcing cancelation of infinities from different Feynman diagrams can improve ultraviolet behaviour of the amplitudes even more. The most striking is the case of supersymmetric theories in which to each fermion field there is a superpartner bosonic field and vice versa and in which supersymmetry ensures some specific relations between coupling constants of different interaction terms; as a result, in diagrams some divergences due to bosonic loops are canceled by fermionic loops owing to their extra minus sign. Because of such cancellations, supersymmetric theories exhibit milder ultraviolet behaviour than would follow from considering the degrees of superficial divergence of individual Feynman graphs. In particular, $N = 4$ supersymmetric gauge theory (with 4 Noether charges of supersymmetry) is completely finite.

As is easy to check, for spin $3/2$ particles naive power counting (i.e. analysis based on degrees of superficial divergences $D$) indicates that no renormalizable interactions can exist. Only a symmetry could help in this case. It was found, that the simplest version of the local supersymmetry realized in supergravity theories (in which spin $3/2$ particles are related by supersymmetry transformations to the graviton states) with $N = 1$ Noether-like supersymmetric charge (the supercharge) is not sufficient to make the theory renormalizable. However supergravity theory with $N = 8$ in four space-time dimensions still offers hopes for a completely finite quantum field theory of gravitons and spin $3/2$ gravitinos!

It is interesting also to ask whether renormalizability is really indispensable feature of quantum field theory models. Until relatively recently the prevailing view was that physically only renormalizable theories can make sense. The requirement of renormalizability played important role in formulating the Standard Model of elementary interactions.

Today we see the issue differently. All quantum field theories are regarded as effective theories whose applicability to description of physical phenomena is limited to a finite range of energies. Because of this, probably no

\footnote{At first sight, for a spin 1 particle the maximal value of $\Delta_i$ is $4 - 0 - 2(0+1) - 1(1+1) = 0$ (minimal number of derivatives is 0, any genuine interaction vertex must be built from at least fields - here 2 scalars and 1 vector), but the Lorentz index of the vector field describing spin 1 particle cannot then get contracted; hence in fact maximal $\Delta_i < 0$.}
quantum field theory can be considered a closed mathematical construction. Lagrangian of any model describing real physics contains in fact infinitely many terms with negative factors $\Delta$. The coefficients (coupling constants) of nonrenormalizable terms have negative dimensions (in units of mass) and must therefore be proportional to inverse powers of some mass scale $M$. It is natural to identify this mass scale $M$ with a mass scale characteristic for a more fundamental theory (not known yet), that is, with some scale above which new physics intervenes. If a vertex $i$ has $\Delta_i < 0$ its coupling constant $g_i$ is

$$g_i \propto 1/M^{-\Delta_i}.$$ (14.59)

For processes in which all participating particles have masses much smaller than $M$, and whose energies are bounded by some characteristic energy scale $E$ much lower than the mass scale $M$, nonrenormalizable vertices usually can be neglected: after renormalization contributions of Feynman diagrams containing $n_i$ vertices with $\Delta_i < 0$ and couplings $g_i$ to amplitudes of such processes are suppressed with respect to contributions of diagrams without $\Delta_i < 0$ vertices by a factor $(E/M)^a \ll 1$ where $a = -\sum_i n_i \Delta_i \geq 1$. Renormalizable models describing physics up to some energy scale $M$ are singled out from all possible nonrenormalizable theories only if the next physical scale $M$ (at which new physics enters) is much higher than the energy range probed experimentally. Thus, the apparent success of the Standard Model is most naturally explained if we assume that the physical scale (next to the electroweak one) is quite high. Recent discovery that neutrinos are not massless is perhaps the first experimental manifestation of the interactions suppressed by $1/M$ which have to be added to the SM Lagrangian. The scale $M$ suggested by the values of the neutrino masses (plus some theoretical assumptions) turns out to be quite high: $M \sim 10^{12-14}$ GeV. In any case, once one accepts that the effective theory Lagrangian contains infinitely many terms (constrained only by some symmetries) nonrenormalizable in the traditional sense models become renormalizable just as renormalizable ones.

Nonrenormalizable $\Delta_i < 0$ interactions can become important at energies $E \ll M$ if they give rise to processes forbidden at the renormalizable level (i.e. when no Feynman diagram contributing to the amplitude of this process and consisting of only $\Delta_i \geq 0$ vertices can be drawn). This is for example the case of weak interactions probed at energies $\sim$ a few GeV: the renormalizable theory describing physical processes at energies $\ll 100$ GeV is quantum chromodynamics (QCD) plus quantum electrodynamics (QED). Interactions of this renormalizable (in the traditional sense) theory all conserve separately the numbers\textsuperscript{19} of muons, electrons, quarks of a given type, i.e. conserve the fermion flavour. Weak interactions are described at such

\textsuperscript{19}Counting antiparticle as minus one particle, of course.
low energies by the 4-Fermi “current×current” interaction (known since the thirties of the XX century - see Section 12) which has \( \Delta = -2 \) and some other operators, some of which have \( \Delta = -1 \). Because they give rise to processes like \( \mu \) decay, \( \bar{B} \to X_s \gamma \) (\( \bar{B} \) is a meson containing the \( b \) quark and \( X_s \) is any hadronic state containing a single quark \( s \)), etc. which are forbidden in QCD+QED, their presence in the Lagrangian of the effective low energy theory can be experimentally detected.

Sometimes a symmetry forbids renormalizable vertices altogether. To show that in this case perturbation calculations can nevertheless be organized in a sensible way (that is, that a small parameter of the expansion can be identified) let us consider a theory of interacting spin zero particles assuming that the Lagrangian is invariant under arbitrary shifts of their scalar field \( \varphi \to \varphi + \kappa \). It is then clear that \( \mathcal{L}(\varphi) = \tilde{\mathcal{L}}(\partial \varphi, \partial \partial \varphi, \ldots) \). The most general such Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \partial \varphi \partial \varphi + \frac{g_1}{4} \partial \varphi \partial \varphi \partial \varphi \partial \varphi + \frac{g_2}{4} \partial^2 \varphi \partial^2 \varphi \partial \varphi \partial \varphi + \frac{g_2}{24} \partial \varphi \partial \varphi \partial \varphi \partial \varphi \partial \varphi \partial \varphi \partial \varphi + \ldots
\]

where in fact each term (apart from the first one) stands for a collection of terms (with different coupling constants) with the same number of fields and the same number of differently contracted derivatives and the dots denote terms with more fields and/or more derivatives. Since contributions of all diagrams to the amplitude of a given process must have the same physical dimension, contributions of different diagrams must differ by factors \( \left( \frac{k}{M} \right)^\nu \) where \( k \) is a characteristic momentum scale of the considered process, and \( M \) is the overall mass scale of all coupling constants \( g_i \) as in (14.59). A diagram with \( V_i \) vertices of type \( i \) has therefore\(^{20}\)

\[
\nu = - \sum_i V_i \Delta_i = \sum_i V_i (d_i + n_i - 4)
\]  

(14.60)

(since there is only one field \( \varphi \), \( n_i f \equiv n_i \)). The topological relation (14.47) and another one \(^{21}\)

\[
\sum_i V_i = I - L + 1,
\]

(14.61)

allow to rewrite (14.60) in the form

\[
\nu = 2E - 4 + 4L + \sum_i V_i (d_i - n_i),
\]

\(^{20}\)Just by counting the powers of \( M \) introduced by the vertices.

\(^{21}\)This follows from counting the number of independent integrations over momenta in a connected Feynman diagram: \( L \) - the number of loops - is just a number of independent integrals left after using up all four-momentum delta functions associated with each of the \( V_i \) vertices.
where $d_i - n_i > 0$ because of the assumed shift symmetry. For low energy processes with $k/M \ll 1$ this allows to organize the calculated amplitudes in power series in the suppression factor $k/M$: In the first order to the amplitude contribute diagrams with no loops and vertices with lowest number of derivatives; in the next order contribute one loop diagrams constructed from vertices with lowest number of derivatives and tree level diagrams with one vertex with next-to lowest number of derivatives and so on. It is easy to see that at a given order only a finite number of terms in $\mathcal{L}$ contributes. This way of organizing perturbation series allows to make sense out of non-renormalizable theories. The most important example here is the low energy chiral effective theory of strong interactions of pions - see section ??.

14.7 Exercise in renormalization at one-loop

To get some experience in renormalization we consider now the example of the Yukawa theory describing interactions of a fermion $f$ and its antifermion $\bar{f}$ with the spinless particle $S$. This theory, unlike the $\phi^4$ one, is already sufficiently complicated to allow for illustration of some important aspects of the renormalization procedure.

The Lagrangian with the counterterms reads

$$\mathcal{L} = Z_{\psi} \bar{\psi} \psi - Z_{\psi}(m + \delta m) \bar{\psi} \psi + \frac{1}{2} Z_{\phi} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} Z_{\phi}(M^2 + \delta M^2) \phi^2$$

$$- Z_{\phi} Z_{\phi}^{1/2}(h + \delta h) i \bar{\psi} \gamma^5 \psi \phi - \frac{1}{4!} Z_{\phi}^2 (\lambda + \delta \lambda) \phi^4.$$  

We have dropped the subscript R (renormalized) from the parameters. The Feynman rules following from this Lagrangian are given in figure 14.2 (with $M^2_R$ there renamed to $M^2$, and $Z$ by $Z_{\phi}$) and in figure 14.7.

Up to one loop order the Feynman rules for counterterms given in figures 14.2 and 14.7 can be written as $i \delta Z_{\phi} \bar{\psi} \psi - i (M^2 \delta Z_{\phi} + \delta M^2)$ for the counterterm.
to the scalar particle propagator, \( i\delta Z_\psi \not p - i(m\delta Z_\psi + \delta m) \) for the counterterm to the fermion propagator, \( -i(2\delta Z_\varphi \lambda + \delta \lambda) \) for the counterterm to the scalar self interaction and \( (\delta Z_\psi + \frac{1}{2}\delta Z_\varphi)h + \delta h \) for the counterterm to the Yukawa interaction.

Still without specifying the renormalization scheme (the renormalization conditions) we can write the expressions for renormalized propagators obtained after summing the geometric series corresponding to the blobs shown in figure 13.3. For the scalar particle we have

\[
\frac{i}{p^2 - M^2 - \Sigma^\varphi_R(k^2)}, \quad (14.63)
\]

where \( \Sigma^\varphi_R(k^2) = \Sigma^\varphi(k^2) - \delta Z_\varphi p^2 + (M^2\delta Z_\varphi + \delta M^2) \). Expanding the denominator of (14.63) around its zero at \( k^2 = M^2_{ph} \) where \( M^2_{ph} \) is determined by the solution of the equation\(^{22}\)

\[
M^2_{ph} = M^2 + \Sigma^\varphi_R(k^2 = M^2_{ph}, M^2, m, \lambda, h), \quad (14.64)
\]

we find (prime denotes differentiation with respect to \( k^2 \))

\[
Z_\varphi = [1 - \Sigma^\varphi_R'(k^2 = M^2_{ph}, M^2, m, \lambda, h)]^{-1} 

\approx 1 + \Sigma^\varphi_R(k^2 = M^2_{ph}, M^2, m, \lambda, h). \quad (14.65)
\]

For the series defining the full fermion propagator we get\(^{23}\)

\[
\frac{i}{\not p - m} + \frac{i}{\not p - m}(-i\Sigma^\psi_R) \frac{i}{\not p - m} + \frac{i}{\not p - m}(-i\Sigma^\psi_R) \frac{i}{\not p - m}(-i\Sigma^\psi_R) \frac{i}{\not p - m} + \ldots 

= \frac{i}{\not p - M - \Sigma^\psi_R(\not p)}, \quad (14.66)
\]

where \( \Sigma^\psi_R(\not p) = \Sigma^\psi(\not p) - \delta Z_\psi \not p + (m\delta Z_\psi + \delta m) \). We have written \( \not p \) as the argument of \( \Sigma^\psi_R \) to underline its matrix structure:

\[
\Sigma^\psi_R(\not p) = \Sigma^V_R(p^2) \not p + \Sigma^m_R(p^2). \quad (14.67)
\]

\(^{22}\)If the spinless particle \( S \) corresponding to the field \( \varphi \) is unstable, \( \Sigma^\varphi_R(k^2) \) is complex (\( \delta M^2 \) and the field renormalization \( Z_\varphi \) must be real for Hermiticity of the Lagrangian) and definition of the pole is not obvious. Strictly speaking, the particle \( S \) is only a resonance which can reveal its presence in e.g. the \( f\bar{f} \rightarrow f\bar{f} \) amplitude, but has no corresponding \( in \) and \( out \) states.

\(^{23}\)The summation is performed by using the formula

\[
(A - B)^{-1} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \ldots
\]

where \( A \) and \( B \) are matrices. To check this formula, it suffices to multiply its both sides from the right by \( A - B \).
Near the pole at $p^2 = m_{ph}^2$ the full propagator (14.66) should take the form

$$
\lim_{p^2 \to m_{ph}^2} \frac{i}{\not{p} - m - \Sigma^R_R(p)} = \frac{i Z_\psi}{\not{p} - m_{ph}}.
$$

(14.68)

To find $m_{ph}$ and the factor $Z_\psi$ using (14.67) we write

$$
\frac{i}{\not{p} - m - \Sigma^R_R(p)} = i \frac{(1 - \Sigma^V_R) \not{p} + M + \Sigma^m_R}{p^2 (1 - \Sigma^V_R)^2 - (m + \Sigma^m_R)^2}.
$$

(14.69)

The pole is at $p^2 = (m + \Sigma^m_R)^2/(1 - \Sigma^V_R)^2$, so that, in full generality, $m_{ph}$ is the solution of the equation

$$
m_{ph} = \frac{m + \Sigma^m_R(p^2 = m_{ph}^2, m, M^2, \lambda, \hbar)}{1 - \Sigma^V_R(p^2 = m_{ph}^2, m, M^2, \lambda, \hbar)}.
$$

(14.70)

The factor $Z_\psi$ can be extracted by expanding the denominator of (14.69) around $p^2 = m_{ph}^2$ and by looking at the coefficient of $\not{p}$ in the numerator:

$$
Z_\psi = 1 - \frac{\Sigma^V_R}{(1 - \Sigma^V_R)^2 - 2m_{ph}^2 \Sigma^V_R(1 - \Sigma^V_R) - 2(m + \Sigma^m_R)\Sigma^m_R} = \frac{1}{1 - \Sigma^V_R - 2m_{ph}^2 \Sigma^V_R(1 - \Sigma^V_R) - 2m_{ph} \Sigma^m_R},
$$

(14.71)

where the arguments of the self energy functions and of their derivatives with respect to $p^2$ are as in (14.70). (To obtain the second line we have divided the numerator and the denominator by $1 - \Sigma^V_R$ and then in the last term of the denominator we have used the equation (14.70) to get $m_{ph}$.)

It is instructive to recover the same results by using the effective field theory technique introduced at the end of subsection 13.5. This allows for easy treatment of the most general case of fermionic self energies in theories in which left and right-chiral fermions couple differently to other fields. In such cases the decomposition (14.67) has to be generalized to

$$
\Sigma^R_R(\not{p}) = p_L \Sigma^V_L(p^2) + p_R \Sigma^V_R(p^2) + p_L \Sigma^m_L(p^2) + p_R \Sigma^m_R(p^2).
$$

(14.72)

When expanded (using $\partial f(p^2)/\partial \not{p} = 2 \not{p}[\partial f(p^2)/\partial p^2]$) around $\not{p} = m_{ph}$ this is

$$
\Sigma^R_R(\not{p}) = \left\{ m_{ph} \Sigma^V_L(m_{ph}^2) + \Sigma^m_L(m_{ph}^2) \\
+ \left[ \Sigma^V_L(m_{ph}^2) + 2m_{ph}^2 \Sigma^V_R(m_{ph}^2) + 2m_{ph} \Sigma^m_R(m_{ph}^2) \right] (\not{p} - m_{ph}) \right\} p_L
+ (L \to R).
$$

625
This has to be reproduced by the effective Lagrangian

\[
L_{\text{eff}} = (1 + \delta z_L^\phi) \bar{\psi} P_L \psi - \Delta m_L \bar{\psi} P_L \psi + (1 + \delta z_R^\psi) \bar{\psi} P_R \psi - \Delta m_R \bar{\psi} P_R \psi + \text{interaction terms.}
\] (14.73)

The factors \(\delta z_L^\psi, \delta z_R^\psi, \Delta m_L, \Delta m_R\) are not counterterms. They must be chosen in such a way that the tree level diagram computed by using terms with \(\delta z_L^\psi\) and \(\delta z_R^\psi\) in (14.73) as interactions reproduces for \(\not{p} \to m_{\text{ph}}\) the full 1PI Green’s function \(\tilde{\Gamma}^{(2)}\):

\[
i\delta z_L^\psi \not{p} P_L - i(\Delta m_L - m)P_L + (L \to R) = -i \Sigma_L^\psi(p) .
\] (14.74)

This gives

\[
\delta z_L^\psi = - \left[ \Sigma_{VL}^R(m_{\text{ph}}^2) + 2m_{\text{ph}}^2 \Sigma_{VL}^L(m_{\text{ph}}^2) + 2m_{\text{ph}} \Sigma_{VL}^L(m_{\text{ph}}^2) \right] .
\] (14.75)

To bring the kinetic term of the effective Lagrangian (14.73) to the canonical form the fields \(P_{L,R}\bar{\psi}\) and \(\bar{\psi} P_{R,L}\) have to be rescaled by the factor \((1 + \delta z_L^\psi)^{-1/2}\). The other terms of the effective Lagrangian will be then multiplied by \((1 + \delta z_L^\psi)^{-1/2}\) per each \(P_{L,R}\bar{\psi}\) and \(\bar{\psi} P_{R,L}\) appearing in the vertex. Hence, the \(Z_{L,R}^\psi\) are just given by \(1/(1 + \delta z_L^\psi)\) in agreement with (14.71). For \(\Delta m_{L,R} - m\) we find

\[
\Delta m_L - m = \Sigma_{RL}^L(m_{\text{ph}}^2) - 2m_{\text{ph}}^3 \Sigma_{RL}^L(m_{\text{ph}}^2) - 2m_{\text{ph}}^2 \Sigma_{RL}^L(m_{\text{ph}}^2) ;
\]
and similarly for \(\Delta m_R - m\). After rescaling the fields \(P_{L,R}\bar{\psi}\) and \(\bar{\psi} P_{R,L}\) the physical mass is then\textsuperscript{24}

\[
m_{\text{ph}} = \frac{1 + \delta z_L^\psi}{\Delta m_L} = \frac{1 + \Sigma_{RL}^L - 2m_{\text{ph}}^3 \Sigma_{RL}^L - 2m_{\text{ph}}^2 \Sigma_{RL}^L}{1 - \Sigma_{RL}^L - 2m_{\text{ph}}^3 \Sigma_{RL}^L - 2m_{\text{ph}}^2 \Sigma_{RL}^L} .
\]

This looks more complicated than (14.70) but in fact is equivalent to it, as can be easily checked. Thus, we have recovered previous results in a way which is applicable also to chiral fermions.

In the one-loop approximation these results simplify to

\[
m_{\text{ph}} \approx m + \Sigma_{R}^L + m \Sigma_{R}^L ;
\]

\[
\mathcal{Z}_\psi \approx 1 + \Sigma_{R}^L + 2m^2 \Sigma_{R}^L + 2m \Sigma_{R}^L ,
\] (14.76)

\textsuperscript{24}The physical masses \(m_{\text{ph}}\) obtained from \(\Delta m_L/(1 + \delta z_L^\psi)\) and from \(\Delta m_R/(1 + \delta z_R^\psi)\) must be equal up to a complex conjugation, if the fermions are to preserve their Dirac character (and not be split by radiative corrections into a pair of Majorana fermions of different masses); if they are complex and have opposite phases, additional chiral rotations are required to make the mass \(m_{\text{ph}}\) real.
where in this approximation \( p^2 = m^2 \) and \( m_{\text{ph}} = m \) in the one-loop terms.

**Tu nawiazać do LSZ**

Our aim in this section is to show on an explicit example that theoretical predictions for physical (measurable) quantities are independent of the renormalization scheme used, up to terms which are of higher order (than the order to which we are computing them) provided the results are expressed in terms of (the same for all renormalization schemes) other physical quantities. We will compare here the On-Shell and the MS schemes described in subsection 14.2. Assuming that \( M_{\text{ph}} > 2m_{\text{ph}} \) we will use in both schemes the decay \( S \to f \bar{f} \) of the scalar into the fermion-antifermion pair to define the physical Yukawa coupling \( h_{\text{ph}} \). At one-loop the amplitude of this decay is given by

\[
-i M(S \to f \bar{f}) = Z_\psi Z_{\varphi}^{1/2} \bar{u} \left\{ h \gamma^5 + h \gamma^5 \Delta \Gamma_R^{(3)} \left( p_1^2 = m^2, p_2^2 = m^2, p_1 \cdot p_2 = \frac{1}{2} M^2 - m^2 \right) \right\} v,
\]

(in the one-loop term we approximate \( M_{\text{ph}} \) by \( M \) and \( m_{\text{ph}} \) by \( m \)) where we have factorized \( h \gamma^5 \) from the vertex correction and

\[
\Delta \Gamma_R^{(3)} = \Delta \Gamma^{(3)} + \delta Z_\psi + \frac{1}{2} \delta Z_{\varphi} + \frac{\delta h}{h}.
\]

(14.77)

We now specify the formulae to the two chosen renormalization schemes.

In the On-Shell scheme we impose the following conditions

\[
M^2 = M_{\text{ph}}^2, \quad Z_\varphi = 1, \quad m = m_{\text{ph}}, \quad Z_\psi = 1,
\]

(14.78)

In addition we require that the renormalized coupling constant \( h \) equals the physical coupling defined through the \( S \to f \bar{f} \) decay amplitude. This means that we require that

\[
\Delta \Gamma_R^{(3)} \left( p_1^2 = m^2, p_2^2 = m^2, p_1 \cdot p_2 = \frac{1}{2} M^2 - m^2 \right) = 0.
\]

(14.79)

The above conditions determine the field renormalization constants \( Z_\varphi, Z_\psi \) and the factors \( \delta M^2, \delta m, \delta h \)

\[
\delta Z_\varphi = \Sigma_{\varphi}(p^2 = M^2, M, m), \quad \delta Z_\psi = \Sigma + 2m_2 \Sigma + 2m \Sigma m', \quad \delta M^2 = -\Sigma(0^2 = M^2, M, m), \quad \delta m = -(m \Sigma + \Sigma m),
\]

(14.80)

(the momentum arguments are fixed to \( M_{\text{ph}}^2 \) or \( m_{\text{ph}} \)) and

\[
\frac{\delta h}{h} = -\Delta \Gamma_R^{(3)} - \delta Z_\psi - \frac{1}{2} \delta Z_\varphi.
\]

(14.81)
Figure 14.8: Diagrams for elastic $Sf \rightarrow Sf$ scattering in the one-loop approximation. For all diagrams except the last one there is a “crossed diagram” with the two scalar lines interchanged.

where $\Delta \Gamma^{(3)}_{\ast} \equiv \Delta \Gamma^{(3)}(p_1^2 = m^2, p_2^2 = m^2, p_1 \cdot p_2 = \frac{1}{2}M^2 - m^2)$.

We will now compute at one loop approximation the amplitude for the elastic $Sf \rightarrow Sf$ scattering expressing it in terms of physical masses and physical coupling $h_{ph}$. The diagrams to be computed are shown in figure 14.8.

Since in the On-Shell scheme $Z_\psi = Z_\varphi = 1$ the elastic scattering amplitude takes in this scheme the form

$$-iM^{OS} = h_{ph}^2 \bar{u} \gamma^5 \frac{i}{q - m_{ph}} \gamma^5 u + h_{ph}^2 \bar{u} \gamma^5 \frac{i}{q - m_{ph}} (-i\Sigma^\psi_{\ast}) \frac{i}{q - m_{ph}} \gamma^5 u$$

$$+ h_{ph}^2 \bar{u} \gamma^5 \frac{i}{q - m_{ph}} \Delta \Gamma_{R}^{(3)} \gamma^5 u + h_{ph}^2 \bar{u} \gamma^5 \Delta \Gamma_{R}^{(3)} \frac{i}{q - m_{ph}} \gamma^5 u + \ldots$$

The ellipses stand for the crossed diagrams which form a separate class and the whole calculation can be repeated for them, and for the remaining diagrams which are finite and (to one loop accuracy) are the same an both schemes. We do not write explicitly momentum arguments of $\Sigma^\psi_{R}$ or $\Delta \Gamma_{R}^{(3)}$. In terms of bare (i.e. unsubtracted, that is unrenormalized) self-energy functions and $\Delta \Gamma^{(3)}$ this is

$$-iM^{OS} = h_{ph}^2 \bar{u} \gamma^5 \frac{i}{q - m_{ph}} \gamma^5 u + h_{ph}^2 \bar{u} \gamma^5 \frac{i}{q - m_{ph}} (-i) \left\{ \Sigma^\psi \\ - (\Sigma^V + 2m_{ph}^2 \Sigma^V') - (m_{ph} \Sigma^V + \Sigma^m) \right\} \frac{i}{q - m_{ph}} \gamma^5 u$$

$$+ \left( m_{ph} \Sigma^V + 2m_{ph}^3 \Sigma^V' + 2m_{ph}^2 \Sigma^m \right) \frac{i}{q - m_{ph}} \gamma^5 u$$
\[ +h_{ph}^2 \bar{u} \gamma^5 \frac{i}{\not{q} - m_{ph}} \left\{ \Delta \Gamma^{(3)} - \Delta \Gamma_*^{(3)} \right\} \gamma^5 u \]  
(14.82)

where we have denoted by an asterisk (*) the functions for fixed kinematical variables (as they follow from the renormalization conditions (14.80) and (14.81)).

We now want to compare this with the amplitude of elastic \( S f \rightarrow S f \) scattering computed in the MS scheme. In the MS scheme adding counterterms to the self energy \( \Sigma^\psi \) and to the three point 1PI vertex function \( \Delta \Gamma^{(3)} \) amounts to removing only their pole parts. As usually we will denote these quantities renormalized in the MS scheme by a hat: \( \hat{\Sigma}^\psi \) and \( \hat{\Delta} \Gamma^{(3)} \) (similarly as the renormalized parameters: \( \hat{h}, \hat{m} \) etc.). It is however important to remember that they differ from the bare \( \Sigma^\psi \) and \( \Delta \Gamma^{(3)} \) only by the absence of the \( 1/\epsilon \) terms. Since in the expression (14.82) for \( -i M^{\text{OS}} \) the infinities cancel out, one can simply replace there \( \Sigma^\psi \) and \( \Delta \Gamma^{(3)} \) by \( \hat{\Sigma}^\psi \) and \( \hat{\Delta} \Gamma^{(3)} \).

Since in the MS scheme \( Z_\psi \neq 1 \), \( Z_\varphi \neq 1 \) we have

\[ -i M^{\text{MS}} = Z_\varphi Z_\psi \left\{ \begin{array}{l}
h^2 \bar{u} \gamma^5 \frac{i}{\not{q} - \hat{m}} \gamma^5 u + h^2 \bar{u} \gamma^5 \frac{i}{\not{q} - \hat{m}} (-i \hat{\Sigma}^\psi) \frac{i}{\not{q} - \hat{m}} \gamma^5 u \\
+ \hat{h}^2 \bar{u} \gamma^5 \frac{i}{\not{q} - \hat{m}} \hat{\Delta} \Gamma^{(3)} \gamma^5 u + \hat{h}^2 \bar{u} \gamma^5 \hat{\Delta} \Gamma^{(3)} \frac{i}{\not{q} - \hat{m}} \gamma^5 u + \ldots \end{array} \right\}. \]

With one loop accuracy, in all terms but the first one the renormalized parameters \( \hat{m}, \hat{h} \) can be replaced by the physical ones: \( m_{ph} \) and \( h_{ph} \). The difference between \( -i M^{\text{MS}} \) and \( -i M^{\text{OS}} \) stems therefore from: \( i) \) the presence of the \( Z \) factors, \( ii) \) the presence of \( \hat{m} \) and \( \hat{h} \) instead of \( m_{ph} \) and \( h_{ph} \) in the first term, \( iii) \) absence of some terms in the one loop part of \( -i M^{\text{MS}} \).

To one loop accuracy the factors \( Z_\varphi \) and \( Z_\psi \) are given in eqs (14.65) and (14.76), respectively.\(^{25}\) The one loop relation between \( m_{ph} \) and \( \hat{m} \) is also given in (14.76). To relate and \( \hat{h} \) to \( h_{ph} \) we have to compute in the MS scheme the amplitude of the \( \varphi \rightarrow f \bar{f} \) decay which defines \( h_{ph} \):

\[ -i M^{\text{MS}} (\varphi \rightarrow f \bar{f}) = Z_{\varphi}^{1/2} Z_\psi \hat{h} \bar{u} \gamma^5 \left( 1 + \hat{\Delta} \Gamma_*^{(3)} \right) v \]

\[ = \hat{h} \bar{u} \gamma^5 \left( 1 + \hat{\Delta} \Gamma_*^{(3)} + \frac{1}{2} \hat{\Sigma}^\varphi + \hat{\Sigma}^\psi + 2 m_{ph}^2 \hat{\Sigma}^\psi V + 2 m_{ph} \hat{\Sigma}^\psi V' \right) v, \]

\(^{25}\)In the notation used here the self energies appearing in these formulae are denoted with the hat (because they are renormalized in the MS scheme) and by an asterisk (*).
Hence, to one loop accuracy, the required relation reads

\[ \hat{h} = h_{ph} \left( 1 - \Delta \Gamma^{(3)}_* - \frac{1}{2} \hat{\Sigma}_{s}^\phi - \hat{\Sigma}_{s}^V - 2m_{ph}^2 \hat{\Sigma}_{s}^{V'} - 2m_{ph} \hat{\Sigma}_{s}^{m'} \right). \]  \hspace{1cm} (14.83)

It remains to work out the first term in the expression for \(-iM^{MS}\). To one loop accuracy, inserting there expressions (14.83) for \(\hat{h}\), (14.65) and (14.76) for \(Z\_\phi\), \(Z\_\psi\) and \(\hat{m}\) expressed in terms of \(m_{ph}\), we obtain:

\[
Z\_\phi Z\_\psi \hat{h}^2 \frac{i}{q - \hat{m}} = \left( 1 + \hat{\Sigma}_{s}^\phi + \hat{\Sigma}_{s}^V + 2m_{ph}^2 \hat{\Sigma}_{s}^{V'} + 2m_{ph} \hat{\Sigma}_{s}^{m'} \right) \\
\times h_{ph}^2 \left( 1 - 2\Delta \Gamma^{(3)}_* - \hat{\Sigma}_{s}^\phi - 2\hat{\Sigma}_{s}^V - 4m_{ph}^2 \hat{\Sigma}_{s}^{V'} - 4m_{ph} \hat{\Sigma}_{s}^{m'} \right) \\
\times \left\{ \frac{i}{q - m_{ph}} + \frac{i}{q - m_{ph}}(+ i) \left[ \hat{\Sigma}_{s}^m + m_{ph} \hat{\Sigma}_{s}^V \right] \frac{i}{q - m_{ph}} \right\} ,
\]

keeping only tree and one loop terms we get

\[
Z\_\phi Z\_\psi \hat{h}^2 \frac{i}{q - \hat{m}} = h_{ph}^2 \frac{i}{q - m_{ph}} \\
+ h_{ph}^2 \frac{i}{q - m_{ph}} \left( -\hat{\Sigma}_{s}^V - 2m_{ph}^2 \hat{\Sigma}_{s}^{V'} - 2m_{ph} \hat{\Sigma}_{s}^{m'} - 2\Delta \Gamma^{(3)}_* \right) \\
+ h_{ph}^2 \frac{i}{q - m_{ph}} (-i) \left[ -\hat{\Sigma}_{s}^m - m_{ph} \hat{\Sigma}_{s}^V \right] \frac{i}{q - m_{ph}} .
\]

Rearranging terms a little bit this is

\[
Z\_\phi Z\_\psi \hat{h}^2 \frac{i}{q - \hat{m}} = h_{ph}^2 \frac{i}{q - m_{ph}} \\
+ h_{ph}^2 \frac{i}{q - m_{ph}} \left( -\Delta \Gamma^{(3)}_* \right) + h_{ph}^2 \frac{i}{q - m_{ph}} \left( -\Delta \Gamma^{(3)}_* \right) \\
+ h_{ph}^2 \frac{i}{q - m_{ph}} (-i) \left[ -\hat{\Sigma}_{s}^m - m_{ph} \hat{\Sigma}_{s}^V \right] \frac{i}{q - m_{ph}} \\
- (q - m_{ph})^2 \left( \hat{\Sigma}_{s}^V + 2m_{ph}^2 \hat{\Sigma}_{s}^{V'} + 2m_{ph} \hat{\Sigma}_{s}^{m'} \right) \frac{i}{q - m_{ph}} .
\]

These are precisely the terms which are missing in \(-iM^{SM}\) compared to \(-iM^{OS}\)!

Thus, the expressions for the amplitude of the elastic \(Sf \rightarrow Sf\) scattering expressed in terms of observable quantities \(m_{ph}\) and \(h_{ph}\) and bare self energies and vertex functions\(^{26}\) is the same in both schemes up to higher order terms. As we have seen, the proof that they indeed coincide is complicated already in the one-loop order and one can only imagine how difficult it would be for the two-loop computations!

\(^{26}\)Recall, that in the expression for \(-iM^{OS}\) the infinities cancel out and one can add freely “hats” on self energies and vertex functions.