Problem I.1
A vector $V$ rotated by an angle $\phi$ around the axis $n$ (where $|n| = 1$) can be written as

$$V' = V \cos \phi + n(n \cdot V)(1 - \cos \phi) + n \times V \sin \phi$$

$$\approx V + \phi \times V - \frac{1}{2} \phi^2 V + \frac{1}{2} \phi(\phi \cdot V) + \ldots$$

where $\phi \equiv \phi n$. Justify this formula. Find the vector $\phi$ corresponding to the composition of two successive infinitesimal rotations characterized by $\phi_1$ and $\phi_2$ of a vector $V$. Using the result find the structure constants of the rotation group. Show also that the matrix

$$R^{ij}(\phi) = \delta^{ij} \cos \phi + (1 - \cos \phi)n^i n^j + \epsilon^{ikj} n^k \sin \phi,$$

such that $V'' = R^{ij}(\phi)V^j$, is just the matrix $\exp(-i\phi^k J^k_{vec})$, where $(J^k_{vec})^{ij} = i\epsilon^{ikj}$ are the rotation group generators in the defining (vector) representation.

Problem I.2
A left-invariant measure $d\mu(g)$ on a group $G$ has the property

$$\int d\mu(g) f(g) = \int d\mu(g) f(g'g)$$

($g$ denotes an element of $G$ and $f(g)$ is a function defined on the group $G$). In a concrete parametrization $g = g(\theta)$ of the group elements by some parameters $\theta^a$, $a = 1, \ldots, n$ (where $n$ is the dimension of the Lie algebra of $G$) the measure is given by $d\mu(g) = d^n \theta \rho(\theta)$. Find the left-invariant measure (i.e. the density $\rho$) on the rotation group $SO(3)$ in the parametrization given by the components of the vector $\phi = (\phi^x, \phi^y, \phi^z)$ defined in Problem I.1. To this end justify and exploit the formula

$$\rho(\theta) = \rho(0) \det^{-1}\left(\frac{\partial h(\theta, \tilde{\theta})}{\partial \tilde{\theta}}\right)_{\tilde{\theta}=0},$$

in which $h(\theta, \tilde{\theta})$ is the group composition function appropriate for the chosen parametrization $g = g(\theta)$ of the group elements. Compute the rotation group volume assuming that the measure is normalized so that $\rho(0) = 1$. 

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**Problem I.3**
Prove that if the dimension \( n \) of the group \( G \) is odd the formula \( d\mu(g) \equiv d^n\rho(\xi^1, \ldots, \xi^n) \) with

\[
\rho(\xi^1, \ldots, \xi^n) \propto e^{i\xi^1_1 \cdots \xi^n_n} \text{tr} \left( O^{-1} \left( O^{-1} \frac{\partial O}{\partial \xi^1_1} \cdot O^{-1} \frac{\partial O}{\partial \xi^2_2} \cdot \ldots \right) \right),
\]

where \( O(\xi^1, \ldots, \xi^n) \) is a matrix representation of the group element parametrized by the parameters \( \xi^i \), defines on \( G \) a left-invariant measure (if \( n \) even the measure defined in this way vanishes as a result of the antisymmetry of \( e^{i\xi^1_1 \cdots \xi^n_n} \) and the cyclicity of the trace). Use this result to find explicitly the density \( \rho(\alpha, \beta, \gamma) \) of the left-invariant measure on the \( SO(3) \) and \( SU(2) \) groups parametrized by the three Euler angles \( \alpha, \beta \) and \( \gamma \).

**Problem I.4**
Prove the following expansion

\[
e^{-A}B e^A = B - [A, B] + \frac{1}{2!} [A, [A, B]] - \frac{1}{3!} [A, [A, [A, B]]] + \ldots
\]

Prove also the Baker-Hausdorff operator identity

\[
e^{A+B} = e^A e^{B} e^{-\frac{1}{2} [A, B]},
\]

holding for operators \( A \) and \( B \) commuting with \([A, B]\). Finally, prove the general formula,\(^1\)

\[
e^{t(A+B)} = e^{tA} T \exp \left( \int_0^t d\tau e^{-\tau A} B e^{\tau A} \right),
\]

valid for any two operators \( A \) and \( B \), in which \( T \) denotes the “time” ordered product.

**Hints:** To prove the expansion solve iteratively the differential equation satisfied by the operator function \( C(\lambda) = e^{-\lambda A} B e^{\lambda A} \). Similarly, to prove the Baker-Hausdorff formula consider the function \( F(\lambda) = e^{-\lambda B} e^{-\lambda A} e^{\lambda (A+B)} \) and simplify the differential equation satisfied by it using the fact that owing to the assumption, in the expansion of \( e^B A e^{-B} \) in powers of the operator \( B \) only two first terms are nonvanishing.

**Problem I.5**
Using the commutation rules of the rotation group generators \( J^k \) with \( k = x, y, z \) show that

\[
e^{-i\phi J^x} e^{-i\theta J^y} e^{+i\phi J^x} = e^{-i\theta (J^y \cos \phi - J^x \sin \phi)}.
\]

\(^1\) The Baker-Hausdorff formula is its special case for \( t = 1 \) and \([A, [A, B]] = [B, [A, B]] = 0\).  

---
Problem I.6
Using the definition of the creation and annihilation operators show that \(a(\varphi_1)\) and \(a^\dagger(\varphi_2)\) associated respectively with the one-particle states \(|\varphi_1\rangle\) and \(|\varphi_2\rangle\) satisfy the (anti)commutation rule
\[
[a(\varphi_1), a^\dagger(\varphi_2)]_{\mp} = \langle \varphi_1 | \varphi_2 \rangle.
\]
\([\cdot, \cdot]_{\mp}\) denotes the commutator for bosons and the anticommutator for fermions.

Problem I.7
Show that the action of the two-particle operator
\[
V_{\text{int}} = \frac{1}{2} \int d^3x \int d^3y \ a^\dagger(x) a^\dagger(y) V(x, y) a(y) a(x)
\]
on the states \(|x_1, x_2, \ldots, x_N\rangle\) of the position basis is
\[
V_{\text{int}} |x_1, x_2, \ldots, x_N\rangle = \sum_{i<j} V(x_i, x_j) |x_1, x_2, \ldots, x_N\rangle.
\]

Problem I.8
Check that for both, bosons and fermions, two-particle operators of the general form
\[
O = \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} f(k_1, k_2, k_3, k_4) a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4},
\]
commute with the total particle number operator \(\hat{N} = \sum_k a^\dagger_k a_k\). (The same can also be shown in the case of the continuous normalization of the states with the sums replaced by the appropriate integrals.)

Problem I.9
Show that the operator of two-particle interactions
\[
V_{\text{int}} = \frac{1}{2} \int d^3x \ d^3y \ a^\dagger(x) a^\dagger(y) V(x - y) a(y) a(x),
\]
commutes (for bosons as well as for fermions) with the total momentum operator
\[
\hat{P} = \int \frac{d^3k}{(2\pi)^3} \hbar k \ a^\dagger(k) a(k).
\]

Hint: It is better to write \(V_{\text{int}}\) in terms of the momentum space creation/annihilation operators.
Problem I.10
Using the general prescription for expressing one-particle operators through creation
and annihilation operators, construct for the system of many identical spinless par-
ticles of mass $m$ three operators $\hat{J} = (J^x, J^y, J^z)$ satisfying the commutation rules
\[
\left[ J^i, J^j \right] = i\hbar \epsilon^{ijk} J^k,
\]
\[
\left[ J^i, P^j \right] = i\hbar \epsilon^{ijk} P^k,
\]
and show that the operators $J^i$ commute with the Hamiltonian
\[
H = \int \frac{d^3p}{(2\pi)^3} \frac{\hbar^2 p^2}{2m} a^\dagger(p)a(p) + \frac{1}{2} \int d^3x \int d^3y \, a^\dagger(x)a^\dagger(y)V(x, y)a(y)a(x),
\]
provided $V(x, y) = V(|x - y|)$.

Problem I.11
A system of $N$ mutually noninteracting identical bosons is enclosed in a box of
finite volume $V$ (or is placed in in an external confining potential), so that the one-
particle energy eigenstates can be labeled with a discrete index $l$: $H^{(1)}|l\rangle = \epsilon_l|l\rangle$
(the one-particle states $|l\rangle$ correspond to energies $\epsilon_l$). Using the occupation num-
ber representation write down the expression for matrix elements of the Canonical
Ensemble density operator
\[
\rho = \frac{1}{Q} \exp\left(-\frac{H}{k_B T}\right), \quad \text{with} \quad Q = \Tr \exp\left(-\frac{H}{k_B T}\right),
\]
where $k_B$ is the Boltzmann constant and $T$ the temperature. Why it is in general
impossible to evaluate $Q$ directly?

Compute therefore the Grand Canonical Ensemble statistical sum $\Xi$
\[
\Xi(T, V, \mu) = \sum_{N=0}^{\infty} e^{\mu N/k_B T} \Tr \exp\left(-\frac{H}{k_B T}\right),
\]
where $\mu$ is the chemical potential and $N$ the particle number operator, and the
average occupation $\bar{n}_l$ of the $l$-th one-particle energy eigenstate
\[
\bar{n}_l = \frac{1}{\Xi} \sum_{N=0}^{\infty} e^{\mu N/k_B T} \Tr \left(a_l^\dagger a_l e^{-H/k_B T}\right),
\]
where $a_l^\dagger$ and $a_l$ are the creation and annihilation operators associated with the one-
particle energy eigenstate $|l\rangle$. Is there any constraint on the chemical potential $\mu$?
Do the same for $N$ identical fermions.
Show also that
\[ \bar{n}_l^2 - (\bar{n}_l)^2 = \bar{n}_l (1 + \zeta \bar{n}_l), \]
where \( \zeta = +1 \) if the particles are bosons and \( \zeta = -1 \) if they are fermions. What is the probability that the one-particle state \( |l\rangle \) is occupied by exactly \( n_l \) particles?

**Problem I.12**
Consider a system of noninteracting particles of spin \( s \) enclosed in a box of volume \( V = L^3 \), so that the one-particle energies are \( \varepsilon_k = \hbar^2 k^2 / 2m \), where \( k^2 = (2\pi / L) n^2 \) etc. Use the rule
\[ \sum_k \to \frac{V}{(2\pi)^3} \int d^3k, \]
to compute the thermodynamical potential \( \Omega(T, V, \mu) = -k_B T \ln \Xi(T, V, \mu) \) and relate it to the mean internal energy
\[ U = \sum_{k, s} \varepsilon_k \bar{n}_{k, s}, \]
the pressure \( p \) and the volume \( V \) using the fact that \( \Omega = -p V \). Find also the entropy \( S \) and the mean particle number \( \bar{N} \) from the formulae
\[ S = -\left( \frac{\partial \Omega}{\partial T} \right)_{V, \mu}, \quad \bar{N} = -\left( \frac{\partial \Omega}{\partial \mu} \right)_{T, V}. \]

Show that
\[ \Omega = VT^{5/2} \times f(\mu / T), \]
\[ S = VT^{3/2} \times g(\mu / T), \]
\[ \bar{N} = VT^{3/2} \times h(\mu / T), \]
where \( f(x) \), \( g(x) \) and \( h(x) \) are functions given by some integrals which cannot be evaluated in closed forms. On this basis show that adiabatic processes \( (S = \text{const}, \bar{N} = \text{const}) \) are characterized by the relation \( p V^{5/3} = \text{const} \) (although it is not true that \( c_p / c_v = 5/3 \) nor that \( p V = Nk_B T \)). Finally, show that the equation of state obtained by eliminating \( \mu \) from the expressions for \( \Omega = -p V \) and \( \bar{N} \) is
\[ p V = \bar{N} k_B T \left\{ 1 - \zeta \frac{\bar{N}}{g s V} \frac{1}{2} \left( \frac{\pi \hbar^2}{m k_B T} \right)^{3/2} + \ldots \right\}, \]
where \( g_s = 2s + 1 \) is the number of spin states and the ellipses stand for terms of higher order in the density \( \bar{N}/V \). The second term in the bracket is the quantum correction to the classical perfect gas equation of state.

**Problem I.13**
Prove that the most general solution of the condition

\[
g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\lambda} \frac{\partial x^\nu}{\partial x^\kappa} = g_{\lambda\kappa},
\]

takes the form \( x^\mu = \Lambda^\mu_\nu x^\nu - a^\mu \) with constant \( \Lambda^\mu_\nu \) and \( a^\mu \).

**Problem I.14**
Using the commutation rules of the Poincaré group generators

\[
\begin{align*}
[P^\mu, P^\nu] &= 0, \\
[J^{\lambda\rho}, P^\mu] &= i \left( P^\lambda g^{\rho\mu} - P^\rho g^{\lambda\mu} \right), \\
[J^{\lambda\rho}, J^{\mu\nu}] &= i \left( J^{\lambda\nu} g^{\mu\rho} - J^{\lambda\mu} g^{\rho\nu} - J^{\rho\nu} g^{\lambda\mu} + J^{\rho\mu} g^{\lambda\nu} \right),
\end{align*}
\]
calculate the commutators \([J^{\lambda\rho}, W^\nu], [P^\mu, W^\nu], [W^\mu, W^\nu] \) and \([W^\mu W_\mu, W_\nu] \) of the Pauli-Lubański operator

\[
2 W^\mu = -\frac{1}{2} \epsilon_{\mu\nu\lambda\rho} j^{\nu\lambda} P^\rho.
\]

Using the results show that the operators

\[
P^2 = P_\mu P^\mu \quad \text{and} \quad W^2 = W_\mu W^\mu,
\]

commute with all the generators of the Poincaré group. The conclusion from the results is that \( W^2 \) and \( P^2 \) are the two Racah operators of the Poincaré group (\( P^2 \) being bilinear in the group generators is its Casimir operator) and can serve to label its irreducible representations, while states within a given representation can be labeled by eigenvalues of the operators \( P^i \) and one (linear combination) of the components of \( W^\mu \).

**Hint:** Use the relation

\[
\epsilon^{\mu\nu\lambda\sigma} \det(\Lambda) = \Lambda^\mu_\mu \Lambda^\nu_\nu \Lambda^\lambda_\lambda \Lambda^\sigma_\sigma \epsilon^{\mu\nu\lambda\sigma},
\]

which shows that \( \epsilon^{\mu\nu\lambda\sigma} \) is an invariant tensor with respect to proper ortochroneous Lorentz transformations (which have \( \det(\Lambda) = 1 \)).

---

\(^2\)We use \( \epsilon^{0123} = -\epsilon_{0123} = +1 \).
Problem I.15
Check directly that the matrix $(L_p)^\mu_\nu$ whose elements read

$(L_p)^0_0 = \gamma , \quad (L_p)^i_j = \delta_i^j - (\gamma - 1)\frac{p_i p_j}{|p|^2},$

$(L_p)^0_i = -\frac{p_i}{|p|}\sqrt{\gamma^2 - 1}, \quad (L_p)^i_0 = \frac{p_i}{|p|}\sqrt{\gamma^2 - 1},$

where $\gamma = \sqrt{1 + \frac{p^2}{m^2}} = \frac{E_p}{m} = 1/\sqrt{1 - v^2}$, defines a Lorentz transformation (that is, $L_p$ satisfies the basic condition $L^T_p \cdot g \cdot L_p = g$ where $g$ is the Minkowski space-time metric tensor). Show that $L_p$ is the composition of the following three transformations:

$L_p = R_z(\hat{p}) \cdot B_z(|p|) \cdot R_{\hat{z}}^{-1}(\hat{p}),$

where $B_z(|p|)$ is the Lorentz boost transforming a particle of mass $m$ at rest (in a system $O$) into a particle moving (in a system $O'$) along the $z$-axis with velocity $|p|/E(p)$, and the rotation $R_z(\hat{p}) = R_{\hat{z}}^{-1}(\phi_p) \cdot R_{\hat{y}}^{-1}(\theta_p)$ which transforms a vector pointing in the $z$ direction into a vector pointing in the direction $\hat{p}$ specified by the polar angles $\theta_p$ and $\phi_p$. Show also that $L_p$ given by this composition is just the boost in the direction opposite to the direction of $p$.

Problem I.16
Show by direct calculation that the measure

$d\Gamma_p \equiv \frac{d^3p}{(2\pi)^3 2E(p)},$

is invariant with respect to ortochroneous Lorentz transformations, i.e. that $d^3p'/E' = d^3p/E$.

Problem I.17
Check by direct calculation that the matrices

$\left(J^z_{(j)}\right)_{j'\sigma'j\sigma} = \sigma \delta_{\sigma'\sigma} \delta_{j'j},$

$\left(J^\pm_{(j)} \pm i J^y_{(j)}\right)_{j'\sigma'j\sigma} = \delta_{j'j} \delta_{\sigma'\sigma} \pm 1 \sqrt{(j \mp \sigma)(j \pm \sigma + 1)},$

satisfy the $SU(2)$ algebra commutation rules.
Problem I.18
Using solely the properties (the commutation rules) of the Poincaré group generators show that the operator $U(L_p)$ corresponding to the transformation $L_p$ of a massive particle satisfies the relations

$$\mathcal{P}U(L_p)\mathcal{P}^{-1} = U(L_{P,p}),$$

and

$$\mathcal{T}U(L_p)\mathcal{T}^{-1} = U(L_{P,p}),$$

. 

Problem I.19
Show that the states

$$|\mathbf{p}, \sigma_s\rangle = U(L_p)|\mathbf{0}, \sigma_s\rangle,$$

of a massive particle of mass $m$, where $|\mathbf{0}, \sigma_s\rangle$ is such that $\hat{s} \cdot J|\mathbf{0}, \sigma_s\rangle = \sigma_s|\mathbf{0}, \sigma_s\rangle$ for a three-vector $\hat{s}$ of unit length, are the eigenstates with the same eigenvalues $\sigma_s$ of the operator $-s^\mu_p W_\mu/m$, in which

$$s^\mu_p = (L_p)^\mu_\nu s^\nu_{\text{rest}}, \quad s^\nu_{\text{rest}} = (0, \hat{s}).$$

Find the eigenvalues of $W_\mu W^\mu$ on the states $|\mathbf{p}, \sigma_s\rangle$. Show also that $U(\Lambda)|\mathbf{p}, \sigma_s\rangle$ is the eigenstate of $-(\Lambda \cdot s_p)^\mu W_\mu/m$ with the same eigenvalue $\sigma_s$.

Problem I.20
In a frame $\mathcal{O}_1$ the $W^+$ boson (a spin 1 particle of mass $M = 80.4$ GeV/$c^2$) is in the state $|\mathbf{p}_1, \sigma\rangle$ with $\mathbf{p}_1 = (0, |\mathbf{p}_1|, 0)$, that is, has (in its rest frame) the spin projection onto the $z$ axis equal $\sigma$. In what state will see this $W^+$ an observer $\mathcal{O}_2$ moving with respect to $\mathcal{O}_1$ with velocity $v$ along the $z$-axis? Does the result mean that while in the frame of $\mathcal{O}_1$ a beam of fully polarized $W$’s (assuming for a while they are stable) would not be split by a Stern-Gerlach device (with appropriately oriented magnetic field), it should be split be the same device from the point of view of $\mathcal{O}_2$?

Problem I.21
The helicity states $|\mathbf{p}, \lambda\rangle$ of a massive particle are defined by the formula

$$|\mathbf{p}, \lambda\rangle = U(R_\lambda(p)) U(B_z(|\mathbf{p}|)) |\mathbf{0}, \lambda\rangle,$$
in which \( 0 \) represents the standard four-momentum \( k^\mu = (m, 0) \) and \( \lambda \) is the spin projection onto the \( z \) direction in the particle’s rest frame: \( J^z|0, \lambda\rangle = \lambda|0, \lambda\rangle \) (that is, \( \lambda \) has the same meaning as \( \sigma \) in the definition of the standard states \( |p, \sigma\rangle \)). Show that \( |p, \lambda\rangle \) are eigenstates of the operator

\[
W^0 = J \cdot P,
\]

with the eigenvalues \( \lambda|p| \) and that they are related to the standard states \( |p, \sigma\rangle = U(L_p)|0, \sigma\rangle \equiv U(L_p)|0, \lambda\rangle \), where \( L_p = R_\hat{z}(\hat{p}) \cdot B_z(|p|/\kappa) \cdot R_\hat{z}^{-1}(\hat{p}) \), by

\[
|p, \lambda\rangle = \sum_\sigma |p, \sigma\rangle D^{(s)}_{\sigma \lambda}(R_\hat{z}(\hat{p})) ,
\]

where \( D^{(s)}_{\sigma \lambda}(R_\hat{z}(\hat{p})) \equiv D^{(s)}_{\sigma \lambda}(\phi_p, \theta_p, 0) \), with \( \phi_p, \theta_p \) being the angles specifying the direction \( \hat{p} \).

Give the transformation properties of the helicity states \( |p, \lambda\rangle \) under arbitrary Lorentz transformations. In particular, show how they transform under pure rotations and compare the result with the corresponding transformation properties of the helicity states of massless particles.

**Problem I.22**
The “canonical” helicity state \( |-\hat{z}|p|, \lambda\rangle \), is defined as the limit \( p \to \hat{z}|p| \) (i.e. \( \theta_p \to 0, \phi_p \to 0 \)) of the state \( |-p, \lambda\rangle \). Show that for a massive particle of spin \( s \)

\[
|-\hat{z}|p|, \lambda\rangle = e^{i\pi s} U(B_{-z}(|p|)) |0, -\lambda\rangle ,
\]

where the boost \( B_{-z}(|p|) \) produces the particle moving with the momentum \(-\hat{z}|p|\) out of the particle at rest.

**Problem I.23**
Show that the (helicity) states of a massless particle

\[
|p, \lambda\rangle = U(L_p)|k, \lambda\rangle ,
\]

where \( k^\mu = (\kappa, 0, 0, \kappa) \) and \( L_p = R_\hat{z}(\hat{p}) \cdot B_z(|p|/\kappa) \) are the eigenstates of the operator \( W^0 \) with the eigenvalue \( \lambda|p| \). Find the value of the operator \( W^\mu W_\mu \) on these states.

**Problem I.24**
Write explicitly the formulae for transformations of the helicity state \( |p, \lambda\rangle \), where \( p = (0, |p|, 0) \), under rotations of the reference frame by the angle \( \varphi \) around the \( z, x \) and \( y \) axes. Consider the cases of a massive and massless particle and different ranges of the angle \( \varphi \).
Problem I.25
In the frame $O_1$ a massless particle has momentum $p_1 = (0, |p_1|, 0)$ and helicity $\lambda$. Show by direct calculation that under the action of pure boosts $\Lambda$ along the $z$, $x$ or $y$ axes the helicity state $|p_1, \lambda\rangle$ of the particle transforms according to the rule
$$U(\Lambda)|p_1, \lambda\rangle = |\Lambda p_1, \lambda\rangle,$$
i.e. that the state is unchanged (apart from the trivial change of its momentum).

Problem I.26
In the frame $O_1$ a massive spin $s$ particle has momentum $p_1 = (0, |p_1|, 0)$ and helicity $\lambda$. What is its helicity state in the frame $O_2$ moving with respect to $O_1$ with velocity $v$ along the $z$-axis? Check the limit of vanishing mass of the particle.

Problem I.27
A massive spin $s$ particle has in the frame $O_1$ momentum $p_1 = (0, |p_1|, 0)$ and helicity $\lambda$. What is its helicity state in the frame $O_2$ moving with respect to $O_1$ with velocity $v$ along the $y$-axis? Consider the cases $v < |p_1|/E_1$ and $v > |p_1|/E_1$.

Problem I.28
Find the action of the parity and time reversal operators $P$ and $T$ on the helicity states $|E, l, m\rangle$ of a massive particle (defined in Problem I.21).

Problem I.29
Using the operators $J^i$ constructed in Problem I.10 show by explicit calculation that the states
$$|E, l, m\rangle = \sqrt{\frac{2l + 1}{4\pi}} \int_0^{2\pi} d\phi_p \int_0^\pi d\theta_p \sin \theta_p \, |p\rangle \, D_{m,0}^{(l)}(\phi_p, \theta_p, 0),$$
of a massive spinless particle, where the angles $\phi_p, \theta_p$ characterize the direction of the vector $p$, are eigenstates of $J^z = L^z$ and $J^2 = L^2$. Find the action of the $P$ and $T$ operators on these states.

Problem I.30
Generalize the result of Problem I.29 to particles with arbitrary spin and mass, showing that the states
$$|E, \lambda, j, m_j\rangle = \sqrt{\frac{2j + 1}{4\pi}} \int_0^{2\pi} d\phi_p \int_0^\pi d\theta_p \sin \theta_p \, |p, \lambda\rangle \, D_{m_j,\lambda}^{(j)}(\phi_p, \theta_p, 0),$$
are common eigenstates of the following operators: the Hamiltonian $\hat{H} = P^0$, total angular momentum $J^2$, $J^z$ (and of $W^0$ and $W^\mu W_\mu$). Find also the expansion of
\(|\mathbf{p}, \lambda\rangle\) in terms of the states \(|E, \lambda, j, m_j\rangle\).

**Hint:** Show that the states \(|E, \lambda, j, m_j\rangle\) transform properly when acted upon by the rotation operator \(U(R(\alpha, \beta, \gamma)) \equiv U(R_z^{-1}(\alpha) \cdot R_y^{-1}(\beta) \cdot R_z^{-1}(\gamma))\). To this end introduce a dummy angular variable \(\chi\) to write the integral over \(d\phi d\cos \theta\) in the form \(d\tilde{R} \equiv d\phi d(\cos \theta)\) which will allow to exploit the property of left-invariance, \(d\tilde{R} = d(R \cdot \tilde{R})\), of the measure on the rotation group (Problems I.2 & I.3).

**Problem I.31**
Find the action of the parity and time reversal operators \(\mathcal{P}\) and \(\mathcal{T}\) on the states \(|E, \lambda, j, m_j\rangle\) (of massive and massless particles).

**Problem I.32**
Using the operators \(J^i\) constructed in Problem I.10 show explicitly that the states of two spinless particles in their CMS given by the formula \((0\) stands for vanishing total three-momentum of the system of two particles)

\[ |0, \sqrt{s}, l, m_l\rangle \equiv \int_0^{2\pi} d\phi_\mathbf{p} \int_0^\pi d\theta_\mathbf{p} \sin \theta_\mathbf{p} |0, \sqrt{s}, \hat{\mathbf{p}}\rangle Y_{lm_l}(\theta_\mathbf{p}, \phi_\mathbf{p}) , \]

in which \(\sqrt{s} = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2}\), the angles \(\theta_\mathbf{p}, \phi_\mathbf{p}\) characterize the direction of the vector \(\mathbf{p}\) and

\[ |0, \sqrt{s}, \hat{\mathbf{p}}\rangle \equiv |\mathbf{p}, -\mathbf{p}\rangle = \mathfrak{a}^\dagger(\mathbf{p}) \mathfrak{a}^\dagger(-\mathbf{p}) |\Omega_0\rangle , \]

are eigenstates of the angular momentum operators \(J^z\) and \(J^2\) with the eigenvalues \(m_l\) and \(l(l + 1)\), respectively.

**Problem I.33**
Generalize the result of Problem I.32 to particles with arbitrary spins \(s_1\) and \(s_2\) and masses by showing that the states

\[ |0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle \]

\[ \equiv \sqrt{\frac{2j + 1}{4\pi}} \int_0^{2\pi} d\varphi_\mathbf{p} \int_0^\pi d\theta_\mathbf{p} \sin \theta_\mathbf{p} |0, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle D_{m_j, \lambda_1 - \lambda_2}^{(j)}(\varphi_\mathbf{p}, \theta_\mathbf{p}, 0) , \]

in which for distinct particles

\[ |0, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle \equiv e^{-i\pi s_2} U(R_\mathbf{z}(\hat{\mathbf{p}})) \left( |\hat{\mathbf{p}}, \lambda_1 \rangle \otimes -|\hat{\mathbf{p}}, \lambda_2\rangle \right) , \]

\[ \text{for the case of identical particles and the inverse formula expressing the state } |0, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle \text{ (for distinct and identical particles) through the states } |0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle \text{ - see Problem I.34.} \]
(| ± \hat{z}|p\rangle, \lambda_i) are the “canonical” one-particle states; the extra phase factor $e^{-i\pi s_2}$ ensures symmetrical treatment of both particles) are eigenstates of the operators $J^2$ (the total angular momentum) and $J^z$ with the respective eigenvalues $j(j+1)$ and $m_j$. (Show that they transform properly when acted upon by the rotation operator $U(\hat{R})$ - see the Hint to Problem I.30).

Argue also that the states

$$|0, \sqrt{s}, \sigma_1, \sigma_2, l, m_l\rangle \equiv \int_0^{2\pi} d\varphi_p \int_0^\pi d\vartheta_p \sin \vartheta_p |\mathbf{p}, \sigma_1\rangle \otimes | -\mathbf{p}, \sigma_2\rangle Y_{lm}^l(\vartheta_p, \varphi_p),$$

describe two particles (in their CMS) with spin projections $\sigma_1$ and $\sigma_2$ in the state with relative orbital angular momentum $l$ and its projection onto the $z$-axis equal $m_l$.

**Problem I.34**

Show that for two identical (massive or massless) particles of spin $^4s$

$$|0, \sqrt{s}, \lambda_1, j, m_j\rangle = (-1)^j |0, \sqrt{s}, \lambda_1, j, m_j\rangle .$$

Prove also that for both cases, of distinct and identical particles,

$$|0, \sqrt{s}, \hat{p}, \lambda_1, \lambda_2\rangle = \sum_j \sum_{m_j} \sqrt{2j + 1} |0, \sqrt{s}, \lambda_1, j, m_j\rangle D_{m_j, \lambda_1 - \lambda_2}^{(j)}(\Omega_{\hat{p}}).$$

where $D_{m_j, \lambda_1 - \lambda_2}^{(j)}(\Omega_{\hat{p}}) \equiv D_{m_j, \lambda_1 - \lambda_2}(\varphi_{\hat{p}}, \vartheta_{\hat{p}}, 0)$.

**Problem I.35**

Prove that (for distinct particles)

$$\langle \mathbf{P}', \mathbf{p}', \lambda_1', \lambda_2' | \mathbf{P}, \mathbf{p}, \lambda_1, \lambda_2\rangle \equiv \langle \mathbf{P}', \sqrt{s'}, \hat{\mathbf{p}}', \lambda_1', \lambda_2' | \mathbf{P}, \sqrt{s}, \hat{\mathbf{p}}, \lambda_1, \lambda_2\rangle$$

$$= (2\pi)^4 \delta^{(4)}(\mathbf{P}' - \mathbf{P}) 16\pi^2 \sqrt{s} \frac{\delta^{(2)}(\Omega_{\mathbf{P}'} - \Omega_{\mathbf{P}}) \delta_{\lambda_1', \lambda_1} \delta_{\lambda_2', \lambda_2}}{|\mathbf{P}|},$$

where $\mathbf{p}$ is the momentum of the first particle in the center of mass frame, $\sqrt{s} = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2}$ and $P^\mu = p_1^\mu + p_2^\mu$ is the total four-momentum of the two-particle system; the Laboratory frame four momenta $p_{1,2}^\mu$ are defined by the formula $p_i^\mu = [R_2(\hat{\mathbf{P}}) \cdot B_2(|\mathbf{P}|)]_\mu p_{i,CM}^\mu$. Find also the scalar product of two $|\mathbf{P}, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle$ states constructed in Problem I.33.

[^4s]: denoting the spin of both identical particles should not be confused with their center of mass energy $\sqrt{s}$. 

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Give also modifications of these formulae for states of two identical particles.

**Hint:** Show that

\[
\frac{d^3p_1}{E_1(p_1)} \frac{d^3p_2}{E_2(p_2)} = \frac{|p|}{\sqrt{s}} d^4P d\Omega_\hat{P}.
\]

**Problem I.36**

Show that the states \(|0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle\) constructed in Problem I.33 are eigenstates of the following operators: \(P\mu P_\mu\), \(W\mu W_\mu\) and \(W^3\) with the eigenvalues \(s, -sj(j+1)\) and \(\sqrt{s}m_j\), respectively. Show also that the states \(|P, \sqrt{s}, \hat{p}, \lambda_1, \lambda_2\rangle\) defined as

\[
|P, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle \equiv U(R_z(\hat{P})) U(B_z(|P|)) |0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle,
\]

are the eigenstates of the operators \(P\mu P_\mu\), \(W\mu W_\mu\) and \(W^0\) with the eigenvalues \(s, -sj(j+1)\) and \(|P|m_j\), respectively. For \(P = 0\) (i.e. if the laboratory frame coincides with the center of mass frame) \(j\) is therefore the total angular momentum of the two-particle system and \(m_j\) the total angular momentum projection onto the \(z\) axis.

For \(P \neq 0\) the quantum number \(j\) is called the total spin \(S\) of the system and \(m_j\) acquires the interpretation of the system’s total helicity (it is denoted \(\Lambda\)).

**Problem I.37**

Show that

\[
P|0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle = \eta_1 \eta_2 (-1)^{j-s_1-s_2} |0, \sqrt{s}, -\lambda_1, -\lambda_2, j, m_j\rangle,
\]

\[
T|0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle = \zeta_1 \zeta_2 (-1)^{-m_j} |0, \sqrt{s}, \lambda_1, \lambda_2, j, -m_j\rangle.
\]

where \(\eta_1, \eta_2\) and \(\zeta_1, \zeta_2\) are the intrinsic parities and phase factors related to time reversal\(^5\) of the two particles in the state. Find also the action of the \(P\) and \(T\) operators on the states \(|P, \sqrt{s}, \hat{p}, \lambda_1, \lambda_2\rangle\).

**Problem I.38**

Using the transformation properties of the states with respect to rotations and the spatial reflection formulate the selection rules for the decay of a massive particle of spin \(s = 0\) into two photons (two massless particles of spin 1). Consider different internal parities of the decaying particle. Show also that a massive spin 1 particle

\(^5\)As usually with the action of the antihermitian operator \(T\), the phase factor in the second formula is unphysical and depends crucially on the precise definition of the \(|0, \sqrt{s}, \lambda_1, \lambda_2, j, m_j\rangle\) states specified in Problem I.33.
cannot decay (irrespectively of whether parity is conserved or not by the underlying dynamics) into two photons (the Landau–Yang theorem).\textsuperscript{6}

**Hint:** Take the momenta of the two photons along the $y$-axis. To prove the Landau-Yang theorem consider an arbitrary rotation around the $y$-axis by an angle $\varphi$ and the rotation around the $z$ axis by $\pi$.

**Problem I.39**

Extending Problem I.10 construct explicitly in terms of the creation and annihilation operators all the generators of the Poincaré group acting in the Hilbert space spanned by tensor products of one-particle states of massive spin 0 particles.

**Problem I.40**

Construct explicitly in terms of the creation and annihilation operators the rotation group generators $\hat{J}^k$ acting in the Hilbert space spanned by tensor products of one-particle states of massive spin $\frac{1}{2}$ particles.

**Problem I.41**

One-particle states $|E, \lambda, j, m_j\rangle$ constructed in Problem I.30 are not eigenstates of the parity operator $\mathcal{P}$ (Problem I.31). Composing spin with the orbital angular momentum in the usual manner construct one-particle states $|E, l, j, m_j\rangle$ which are eigenstates of the parity operator $\mathcal{P}$ and transform as a regular representation under rotations (that is, are also eigenstates of the operators $W^\mu W_\mu$ and $W^0$). To make the latter property explicit, express these states as linear combinations of the $|E, \lambda, j, m_j\rangle$ states. Find the action of $\mathcal{P}$ and $\mathcal{T}$ on the constructed states.