Problems in Quantum Field Theory of Fundamental Interactions. I

Problem I.1
Using the definition of the action of the creation and annihilation operators on the state-vectors $|\psi_1, \dots, \psi_N\rangle$, show that $a(\varphi_1)$ and $a^\dagger(\varphi_2)$ associated respectively with the one-particle states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ satisfy the relation

$$\left[a(\varphi_1),\ a^\dagger(\varphi_2)\right]_\mp = \langle\varphi_1|\varphi_2\rangle\,.$$

$[\,\cdot\,,\ \cdot\,]_\mp$ denotes here the commutator if the operators are bosonic and the anticommutator if they are fermionic.

Problem I.2
The Hamiltonian of a system consisting of a fermion interacting with a boson acting in the many particle Hilbert space $\mathcal{H}$ and written in terms of the operators a, $a^\dagger$ satisfying the standard rule, $[a, a^\dagger] = 1$, and the fermionic operators b, $b^\dagger$ satisfying the anticommutation rules

$$\{b,\, b^\dagger\} = 1\,, \qquad \{b,\, b\} = \{b^\dagger,\, b^\dagger\} = 0\,, \qquad [a,\ b] = 0\,,$$

etc., takes the form ($\hbar = 1$, g is a coupling constant having dimension of energy)

$$H = m\, b^\dagger b + \omega\, a^\dagger a + g\, b^\dagger b\,(a + a^\dagger)\,.$$

Find its spectrum and the corresponding eigenvectors.
Hint: One way of solving this problem is to define the Hermitian operator $O = ib^\dagger b(a - a^\dagger)$ and perform on the Hamiltonian the unitary transformation

$$H \to e^{i\kappa O}\, H\, e^{-i\kappa O}\,,$$

chosing appropriately the real parameter κ.

Problem I.3
Check that independently of whether particles are bosons or fermions, two-particle operators of the general form

$$O = \sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\sum_{\mathbf{k}_3}\sum_{\mathbf{k}_4} f(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)\, a^\dagger_{\mathbf{k}_1} a^\dagger_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4}\,,$$

commute with the total particle number operator $\hat{N} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$. (The same can also be shown in the case of the continuous normalization of the states with the sums replaced by the appropriate integrals.)

Problem I.4
Show that the operator of the two-particle interaction[1]

$$V_{\rm int} = \frac{1}{2}\int d^3\mathbf{x}\, d^3\mathbf{y}\; a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) V_{\rm pot}(\mathbf{x} - \mathbf{y})\, a(\mathbf{y}) a(\mathbf{x})\,,$$

[1]Spin indices are not displayed because they are irrelevant to this problem.

commutes (whether particles are bosons or fermions) with the total momentum operator

$$\hat{\mathbf{P}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \hbar\mathbf{k}\, a^\dagger(\mathbf{k}) a(\mathbf{k})\,.$$

Hint: It is better to write $V_{\rm int}$ in terms of the momentum space creation/annihilation operators.

Problem I.5
Using the general prescription for expressing one-particle operators through creation and annihilation operators, construct for the system of many identical spinless particles of mass m three operators $\hat{\mathbf{J}} = (J^x, J^y, J^z)$ satisfying the commutation rules

$$\begin{aligned}\left[J^i,\ J^j\right] &= i\hbar\, \epsilon^{ijk} J^k\,,\\ \left[J^i,\ P^j\right] &= i\hbar\, \epsilon^{ijk} P^k\,,\end{aligned}$$

and show that the operators J^i commute with the Hamiltonian

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{\hbar^2\mathbf{p}^2}{2m}\, a^\dagger(\mathbf{p}) a(\mathbf{p}) + \frac{1}{2}\int d^3\mathbf{x} \int d^3\mathbf{y}\, a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) V_{\rm pot}(\mathbf{x},\mathbf{y}) a(\mathbf{y}) a(\mathbf{x})\,,$$

provided $V_{\rm pot}(\mathbf{x},\mathbf{y}) = V_{\rm pot}(|\mathbf{x}-\mathbf{y}|)$.

Problem I.5′
Consider a system of N spin 1/2 identical fermions of mass m_f interacting with one another through a spin independent potential $V_{\rm pot}(|\mathbf{x}_i - \mathbf{x}_j|)$. If they are nonrelativistic and move in an infnite space, the system is invariant with respect to transformations forming the Galileo group. Using the field operators $\hat{\psi}_\sigma(\mathbf{x})$, $\hat{\psi}^\dagger_\sigma(\mathbf{x})$ of the second quantization formalism construct explicitly the ten generators of this group acting in the systems Hilbert space and check that they satisfy the commutation rules

$$\begin{aligned}
&[\hat{J}^i,\ \hat{J}^j] = i\hbar\, \epsilon^{ijk} \hat{J}^k\,, && [\hat{P}^i,\ \hat{P}^j] = 0\,,\\
&[\hat{J}^i,\ \hat{P}^j] = i\hbar\, \epsilon^{ijk} \hat{P}^k\,, && [\hat{K}^i,\ \hat{K}^j] = 0\,,\\
&[\hat{J}^i,\ \hat{K}^j] = i\hbar\, \epsilon^{ijk} \hat{K}^k\,, && [\hat{K}^i,\ \hat{P}^j] = -i\hbar\, \delta^{ij} \hat{M}\,,\\
&[\hat{J}^i,\ H] = 0\,, && [\hat{P}^i,\ H] = 0\,,\\
& && [\hat{K}^i,\ H] = -i\hbar \hat{P}^i\,.
\end{aligned}$$

in which $\hat{M}$ is the operator of the total mass of the system, by virtue of the anticommutation rules

$$\begin{aligned}\{\hat{\psi}_\alpha(\mathbf{x}),\ \hat{\psi}^\dagger_\beta(\mathbf{y})\} &= \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ \{\hat{\psi}_\alpha(\mathbf{x}),\ \hat{\psi}_\beta(\mathbf{y})\} &= \{\hat{\psi}^\dagger_\alpha(\mathbf{x}),\ \hat{\psi}^\dagger_\beta(\mathbf{y})\} = 0\,.\end{aligned}$$

Hint: The operators $\hat{\mathbf{J}}$ can be constructed using the general prescription for converting one-particle operators into their second-quantized counterparts. The same prescription can be also applied to the operators realizing boosts in quantum mechanics of a single fermion moving freely in infinite space - check that its second-quantized counterpart satisfies the required commutation rules in spite of the presence of the interaction term in the Hamiltionian of the considered system. In checking the commutation rules one has to remember that because $\hat{\psi}_\sigma(\mathbf{x})$, $\hat{\psi}^\dagger_\sigma(\mathbf{x})$ are in fact operator valued distributions, expressions having the form of the space integral of the total (space) derivative of a string of field operators should be treated as zero operators.

Problem I.5″
Consider the second-quantized version of quantum mechanics of the system of N nonrelativistic spin 1/2 identical fermions of mass m_f interacting with one another through a spin independent potential $V_{\rm pot}(|\mathbf{x}_i-\mathbf{x}_j|)$ (but not with an external potential). Check that owing to the normal ordering (with respect to the $|{\rm void}\rangle$ vector) of the Hamiltonian, the state $a^\dagger_\sigma(\mathbf{p})|{\rm void}\rangle$ of a single particle is its eigenvector with the eigenvalue $\hbar^2\mathbf{p}^2/2m_f$ and that the time dependent state-vector

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = e^{-iHt/\hbar}\!\int\! d^3\mathbf{x}\, u_\sigma(\mathbf{x})\, \hat{\psi}^\dagger_\sigma(\mathbf{x})|{\rm void}\rangle\,,$$

in which the profile $u_\sigma(\mathbf{x})$ satisfies the normalization condition $\int\! d^3\mathbf{x}\sum_\sigma |u_\sigma(\mathbf{x})|^2 = 1$, satisfies the Schrödinger equation

$$i\hbar\,\frac{d}{dt}\,|\psi(t)\rangle = \hat{H}|\psi(t)\rangle\,,$$

and that it is properly normalized, i.e. that $\langle\psi(t)|\psi(t)\rangle = 1$. Verify by explicit computation that the state-vector $\exp(-(i/\hbar)\mathbf{V}\cdot\hat{\mathbf{K}})|\psi(t)\rangle$ where $\hat{\mathbf{K}}$ is the boost generator constructed in Problem I.5′ also satisfies the same Schrödinger equation. Check also explicitly that the wave function $\langle\alpha,\mathbf{x}|e^{-\frac{i}{\hbar}\mathbf{V}\cdot\hat{\mathbf{K}}}|\psi(t)\rangle$ of the boosted state is related in the standard textbook way to wave function $\langle\alpha,\mathbf{x}|\psi(t)\rangle$ of the original state.

Problem I.6
A system of N identical spinless (nonrelativistic) bosons of mass m interacting pairwise through a regular potential $V_{\rm pot}(\mathbf{x}_i-\mathbf{x}_j)$ are enclosed in a box of volume $V=L^3$. Write down the expressions for the first and second order corrections to the ground state energy of this system. Argue that if $\int\! d^3\mathbf{x}\,V_{\rm pot}(\mathbf{x})<0$ (and the integral is finite), the system is intrinsically unstable and should collapse.

Problem I.7
Compute in the first nontrivial order of the perturbative expansion the ground state energy of a system of N spin 1/2 fermions of which N_+ have spin projection $+1/2$ onto the z axis and N_- have this spin projection $-1/2$, and express it in terms of the densities $\rho_+ = N_+/V$ and $\rho_- = N_-/V$ of the upwards and downwards polarized particles and the volume V in which the system is enclosed. The fermions interact with one another through the potential $V_{\rm pot}(\mathbf{x}_i-\mathbf{x}_j) = \lambda\,\delta^{(3)}(\mathbf{x}_i-\mathbf{x}_j)$.

Problem I.8
The system consists of N positive ions of charge $e > 0$ located at fixed positions so that in the volume $V = L^3$ their density can be approximated by a homogeneous distribution $n_{\rm io}(\mathbf{x}) = N/V$, and of N (nonrelativistic) electrons (of mass m and charge $-e$) which can move in the entire volume V interacting with one another and with the ions through the Coulomb potential. Find in the first order of the Rayleigh - Schrödinger expansion energy of the ground state of this system. Express the results through the dimensionless mean interelectron distance $\bar{r} = [V/(4\pi a_{\rm B}^3 N)]^{1/3}$ where $a_{\rm B} = \hbar^2/me^2$ is the Bohr radius.
Hint: Introduce the factor $\exp(-\mu|\mathbf{x}_i - \mathbf{x}_j|)$ regularizing the Coulomb interaction. Take the thermodynamic limit before setting $\mu \to 0$.

Problem I.9
Compute in the first nontrivial order of the perturbative expansion the ground state energy of a system of N spin 1/2 electrons of which N_+ have spin projection $+1/2$ onto the z axis and N_- have this spin projection $-1/2$, and express it in terms of the total number $N = N_+ + N_-$ of electrons and the system's polarization $\zeta = (N_+ - N_-)/N$. Electrons are enclosed in the volume V and interact through the Coulomb force with themselves and with the neutralizing the system uniform background of positive ions. Compare the ground state energies of the unpolarized ($\zeta = 0$) and fully polarized ($\zeta = 1$ or -1) systems as a function of the dimensionless parameter $\bar{r}$ defined by the relation $V = (4\pi/3)N a_{\rm B}^3 \bar{r}^3$ ($a_{\rm B} = \hbar^2/me^2$ is the Bohr radius).

Problem I.10
Write down the expressions for the second order corrections to the ground state energy of the system of N (nonrelativistic) electrons moving in the volume V on a homogeneous background of N positive ions. Are these corrections finite?

Problem I.11
Use the Gell-Mann - Low formula

$$E_\Omega - E_{\Omega_0} = \lim_{\varepsilon\to 0} \frac{\langle\Omega_0|V_{\rm int}\, U_I^\varepsilon(0,-\infty)|\Omega_0\rangle}{\langle\Omega_0|U_I^\varepsilon(0,-\infty)|\Omega_0\rangle}\,,$$

to compute, up to the second order in λ the ground state energy E_Ω of the one dimensional harmonic oscillator of frequency ω perturbed by the interaction $V_{\rm int} = (\lambda/4)(a^\dagger + a)^4 \equiv \lambda(\hat{x}/l)^4$, where $l = (\hbar/m\omega)^{1/2}$. To evaluate the numerator and the denominator use the Wick theorem. Compare with the result obtained using the standard Rayleigh-Schrödinger perturbative expansion.

Problem I.12
Treating the term proportional to λ^2 in the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m(\omega^2 + \lambda^2)\hat{x}^2 = H_0 + V_{\rm int}\,,$$

as a perturbation compute corrections to the ground state energy up to the third order in λ^2 using the Gell-Mann - Low formula evaluating only the connected contributions to its numerator in two ways: applying the Wick theorem to the formula involving the chronological product (the Dyson way) and alternatively by replacing the chronological product by the nested integrals over interaction times and inserting the complete sets of H_0 eigenvectors between the operators (the Goldstone way); in this case "connected contributions" are commonly taken to mean the contributions which do not involve the ground state as an intemediate state). Which result is correct? Why?

Problem I.13
Prove that the most general solution of the condition

$$g_{\mu\nu}\,\frac{\partial x^{\mu\prime}}{\partial x^{\lambda}}\frac{\partial x^{\nu\prime}}{\partial x^{\kappa}} = g_{\lambda\kappa}\,,$$

takes the form $x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$ with constant $\Lambda^{\mu}{}_{\nu}$ and a^{μ}.

Problem I.14
Using the commutation rules of the Poincaré group generators

$$\begin{aligned}
[P^{\mu},\ P^{\nu}] &= 0\,,\\
[J^{\lambda\rho},\ P^{\mu}] &= i\left(P^{\lambda}g^{\rho\mu} - P^{\rho}g^{\lambda\mu}\right),\\
[J^{\lambda\rho},\ J^{\mu\nu}] &= i\left(J^{\lambda\nu}g^{\mu\rho} - J^{\lambda\mu}g^{\rho\nu} - J^{\rho\nu}g^{\lambda\mu} + J^{\rho\mu}g^{\lambda\nu}\right),
\end{aligned}$$

calculate the commutators $[J^{\lambda\rho},\ W^{\nu}]$, $[P^{\mu},\ W^{\nu}]$, $[W^{\mu},\ W^{\nu}]$ and $[W^{\mu}W_{\mu},\ W^{\nu}]$ of the Pauli-Lubański operator[2] $W_{\mu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}J^{\nu\lambda}P^{\rho}$. Using the results show that the operators

$$P^2 = P_{\mu}P^{\mu} \qquad \text{and} \qquad W^2 = W_{\mu}W^{\mu}\,,$$

commute with all the generators of the Poincaré group. The conclusion from the results is that W^2 and P^2 are the two Racah operators of the Poincaré group (P^2 being bilinear in the group generators is its Casimir operator) and can serve to label its irreducible representations, while states within a given representation can be labeled by eigenvalues of the operators P^i and one (linear combination) of the components of W^{μ}.
Hint: Use the relation

$$\epsilon^{\mu\nu\lambda\sigma}\,\det(\Lambda) = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\lambda}{}_{\lambda'}\Lambda^{\sigma}{}_{\sigma'}\,\epsilon^{\mu'\nu'\lambda'\sigma'}\,,$$

which shows that $\epsilon^{\mu\nu\lambda\sigma}$ is an invariant tensor with respect to proper ortochronous Lorentz transformations (which have $\det(\Lambda) = 1$.

Problem I.15
Check directly that the matrix $(L_p)^{\mu}{}_{\nu}$ the elements of which read

$$(L_p)^{0}{}_{0} = \gamma\,, \qquad (L_p)^{i}{}_{j} = \delta^{i}{}_{j} - (\gamma-1)\frac{p^i p_j}{|\mathbf{p}|^2}\,,$$
$$(L_p)^{0}{}_{i} = -\frac{p_i}{|\mathbf{p}|}\sqrt{\gamma^2-1}\,, \qquad (L_p)^{i}{}_{0} = \frac{p^i}{|\mathbf{p}|}\sqrt{\gamma^2-1}\,,$$

[2] We use $\epsilon^{0123} = -\epsilon_{0123} = +1$.

where $\gamma = \sqrt{1 + \mathbf{p}^2/m^2} = E_{\mathbf{p}}/m = 1/\sqrt{1 - \mathbf{v}^2}$, defines a Lorentz transformation (that is, L_p satisfies the basic condition $L_p^T \cdot g \cdot L_p = g$ where g is the Minkowski space-time metric tensor). Show that L_p is the composition of the following three transformations:

$$L_p = R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) \cdot B_z(|\mathbf{p}|) \cdot R_{\hat{\mathbf{z}}}^{-1}(\hat{\mathbf{p}}) \, ,$$

where $B_z(|\mathbf{p}|)$ is the Lorentz boost transforming a particle of mass m at rest (in a system $\mathcal{O}$) into a particle moving (in a system $\mathcal{O}'$) along the z-axis with velocity $|\mathbf{p}|/E(\mathbf{p})$, and the rotation $R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) = O(\phi_{\mathbf{p}}, \mathbf{e}_z) \cdot O(\theta_{\mathbf{p}}, \mathbf{e}_y)$ which transforms a vector pointing in the z-direction into a vector pointing in the direction $\hat{\mathbf{p}}$ specified by the polar angles $\theta_{\mathbf{p}}$ and $\phi_{\mathbf{p}}$. Show also that L_p given by this composition is just the boost in the direction opposite to the direction of $\mathbf{p}$.

Problem I.16
Show by direct calculation that the measure

$$d\Gamma_{\mathbf{p}} \equiv \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \, ,$$

is invariant with respect to ortochroneous Lorentz transformations that is, that if E' and $\mathbf{p}'$ are related to E and $\mathbf{p}$ by a Lorentz transformation, then $d^3\mathbf{p}'/E' = d^3\mathbf{p}/E$.

Problem I.17
Check by direct calculation that the matrices

$$\begin{aligned}
\left(J^z_{(j)}\right)_{j'\sigma',j\sigma} &= \sigma \, \delta_{\sigma'\sigma}\delta_{j'j} \, , \\
\left(J^x_{(j)} \pm iJ^y_{(j)}\right)_{j'\sigma',j\sigma} &= \delta_{j'j}\delta_{\sigma'\sigma\pm1}\sqrt{(j \mp \sigma)(j \pm \sigma + 1)} \, ,
\end{aligned}$$

satisfy the $SU(2)$ algebra commutation rules.

Problem I.18
Using solely the properties (the commutation rules) of the Poincaré group generators show that the operator $U(L_p)$ corresponding to the transformation L_p of a massive particle satisfies the relations

$$\mathcal{P}U(L_p)\mathcal{P}^{-1} = U(L_{P\cdot p}) \, ,$$

and

$$\mathcal{T}U(L_p)\mathcal{T}^{-1} = U(L_{P\cdot p}) \, ,$$

.

Problem I.19
Show that the states

$$|\mathbf{p}, \sigma_s\rangle = U(L_p)|\mathbf{0}, \sigma_s\rangle \, ,$$

of a massive particle of mass m, where $|\mathbf{0},\sigma_s\rangle$ is such that $\hat{\mathbf{s}}\cdot\mathbf{J}|\mathbf{0},\sigma_s\rangle = \sigma_s|\mathbf{0},\sigma_s\rangle$ for a three-vector $\hat{\mathbf{s}}$ of unit length, are the eigenstates with the same eigenvalues σ_s of the operator $-s_p^\mu W_\mu/m$, in which

$$s_p^\mu = (L_p)^\mu{}_\nu\, s_{\rm rest}^\nu\,, \qquad\qquad s_{\rm rest}^\nu = (0,\hat{\mathbf{s}})\,.$$

Find the eigenvalues of $W_\mu W^\mu$ on the states $|\mathbf{p},\sigma_s\rangle$. Show also that $U(\Lambda)|\mathbf{p},\sigma_s\rangle$ is the eigenstate of $-(\Lambda\cdot s_p)^\mu W_\mu/m$ with the same eigenvalue σ_s.

Problem I.20

In a frame $\mathcal{O}_1$ the W^+ boson (a spin 1 particle of mass $M = 80.4$ GeV/c^2) is in the state $|\mathbf{p}_1,\sigma\rangle$ with $\mathbf{p}_1 = (0,|\mathbf{p}_1|,0)$, that is, has (in its rest frame) the spin projection onto the z axis equal σ. In what state will this W^+ be seen by an observer $\mathcal{O}_2$ moving with respect to $\mathcal{O}_1$ with the velocity v along the z-axis? Does the result mean that while in the frame of $\mathcal{O}_1$ a beam of fully polarized W's (assuming for a while they are stable) would not be split by a Stern-Gerlach device (with appropriately oriented magnetic field), it should be split by the same device from the point of view of the obsever $\mathcal{O}_2$?

Problem I.21

The helicity states $|\mathbf{p},\lambda\rangle$ of a massive particle are defined by the formula

$$|\mathbf{p},\lambda\rangle = U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,U(B_z(|\mathbf{p}|))\,|\mathbf{0},\lambda\rangle\,,$$

in which $\mathbf{0}$ represents the standard four-momentum $k^\mu = (m,\mathbf{0})$ and λ is the spin projection onto the z direction in the particle's rest frame: $J^z|\mathbf{0},\lambda\rangle = \lambda|\mathbf{0},\lambda\rangle$ (that is, λ has the same meaning as σ in the definition of the standard states $|\mathbf{p},\sigma\rangle$). Show that $|\mathbf{p},\lambda\rangle$ are eigenstates of the operator

$$W^0 = \mathbf{J}\cdot\mathbf{P}\,,$$

with the eigenvalues $\lambda|\mathbf{p}|$ and that they are related to the standard states $|\mathbf{p},\sigma\rangle = U(L_p)|\mathbf{0},\sigma\rangle \equiv U(L_p)|\mathbf{0},\lambda\rangle$, where $L_p = R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\cdot B_z(|\mathbf{p}|)\cdot R_{\hat{\mathbf{z}}}^{-1}(\hat{\mathbf{p}})$, by

$$|\mathbf{p},\lambda\rangle = \sum_\sigma |\mathbf{p},\sigma\rangle\, D^{(s)}_{\sigma\lambda}(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\,,$$

where $D^{(s)}_{\sigma\lambda}(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})) \equiv D^{(s)}_{\sigma\lambda}(\phi_{\mathbf{p}},\theta_{\mathbf{p}},0)$, with the angles $\phi_{\mathbf{p}}$, $\theta_{\mathbf{p}}$ specifying the direction $\hat{\mathbf{p}}$.

Give the transformation properties of the helicity states $|\mathbf{p},\lambda\rangle$ under arbitrary Lorentz transformations. In particular, show how they transform under pure rotations and compare the result with the corresponding transformation properties of the helicity states of massless particles.

Problem I.22

The "canonical" helicity state $|-\hat{\mathbf{z}}|\mathbf{p}|,\lambda\rangle$ of a massive (or massless) particle is obtained as the limit $\mathbf{p}\to\hat{\mathbf{z}}|\mathbf{p}|$ (i.e. $\theta_{\mathbf{p}}\to 0$, $\phi_{\mathbf{p}}\to 0$) of the state $|-\mathbf{p},\lambda\rangle$ (the polar angles corresponding to $-\mathbf{p}$ are $\pi-\theta_{\mathbf{p}}$, $\phi_{\mathbf{p}}\pm\pi$). Show that if the spin s particle is massive

$$|-\hat{\mathbf{z}}|\mathbf{p}|,\lambda\rangle = e^{i\pi s}\,U(B_{-z}(|\mathbf{p}|))\,|\mathbf{0},-\lambda\rangle\,.$$

where the boost $B_{-z}(|\mathbf{p}|)$ produces the particle moving with the momentum $-\hat{\mathbf{z}}|\mathbf{p}|$ out of the particle at rest.

Problem I.23
Show that the helicity states of a massless particle

$$|\mathbf{p},\lambda\rangle = U(L_p)|\mathbf{k},\lambda\rangle\,,$$

where $k^\mu = (\kappa,0,0,\kappa)$ and $L_p = R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}})\cdot B_z(|\mathbf{p}|/\kappa)$ are the eigenstates of the operator W^0 with the eigenvalue $\lambda|\mathbf{p}|$. Find the value of the operator $W^\mu W_\mu$ on these states.

Problem I.24
Write explicitly the formulae for transformations of the helicity state $|\mathbf{p},\lambda\rangle$, where $\mathbf{p} = (0,|\mathbf{p}|,0)$, under rotations of the reference frame by the angle φ around the z, x and y axes. Consider the cases of a massive and massless particle and different ranges of the angle φ.

Problem I.25
In the frame $\mathcal{O}_1$ a massless particle has momentum $\mathbf{p}_1 = (0,|\mathbf{p}_1|,0)$ and helicity λ. Show by direct calculation that under the action of pure boosts Λ along the z, x or y axes the helicity state $|\mathbf{p}_1,\lambda\rangle$ of the particle transforms according to the rule

$$U(\Lambda)|\mathbf{p}_1,\lambda\rangle = |{}^\Lambda\mathbf{p}_1,\lambda\rangle\,,$$

i.e. that the state is unchanged (apart from the trivial change of its momentum).

Problem I.26
In the frame $\mathcal{O}_1$ a massive spin s particle has momentum $\mathbf{p}_1 = (0,|\mathbf{p}_1|,0)$ and helicity λ. What is its helicity state in the frame $\mathcal{O}_2$ moving with respect to $\mathcal{O}_1$ with velocity v along the z-axis? Check the limit of vanishing mass of the particle.

Problem I.27
A massive spin s particle has in the frame $\mathcal{O}_1$ momentum $\mathbf{p}_1 = (0,|\mathbf{p}_1|,0)$ and helicity λ. What is its helicity state in the frame $\mathcal{O}_2$ moving with respect to $\mathcal{O}_1$ with velocity v along the y-axis? Consider the cases $v < |\mathbf{p}_1|/E_1$ and $v > |\mathbf{p}_1|/E_1$.

Problem I.28
Find the action of the parity and time reversal operators $\mathcal{P}$ and $\mathcal{T}$ on the helicity states $|\mathbf{p},\lambda\rangle$ of a massive particle (defined in Problem I.21).

Problem I.29
Using the operators J^i constructed in Problem I.6 show by explicit calculation that the states

$$|E,l,m_l\rangle = \sqrt{\frac{2l+1}{4\pi}}\int_0^{2\pi} d\phi_{\mathbf{p}}\int_0^{\pi} d\theta_{\mathbf{p}}\sin\theta_{\mathbf{p}}\,|\mathbf{p}\rangle\, D^{(l)*}_{m_l 0}(\phi_{\mathbf{p}},\theta_{\mathbf{p}},0)\,,$$

of a massive spinless particle, where the angles $\phi_{\mathbf{p}}$, $\theta_{\mathbf{p}}$ characterize the direction of the vector $\mathbf{p}$, are eigenstates of $J^z = L^z$ and $\mathbf{J}^2 = \mathbf{L}^2$. Find the action of the $\mathcal{P}$ and $\mathcal{T}$ operators on these states.

Problem I.30
Generalize the result of Problem I.29 to particles with arbitrary spin and mass, showing that the states

$$|E, \lambda, j, m_j\rangle = \sqrt{\frac{2j+1}{4\pi}} \int_0^{2\pi} d\phi_{\mathbf{p}} \int_0^{\pi} d\theta_{\mathbf{p}} \sin\theta_{\mathbf{p}} \, |\mathbf{p}, \lambda\rangle \, D^{(j)*}_{m_j \lambda}(\phi_{\mathbf{p}}, \theta_{\mathbf{p}}, 0) \, ,$$

are common eigenstates of the following operators: the Hamiltonian $\hat{H} = P^0$, total angular momentum $\mathbf{J}^2$, J^z (and of W^0 and $W^\mu W_\mu$). Find also the expansion of $|\mathbf{p}, \lambda\rangle$ in terms of the states $|E, \lambda, j, m_j\rangle$.
Hint: Show that the states $|E, \lambda, j, m_j\rangle$ transform properly when acted upon by the rotation operator $U(O(\alpha, \beta, \gamma)) \equiv U(O(\alpha, \mathbf{e}_z) \cdot O(\beta, \mathbf{e}_y) \cdot O(\gamma, \mathbf{e}_z))$. To this end introduce a dummy angular variable χ to write the integral over $d\phi_{\mathbf{p}} \, d(\cos\theta_{\mathbf{p}})$ in the form $d\tilde{O} \equiv d\phi_{\mathbf{p}} \, d(\cos\theta_{\mathbf{p}}) \, d\chi$ which will allow to exploit the property of left-invariance, $d\tilde{O} = d(O \cdot \tilde{O})$, of the measure on the rotation group (Problems 0.44 & 0.45).

Problem I.31
Find the action of the parity and time reversal operators $\mathcal{P}$ and $\mathcal{T}$ on the states $|E, \lambda, j, m_j\rangle$ (of massive and massless particles).

Problem I.32
Using the operators J^i constructed in Problem I.5 show explicitly that the states of two spinless particles in their CMS given by the formula ($\mathbf{0}$ stands for vanishing total three-momentum of the system of two particles)

$$|\mathbf{0}, \sqrt{s}, l, m_l\rangle \equiv \int_0^{2\pi} d\phi_{\mathbf{p}} \int_0^{\pi} d\theta_{\mathbf{p}} \sin\theta_{\mathbf{p}} \, |\mathbf{0}, \sqrt{s}, \hat{\mathbf{p}}\rangle \, Y_{l m_l}(\theta_{\mathbf{p}}, \phi_{\mathbf{p}}) \, ,$$

in which $\sqrt{s} = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2}$, the angles $\theta_{\mathbf{p}}$, $\phi_{\mathbf{p}}$ characterize the direction of the vector $\mathbf{p}$ and

$$|\mathbf{0}, \sqrt{s}, \hat{\mathbf{p}}\rangle \equiv |\mathbf{p}, -\mathbf{p}\rangle = a^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p}) |\Omega_0\rangle \, ,$$

are eigenstates of the angular momentum operators J^z and $\mathbf{J}^2$ with the eigenvalues m_l and $l\,(l+1)$, respectively.

Problem I.33
Generalize the result of Problem I.32 to particles with arbitrary spins s_1 and s_2 and masses by showing that the states

$$|\mathbf{0}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle \equiv \sqrt{\frac{2j+1}{4\pi}} \int_0^{2\pi} d\varphi_{\mathbf{p}} \int_0^{\pi} d\vartheta_{\mathbf{p}} \sin\vartheta_{\mathbf{p}} \, |\mathbf{0}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle \, D^{(j)*}_{m_j, \lambda_1 - \lambda_2}(\varphi_{\mathbf{p}}, \vartheta_{\mathbf{p}}, 0) \, ,$$

in which for distinct particles[3]

$$|\mathbf{0},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle \equiv e^{-i\pi s_2}U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))\left(|\,\hat{\mathbf{z}}|\mathbf{p}|,\lambda_1\rangle\otimes|-\hat{\mathbf{z}}|\mathbf{p}|,\lambda_2\rangle\right),$$

($|\pm\hat{\mathbf{z}}|\mathbf{p}|,\lambda_i\rangle$ are the "canonical" one-particle states; the extra phase factor $e^{-i\pi s_2}$ ensures symmetrical treatment of both particles) are eigenstates of the operators $\mathbf{J}^2$ (the total angular momentum) and J^z with the respective eigenvalues $j(j+1)$ and m_j. (Show that they transform properly when acted upon by the operator $U(\tilde{O})$ realizing a rotation O - see the Hint to Problem I.30).

Argue also that the states

$$|\mathbf{0},\sqrt{s},\sigma_1,\sigma_2,l,m_l\rangle \equiv \int_0^{2\pi}d\varphi_{\mathbf{p}}\int_0^{\pi}d\vartheta_{\mathbf{p}}\sin\vartheta_{\mathbf{p}}\,|\mathbf{p},\sigma_1\rangle\otimes|-\mathbf{p},\sigma_2\rangle\,Y_{lm_l}(\vartheta_{\mathbf{p}},\varphi_{\mathbf{p}}),$$

represent two particles (in their CMS) with the spin projections σ_1 and σ_2 in the state with relative orbital angular momentum l and its projection onto the z-axis equal m_l.

Problem I.34
Show that for two identical (massive or massless) particles of spin[4] s

$$|\mathbf{0},\sqrt{s},\lambda_2,\lambda_1,j,m_j\rangle = (-1)^j|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle\,.$$

Prove also that for both cases, of distinct and identical particles,

$$|\mathbf{0},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle = \sum_j\sum_{m_j}\sqrt{\frac{2j+1}{4\pi}}\,|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle\,D^{(j)}_{m_j,\lambda_1-\lambda_2}(\Omega_{\hat{\mathbf{p}}})\,.$$

where $D^{(j)}_{m_j,\lambda_1-\lambda_2}(\Omega_{\hat{\mathbf{p}}}) \equiv D^{(j)}_{m_j,\lambda_1-\lambda_2}(\varphi_{\mathbf{p}},\vartheta_{\mathbf{p}},0)$.

Problem I.35
Prove that (for distinct particles)

$$\begin{aligned}\langle\mathbf{P}',\mathbf{p}',\lambda_1',\lambda_2'|\mathbf{P},\mathbf{p},\lambda_1,\lambda_2\rangle &\equiv \langle\mathbf{P}',\sqrt{s'},\hat{\mathbf{p}}',\lambda_1',\lambda_2'|\mathbf{P},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle\\ &= (2\pi)^4\delta^{(4)}(P'-P)\,16\pi^2\,\frac{\sqrt{s}}{|\mathbf{p}|}\,\delta^{(2)}(\Omega_{\hat{\mathbf{p}}'}-\Omega_{\hat{\mathbf{p}}})\,\delta_{\lambda_1',\lambda_1}\delta_{\lambda_2',\lambda_2}\,,\end{aligned}$$

where $\mathbf{p}$ is the momentum of the first particle in the center of mass frame, $\sqrt{s} = \sqrt{\mathbf{p}^2+m_1^2}+\sqrt{\mathbf{p}^2+m_2^2}$ and $P^\mu = p_1^\mu + p_2^\mu$ is the total four-momentum of the two-particle system; the Laboratory frame four momenta $p^\mu_{1,2}$ are defined by the formula $p_i^\mu = [R_{\hat{\mathbf{z}}}(\hat{\mathbf{P}})\cdot B_z(|\mathbf{P}|)]^\mu{}_\nu p^\nu_{i,\mathrm{CM}}$. Find also the scalar product of two $|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ states constructed in Problem I.33.

[3]For the case of identical particles and for the inverse formula expressing the state $|\mathbf{0},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle$ (for distinct and identical particles) through the states $|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ - see Problem I.34.

[4]s denoting spin of both identical particles should not be confused with their center of mass energy $\sqrt{s}$.

Give also modifications of these formulae for states of two identical particles.
Hint: Show that

$$\frac{d^3\mathbf{p}_1}{E_1(\mathbf{p}_1)}\,\frac{d^3\mathbf{p}_2}{E_2(\mathbf{p}_2)} = \frac{|\mathbf{p}|}{\sqrt{s}}\,d^4P\,d\Omega_{\hat{\mathbf{p}}}\,.$$

Problem I.36
Show that the states $|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ constructed in Problem I.33 are eigenstates of the following operators: $P^\mu P_\mu$, $W^\mu W_\mu$ and W^3 with the eigenvalues s, $-s\,j(j+1)$ and $\sqrt{s}\,m_j$, respectively. Show also that the states $|\mathbf{P},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle$ defined as

$$|\mathbf{P},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle \equiv U(R_{\hat{\mathbf{z}}}(\hat{\mathbf{P}}))\,U(B_z(|\mathbf{P}|))|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle\,,$$

are the eigenstates of the operators $P^\mu P_\mu$, $W^\mu W_\mu$ and W^0 with the eigenvalues s, $-s\,j(j+1)$ and $|\mathbf{P}|m_j$, respectively. For $\mathbf{P}=\mathbf{0}$ (i.e. if the laboratory frame coincides with the center of mass frame) j is therefore the total angular momentum of the two-particle system and m_j the total angular momentum projection onto the z axis. For $\mathbf{P}\neq\mathbf{0}$ the quantum number j is called the total spin S of the system and m_j acquires the interpretation of the system's total helicity (it is denoted Λ).

Problem I.37
Show that

$$\mathcal{P}|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle = \eta_1\,\eta_2\,(-1)^{j-s_1-s_2}|\mathbf{0},\sqrt{s},-\lambda_1,-\lambda_2,j,m_j\rangle\,,$$

$$\mathcal{T}|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle = \zeta_1\,\zeta_2\,(-1)^{j-m_j}|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,-m_j\rangle\,.$$

where η_1, η_2 and ζ_1, ζ_2 are the intrinsic parities and phase factors related to time reversal[5] of the two particles in the state. Find also the action of the $\mathcal{P}$ and $\mathcal{T}$ operators on the states $|\mathbf{P},\sqrt{s},\hat{\mathbf{p}},\lambda_1,\lambda_2\rangle$.

Problem I.38
Using the transformation properties of the states with respect to rotations and the spatial reflection formulate the selection rules for the decay of a massive particle of spin $s=0$ into two photons (two massless particles of spin 1). Consider different internal partities of the decaying particle. Show also that a massive spin 1 particle cannot decay (irrespectively of whether parity is conserved or not by the underlying dynamics) into two photons (the Landau - Yang theorem).[6]
Hint: Take the momenta of the two photons along the y-axis. To prove the Landau-Yang theorem consider an arbitrary rotation around the y-axis by an angle φ and the rotation around the z axis by π.

[5] As usually with the action of the antihermitian operator $\mathcal{T}$, the phase factor in the second formula is unphysical and depends crucially on the precise definition of the $|\mathbf{0},\sqrt{s},\lambda_1,\lambda_2,j,m_j\rangle$ states specified in Problem I.33.

[6] L.D. Landau, *Dokł. Akad. Nauk USSR* **60** (1948) 207. C.N. Yang, *Phys. Rev.* **77** (1949), 242.

Problem I.39

Extending Problem I.5 construct explicitly in terms of the creation and annihilation operators all the generators of the Poincaré group acting in the Hilbert space spanned by tensor products of one-particle states of massive spin 0 particles.

Problem I.40

Construct explicitly in terms of the creation and annihilation operators the rotation group generators $\hat{J}^k$ acting in the Hilbert space spanned by tensor products of one-particle states of massive spin $\frac{1}{2}$ particles.

Problem I.41

One-particle states $|E, \lambda, j, m_j\rangle$ constructed in Problem I.30 are not eigenstates of the parity operator $\mathcal{P}$ (Problem I.31). Composing spin with the orbital angular momentum in the usual manner construct one-particle states $|E, l, j, m_j\rangle$ which are eigenstates of the parity operator $\mathcal{P}$ and transform as a regular representation under rotations (that is, are also eigenstates of the operators $W^\mu W_\mu$ and W^0). To make the latter property explicit, express these states as linear combinations of the $|E, \lambda, j, m_j\rangle$ states. Find the action of $\mathcal{P}$ and $\mathcal{T}$ on the constructed states.