Problems in Quantum Field Theory of Fundamental Interactions. III

Problem III.1
Consider a classical pointlike particle of mass m moving under the action of a force $\mathbf{F} = \mathbf{F}(t)$ constant in space. Translations in space are obvious symmetries of the problem (if $\mathbf{r}(t)$ is a solution of the Newton's equations of motion, so is $\mathbf{r}'(t) = \mathbf{r}(t) + \mathbf{a}$), yet the momentum $\mathbf{p}$ is not conserved. Clarify this apparent violation of the symmetry - conservation law connection.

Problem III.1′
Consider a pointlike particle of mass m moving under the action of a force $\mathbf{F}$ constant in space but possibly dependent on time. In Problem III.1 it has been argued that space translations are symmetries of this system. Are (Galilean) boosts its symmetries too? If yes, use the Noether theorem to obtain the corresponding conserved charge. Going over to the quantum theory show that the operators of space translations and boosts obtained from the canonical quantization do generate symmetry transformations of the system. Exploiting the Heisenberg picture show that the observables represented by the Hermitian generators of these symmetries are indeed constants of motion even though their do not commute with the Hamiltonian. Compare the algebra of the symmetry generators with the one formed by the generators of the Galileo group and explain why additional central charges are possible in the considered case.

Problem III.2
Consider a set of scalar fields $\phi_i(x)$ which, when the Lorentz frame is changed, transform according to the rule:

$$\phi'_i(x') = \left(e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}\right)_{ij} \phi_j(x) \,,$$

with $(\mathcal{J}^{\mu\nu})_{ij} = (-\mathcal{J}^{\nu\mu})_{ij}$ being the generators of some matrix representation of the Lorentz group. The dynamics of the fields $\phi_i(x)$ is set by a Lagrangian density $\mathcal{L}(\phi, \partial\phi)$ giving rise to the conserved canonical energy momentum tensor $T^{\mu\nu}_{\text{can}}$ associated with spacetime tranlations. Show that the Belinfante energy momentum tensor

$$T^{\mu\nu}_{\text{symm}} = T^{\mu\nu}_{\text{can}} + \partial_\rho H^{\rho\mu\nu} \,,$$

with

$$H^{\rho\mu\nu} = \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i)}(-i\mathcal{J}^{\mu\nu})_{ij}\,\phi_j - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(-i\mathcal{J}^{\rho\nu})_{ij}\,\phi_j - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i)}(-i\mathcal{J}^{\rho\mu})_{ij}\,\phi_j\right],$$

is symmetric. Check that the tensor

$$M^{\mu\nu\kappa} = x^\nu\, T^{\mu\kappa}_{\text{symm}} - x^\kappa\, T^{\mu\nu}_{\text{symm}} \,,$$

where $T^{\mu\nu}_{\text{symm}}$ is the Belinfante energy-momentum tensor differs by a total divergence from the tensor $M^{\mu\nu\kappa}_{\text{can}}$ obtained by using the Noether prescription.

Problem III.3
Consider the Lagrangian density of a free real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 \,.$$

Check that for spacetime translations and Lorentz transformations, $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu - a^\mu$, one has $\mathcal{L}(\varphi'(x'), \partial'\varphi'(x')) = \mathcal{L}(\varphi(x), \partial\varphi(x)) + \partial^\mu \mathcal{X}_\mu(\varphi)$ with $\mathcal{X}_\mu(\varphi) = 0$. Find the conserved canonical tensors $T^{\mu\nu}_{\rm can}$ and $M^{\mu\lambda\kappa}_{\rm can} = x^\lambda\, T^{\mu\kappa}_{\rm can} - x^\kappa\, T^{\mu\lambda}_{\rm can}$. Replace in the obtained classical expressions the time derivatives $\dot\varphi$ by the canonical momentum Π and show by using the canonical commutation rules

$$[\varphi(\mathbf{x}),\ \Pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$

etc., that the operators

$$P^\mu = \int d^3\mathbf{x}\, T^{0\nu}_{\rm can}\,, \qquad\qquad J^{\mu\nu} = \int d^3\mathbf{x}\left(x^\mu\, T^{0\nu}_{\rm can} - x^\nu\, T^{0\mu}_{\rm can}\right),$$

(acting in the Hilbert space of the quantized field φ) satisfy the necessary Poincaré group commutation rules. Go next to the Heisenberg picture and argue that the generators in the Heisenberg picture also satisfy the same commutation rules as in the Schrödinger picture. Using these commutation rules and the general equation for the time evolution of the Heisenberg operators show that the Poincaré group symmetry generators in the Heisenberg picture are time independent. Argue that these results can be straightforwardly extended to the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - V(\varphi_i)\,,$$

of N real fields φ_i with the potential $V(\varphi)$ more general than $\frac{1}{2}M_i^2\varphi_i\varphi_i$.

Problem III.4
Express the Poincaré group generators of the same free field theory as in Problem III.3 in terms of the time dependent Heisenberg (interaction) picture field operators which in turn are expressed through the creation and annihilation operators and verify that the generators are explicitly independent of time. Check their action on one-particle states.

Problem III.5
Using the canonical commutation rules compute in the quantum theory of a single real field φ the commutators

$$\begin{aligned}
&\left[J^{ij},\ \varphi(t,\mathbf{x})\right],\\
&\left[J^{0i},\ \varphi(t,\mathbf{x})\right],\\
&\left[J^{ij},\ \partial^\lambda\varphi(t,\mathbf{x})\right],\\
&\left[J^{0i},\ \partial^\lambda\varphi(t,\mathbf{x})\right],
\end{aligned}$$

where J^{ij} and J^{0i} are the generators of the Lorentz transformations. Using these results argue, that for any field $\phi_i(x)$ which classically satisfies the transformation rule

$$\phi_i'(x') = \left(e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}\right)_{ij}\phi_j(x)\,,$$

where $x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}(\omega)x^{\nu} - a^{\mu}$ and $(\mathcal{J}^{\mu\nu})_{ij}$ are the matrix generators of the Lorentz group in the representation appropriate for the index i of the field ϕ_i, in the quantum theory one has

$$e^{+\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}+ia_{\mu}P^{\mu}}\,\phi_i(x)\,e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}-ia_{\mu}P^{\mu}} = \phi_i'(x)\,.$$

Check this formula for infinitesimal $\omega_{\mu\nu}$ and a_μ, i.e. check that:

$$\begin{aligned}\frac{i}{2}\,\omega_{\mu\nu}\,[J^{\mu\nu},\ \phi_i(x)] &= \delta_0^{\rm Lorentz}\phi_i(x)\,,\\ i\epsilon^{\mu}\,[P^{\mu},\ \phi_i(x)] &= \delta_0^{\rm transl}\phi_i(x)\,,\end{aligned}$$

where $\delta_0^{\rm Lorentz}\phi_i(x)$ and $\delta_0^{\rm transl}\phi_i(x)$ denote the differences $\phi_i'(x) - \phi_i(x)$ for Lorentz transformations and translations, respectively.

Problem III.6

Evaluating the commutator with $J^{\mu\nu}$ show that the charges obtained as integrals of conserved Noether currents

$$Q = \int d^3\mathbf{x}\,j^0(t,\mathbf{x})\,,$$

are Lorentz scalars.

Problem III.7

Let j^{μ}_a, $a = 1,\ldots,N$ be the Noether currents associated with some nonabelian group of symmetry transformations of a Lagrangian density $\mathcal{L}$ (not restricted to be at most quadratic in fields but depending only on fields and their first derivatives):

$$j^{\mu}_a = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_n)}\,(-iT^a)_{nm}\phi_m\,,$$

where $(T^a)_{nm}$, $a = 1,\ldots,N$ are matrix generators of a symmetry group in the representation appropriate for the fields ϕ_n and satisfying the commutation rule $[T^a,\ T^b] = if^{abc}T^c$. Use the canonical commutation rules to prove that

$$\left[j^0_a(t,\mathbf{x}),\ j^0_b(t,\mathbf{y})\right] = if^{abc}j^0_c(t,\mathbf{x})\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$

independently of the dynamics (the precise form of $\mathcal{L}$). Integrate this equality over $d^3\mathbf{y}$ and take the commutator of both its sides with the Lorentz boosts generators $K^i = J^{0i}$. Using the result argue that

$$\left[j^0_a(t,\mathbf{x}),\ j^i_b(t,\mathbf{y})\right] = if^{abc}\,j^i_c(t,\mathbf{x})\,\delta^{(3)}(\mathbf{x}-\mathbf{y}) + S^{ik}_{ab}(t,\mathbf{y})\,\partial^{\mathbf{x}}_k\delta^{(3)}(\mathbf{x}-\mathbf{y}) + \ldots,$$

where the dots stand for terms with more derivatives. The additional terms (whose presence and form does depend on the dynamics) are called Schwinger terms.
Hint: In the second part of the problem use the Jacobi identity.

Problem III.8
Consider quantum theory of fields ϕ_n defined by a Lagrangian density $\mathcal{L}(\phi, \partial\phi)$. Show that the currents

$$j^a_\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_n)} (-iT^a)_{nm}\phi_m \,,$$

(which would be conserved if the transformations $\phi_n \to \phi_n - i(T^a)_{nm}\phi_m$ were symmetry transformations of $\mathcal{L}$) formally[1] satisfy the relation

$$i\left[T^{00}_{\rm can}(t, \mathbf{x}),\ j^a_0(t, \mathbf{y})\right] = \partial^\mu j^a_\mu(x)\, \delta^{(3)}(\mathbf{x}-\mathbf{y}) + j^a_i(x)\, \partial^i_{(\mathbf{x})}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,.$$

Do not assume that $\Pi_n = \partial^0\phi_n = \dot{\phi}_n$, but adopt the general relation $\dot{\phi}_n = \dot{\phi}_n(\Pi, \phi)$.

Problem III.9
Computing the commutator with the energy-momentum operator $T^{00}_{\rm can}(t, \mathbf{z})$ of both sides of the relation

$$\left[j^0_a(t, \mathbf{y}),\ j^0_b(t, \mathbf{x})\right] = i f^{abc}\, j^0_c(t, \mathbf{x})\, \delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$

constrain the form of possible Schwinger terms in the commutator $[j^0_a(t, \mathbf{y}),\ j^i_b(t, \mathbf{x})]$ of the temporal and spatial components of two conserved symmetry currents (considered in Problem III.7) to a single term $S^{ik}_{ab}(t, \mathbf{y})\, \partial^{\mathbf{x}}_k \delta^{(3)}(\mathbf{x}-\mathbf{y})$ with only one derivative of the delta function and such that $S^{ik}_{ab}(t, \mathbf{y}) = S^{ki}_{ba}(t, \mathbf{y})$.
Hint: Use the Jacobi identity and integrate both sides of the obtained equality over $d^3\mathbf{y}$ after multiplying them by y^i.

Problem III.10
Derive the Euler-Lagrange field equations of motion following from a Lagrangian density $\mathcal{L}$ which depends on ϕ_i, $\partial_\mu\phi_i$ and $\partial_\nu\partial_\mu\phi_i$. Assuming that the transformations $\phi'_i(x) = \phi_i(x) + \theta^a\delta^a\phi_i(x)$ are symmetries of the system, such that

$$\mathcal{L}(\phi'_i, \partial_\mu\phi'_i, \partial_\nu\partial_\mu\phi'_i) = \mathcal{L}(\phi_i, \partial_\mu\phi_i, \partial_\nu\partial_\mu\phi_i) + \partial_\lambda \mathcal{X}^\lambda(\phi_i, \partial_\mu\phi_i)\,,$$

derive the corresponding conserved Noether currents.

Consider next the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - V(\phi) + \partial_\mu\phi_i\partial^\mu\varphi_i - \frac{1}{2}M^2\varphi_i^2\,,$$

[1] That is, ignoring possible problems which can arise from multiplying field operators taken at the same space-time point. Whenever relations obtained with formal manipulations based on equal-time (anti)commutators of canonical variables cannot be satisfied in actual calculations (which require introducing some regulators) of Green's functions, we talk about anomalies.

in which $i = 1, \ldots, N$, invariant under the simultaneous transformations $\phi'_i = (e^{-i\theta^a T^a})_{ij}\phi_j$ and $\varphi'_i = (\exp -i\theta^a T^a)_{ij}\varphi_j$, where T^a_{ij} are purely imaginary, antisymmetric $N \times N$ matrices. Write down the corresponding Euler-Lagrange equations of motions satisfied by the fields ϕ_i and φ_i and the conserved Noether currents associated with the symmetry transformations. Eliminate then the fields φ_i (using their equations of motion) from the Euler-Lagrange equations of the ϕ_i fields and from the Noether currents. Find the effective Lagrangian density $\mathcal{L}_{\rm eff}(\phi_i, \partial_\mu\phi_i, \partial_\nu\partial_\mu\phi_i)$ which reproduces the resulting equations of motion of the fields ϕ_i alone and check, that the conserved Noether currents derived form $\mathcal{L}_{\rm eff}$ are the same as the old Noether currents obtained after eliminating of φ_i's.

Problem III.11
Show that if the infinitesimal global transformations $\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta\phi_i$ with $\delta\phi_i = \theta^a F^a_i[\phi]$, where $F^a_i[\phi]$ can be nonlinear functions of ϕ_i, are symmetry transformations of the action in the sense that

$$\mathcal{L}(\phi'_i, \partial\phi'_i) = \mathcal{L}(\phi_i, \partial\phi_i) + \partial^\nu\mathcal{X}_\nu(\phi)\,,$$

with some $\mathcal{X}_\nu(\phi)$, the conserved Noether symmetry currents can be obtained as coefficients of $\partial_\mu\theta^a(x)$ in

$$\delta\mathcal{L} \equiv \mathcal{L}(\phi'_i, \partial\phi'_i) - \mathcal{L}(\phi_i, \partial\phi_i) - \partial^\nu\mathcal{X}_\nu(\phi)\,.$$

after performing the *local* transformation $\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \theta^a(x)F^a_i[\phi]$ with space-time dependent parameters $\theta^a(x)$.

Problem III.12
Consider a theory in which the spinor field ψ transforms as

$$\psi' = e^{-iq\theta}\psi\,,$$

under the action of a $U(1)$ group (as in electrodynamics of spin 1/2 particles). Using the canonical (anti)commutation relations check formally, that is ignoring possible problems with regularization of the composite current operator $j^\lambda(x)$, that the equal time commutators of the time-like and spatial components of the $U(1)$ symmetry Noether current $j^\mu(x) = q\bar\psi(x)\gamma^\mu\psi(x)$ commute:

$$\left[j^0(t,\mathbf{x}),\ j^i(t,\mathbf{y})\right] = 0\,,$$

i.e. that the canonical reasoning would imply vanishing of the possible Schwinger term. Then argue that this commutator cannot (nevertheless) vanish in the theory of interacting particles, which means that the canonical reasoning must be invalidated by any regularization used to properly define $j^\lambda(x)$ as a composite operator.
Hint: To prove that the Schwinger term must be present, take the three-divergence of the vacuum expectation value of the commutator, use the current conservation and replace the time derivative of j^0 by its commutator with the Hamiltonian.

Problem III.13
Construct the canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}}$ of the system consisting of the Dirac spinor field ψ interacting with the real scalar field φ and defined by the Lagrangian density

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4!}\,\varphi^4 - ig\varphi\bar{\psi}\gamma^5\psi\,.$$

The tensor $T^{\mu\nu}_{\text{can}}$ is not symmetric nor does it depend on ψ and $\bar{\psi}$ in a symmetrical way. Use the freedom to add to it a term $\partial_\lambda H^{\lambda\mu\nu}$ where $H^{\mu\lambda\nu} = -H^{\lambda\mu\nu}$ to cure the second defect, find a new Lagrangian density $\tilde{\mathcal{L}}$ equivalent to $\mathcal{L}$ (i.e. leading to the same equations of motion) for which the new energy-momentum tensor is the canonical tensor and then apply the Belinfante prescription (Problem III.2) to obtain the tensor $T^{\mu\nu}_{\text{symm}}$ symmetric in the indices $\mu\nu$.

Problem III.14
Find the canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}}$ of the free electromagnetic field. Show that it is conserved. Construct the symmetric Belinfante energy-momentum tensor $T^{\mu\nu}_{\text{symm}}$ by adding to $T^{\mu\nu}_{\text{can}}$ an appropriate term and show that $T^{\mu\nu}_{\text{symm}}$ is also conserved. Construct the canonical tensor $M^{\mu\nu\lambda}_{\text{can}}$ and show explicitly that it differs by a total divergence from the tensor $M^{\mu\nu\lambda}$ constructed using the Belinfante energy-momentum tensor $T^{\mu\nu}_{\text{symm}}$ (see Problem III.2). Finally, show that the action is invariant also with respect to the scale transformations

$$x^\mu \to x'^\mu = e^\lambda x^\mu\,, \qquad A^\mu(x) \to A'^\mu(x') = e^{-\lambda}A^\mu(x)\,,$$

and derive the associated Noether symmetry current J^μ_{scale}. Show that it can be replaced by the modified current $\tilde{J}^\mu_{\text{scale}} = x_\rho T^{\mu\rho}_{\text{symm}}$ such that $\partial_\mu\tilde{J}^\mu_{\text{scale}} = 0$ follows from $\partial_\mu T^{\mu\rho}_{\text{symm}} = 0$ and $g_{\mu\rho}T^{\mu\rho}_{\text{symm}} = 0$.

Problem III.15
Construct the canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}}$ of charged spin $\frac{1}{2}$ fermions interacting with the electromagnetic field described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - eQ\,\bar{\psi}\,A\!\!\!/\psi\,.$$

Use the same procedure as in Problems III.13 and III.14 to convert $T^{\mu\nu}_{\text{can}}$ into a symmetric gauge invariant tensor $T^{\mu\nu}_{\text{symm}}$.

Problem III.16
Perform the canonical quantization of the free complex scalar field ϕ with the classical Lagrangian density.

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - M^2\phi^*\phi - \mathcal{H}_{\text{int}}(\phi^*\phi)\,.$$

To this end, decompose ϕ into two real fields φ and χ so that $\phi = \frac{1}{\sqrt{2}}(\varphi + i\chi)$ and quantize them separately. Using the Noether theorem find the symmetry current $j^\mu(x)$

and the charge $Q = \int d^3\mathbf{x}\, j^0(\mathbf{x}, t)$ corresponding to the symmetry $\phi \to e^{-iq\theta}\phi$ of the original Lagrangian. Check that the current is conserved. Construct the corresponding quantum operator $\hat{Q}$ acting in the Hilbert space and show that it is time independent and commutes with the Hamiltonian. Construct the creation and annihilation operators corresponding to one-particle states which are common eigenstates of H_0 and $\hat{Q}$. Form the free field operators ϕ and $\phi^\dagger$ such that $[\hat{Q}\ , \phi] = -q\,\phi$ and $[\hat{Q}\ , \phi^\dagger] = +q\,\phi^\dagger$. Finally, using the canonical equal time commutators find the regularization independent contribution to the Schwinger term in the commutator $[j^0(t, \mathbf{x}),\ j^i(t, \mathbf{y})]$.

Problem III.17
Consider a theory of three real scalar fields φ_a, $a = 1, 2, 3$ with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{3}\left(\partial_\mu\varphi_a\partial^\mu\varphi_a - M^2\varphi_a\varphi_a\right) - \mathcal{H}_{\rm int}\left(\boldsymbol{\varphi}^2\right),$$

in which $\boldsymbol{\varphi}^2 \equiv \varphi_1^2 + \varphi_2^2 + \varphi_3^2$. Show that the Lagrangian is invariant under the transformations[2]

$$\varphi_a \to \varphi'_a = \varphi_a - i\theta^b(i\epsilon_{abc})\varphi_c\,.$$

and find the corresponding conserved Noether currents $j^a_\mu(x)$. Show that classically $\partial^\mu j^a_\mu(x) = 0$. Perform the canonical quantization and check that by virtue of the equal time canonical commutation rules, the Noether charge operators $\hat{Q}^a \equiv \int d^3\mathbf{x}\, j^a_0(\mathbf{x})$ commute with the Hamiltonians H (and also with its free part H_0) and satisfy the commutation rule $[\hat{Q}^a,\ \hat{Q}^b] = i\epsilon^{abc}\hat{Q}^c$ appropriate for the algebra of the $SU(2)$ (or $SO(3)$) group. Next express the Noether charges $\hat{Q}^a \equiv \int d^3\mathbf{x}\, j^a_0(\mathbf{x})$ through the creation and annihilation operators diagonalizing H_0 and find the free one-particle states which are eigenstates of H_0, $\hat{\mathbf{Q}}^2 \equiv (\hat{Q}^1)^2 + (\hat{Q}^2)^2 + (\hat{Q}^3)^2$ and $\hat{Q}^3$. These states can be identified with the triplet of pions (the $\pi^\pm$, π^0 mesons). Construct also the free two-particle eigenstates of H_0, $\hat{\mathbf{Q}}^2$ and $\hat{Q}^3$.
Hint: To construct the two particle states the well known Clebsch-Gordan coefficients can be used.

Problem III.18
Combine two Dirac fermion fields ψ_p (proton) and ψ_n (neutron) into a column (the nucleon field)

$$\psi_N \equiv \begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix},$$

and let it transform as a doublet under the internal isospin $SU(2)_V$ group[3]

$$(\psi'_N)_i = U^\dagger(\theta)(\psi_N)_i\,U(\theta) = \left(e^{-i\theta^a T^a_{(I=\frac{1}{2})}}\right)_{ij}(\psi_N)_j\,,$$

[2]The field components φ_a are labeled by the same type of letters as are the transformation parameters θ^a because φ_a transform as the adjoint representation of the symmetry group.

[3]The subscript V on $SU(2)$ reminds that it is a vector-like symmetry: left- and right-chiral parts of the nucleon field ψ_N are transformed in the same way.

where $U(\theta)$ are the Hilbert space $SU(2)_V$ symmetry operators and $T^a_{(I=\frac{1}{2})} = \frac{1}{2}\tau^a$ are the ordinary three matrix generators of $SU(2)$ in the two-dimensional (isospin $I = \frac{1}{2}$) representation. Knowing that the three meson fields π_a $(a = 1, 2, 3)$ transform as an isospin triplet (see Problem III.17)

$$\pi'_b(x) = U^\dagger(\theta)\,\pi_b(x)\,U(\theta) = \left(e^{-i\theta^a T^a_{(I=1)}}\right)_{bc}\pi_c(x)\,,$$

where $\left(T^a_{(I=1)}\right)_{bc} = i\epsilon^{bac}$ are the three $SU(2)_V$ matrix generators in the three-dimensional (isospin $I = 1$) representation, write down the simplest renormalizable (i.e. having as the operator dimension not higher than 4) isospin and parity conserving interaction, i.e. the interaction satisfying

$$\begin{aligned} U^\dagger(\theta)\mathcal{H}_{\rm int}(x)U(\theta) &= \mathcal{H}_{\rm int}(x)\,, \\ \mathcal{P}^\dagger\mathcal{H}_{\rm int}(t,\mathbf{x})\mathcal{P} &= \mathcal{H}_{\rm int}(t,-\mathbf{x})\,, \end{aligned}$$

coupling the proton-neutron doublet to the triplet of pions. Remember that the intrinsic parity of pions is negative. Construct the Noether current operators $j^a_\mu(x)$ of the isospin symmetry and using the canonical (anti)commutation relations show that the isospin symmetry generators $\hat{Q}^a$ satisfy the commutation rules of the $SU(2)$ algebra. Express these generators as bilinear combinations of the proton, neutron and pion creation and annihilation operators (diagonalizing H_0).

Problem III.19 (Linear σ model)
Extend the $SU(2)_V$ isospin symmetry realized on the nucleon spinor field ψ_N (Problem III.18) to the chiral $SU(2)_L \times SU(2)_R$ symmetry whose $SU(2)_L$ and $SU(2)_R$ factors act respectively only on the left- and right-chiral parts of the nucleon field. Introduce a real scalar isospin singlet field σ (of positive parity) completing the triplet of the π_a $(a = 1, 2, 3)$ fields to a quadruplet (vector) of the $SO(4) \simeq SU(2)_L \times SU(2)_R$ group and couple this quadruplet to the nucleon field in an $SU(2)_L \times SU(2)_R$ invariant way. Observe that the constructed Lagrangian density has an additional internal symmetry. Give its physical interpretation. Find the Noether symmetry currents of all symmetries and check the algebra of charges. Add to the constructed Lagrangian density $\mathcal{L}_{\rm symm}$ a term $\Delta\mathcal{L} = c\,\sigma$ and find, using the equations of motion, the four-divergences of the symmetry currents.

Problem III.20 (Supersymmetry transformations)
Check that the transformations of the two-component Weyl *anticommuting* spinor field ψ and its conjugate $\bar{\psi}$ and of the scalar field A (and its complex conjugate)

$$\begin{aligned} \delta A\ &= \sqrt{2}\,\xi\psi\,, \\ \delta A^\star &= \sqrt{2}\,\bar{\xi}\bar{\psi}\,, \\ \delta\psi\ &= -i\sqrt{2}\,\sigma^\mu\bar{\xi}\partial_\mu A - \sqrt{2}\,m^*\xi A^*\,, \\ \delta\bar{\psi}\ &= -i\sqrt{2}\,\bar{\sigma}^\mu\xi\partial_\mu A^\star - \sqrt{2}\,m\,\bar{\xi}A\,, \end{aligned}$$

in which ξ and $\bar{\xi}$ are Grassmanian i.e. (anticommuting) spinorial transformation parameters, are the symmetry transformations of the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \partial_\mu A^\star\partial^\mu A - \frac{1}{2}m\,\psi\psi - \frac{1}{2}m^*\bar{\psi}\bar{\psi} - |m|^2 A^* A\,.$$

Find the corresponding conserved Noether current (taking into account that $\delta\mathcal{L} = \partial_\mu\mathcal{X}^\mu$). This current should have spinorial character, i.e. from the Noether theorem one should get $\xi^\alpha j^\mu_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{j}^{\mu\dot{\alpha}}$, where j^μ_α is just the Noether current (and $\bar{j}^{\mu\dot{\alpha}}$ its conjugate). Using the field equations of motion check that the current j^μ_α is conserved. Finally, check that in the quantized version of the theory the transformations of the field operators are obtained by taking their commutators with $i\bar{\xi}\bar{Q} + i\xi Q$ where $\bar{Q}$ and Q are spatial integrals of $\bar{j}^0$ and j^0, respectively.

Problem III.21 (Supersymmetry transformations cont'd)
Extend the Problem III.20 to the interacting theory with the Lagrangian density

$$\begin{aligned}\mathcal{L} = \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \partial_\mu A^\star\partial^\mu A - \frac{1}{2}m\,\psi\psi - \frac{1}{2}m^*\bar{\psi}\bar{\psi}\\ -gA\,\psi\psi - g^* A^*\bar{\psi}\bar{\psi} - \left|mA + gA^2\right|^2.\end{aligned}$$

Check that the symmetry transformations are in this case given by

$$\begin{aligned}\delta A\ &= \sqrt{2}\,\xi\psi\,,\\ \delta A^\star &= \sqrt{2}\,\bar{\xi}\bar{\psi}\,,\\ \delta\psi\ &= -i\sqrt{2}\,\sigma^\mu\bar{\xi}\partial_\mu A - \sqrt{2}\,m^*\xi A^* - \sqrt{2}\,g^*\,\xi A^* A^*\,,\\ \delta\bar{\psi}\ &= -i\sqrt{2}\,\bar{\sigma}^\mu\xi\partial_\mu A^\star - \sqrt{2}\,m\,\bar{\xi}A - \sqrt{2}\,g\,\bar{\xi}AA\,,\end{aligned}$$

and construct the corresponding conserved spinorial Noether current (taking into account that $\delta\mathcal{L} = \partial_\mu\mathcal{X}^\mu$). Check its conservation directly.

Problem III.22
Consider the Lagrangian density of the form

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{tr}\big(\partial_\mu\Sigma(x)\,\partial^\mu\Sigma^\dagger(x)\big)\,,$$

in which Σ is a unitary 2×2 matrix: $\Sigma^{-1} = \Sigma^\dagger$, $\det\Sigma = 1$ and f is a constant of dimension of mass. $\mathcal{L}$ describes low energy interactions of π mesons (in the limit $M_\pi = 0$). It is clearly invariant under the global $SU(N)_L\times SU(N)_R$ transformations

$$\Sigma(x) \to \Sigma'(x) = V(\theta_L)\,\Sigma(x)\,V^\dagger(\theta_R)\,,$$

generated by two independent unitary 2×2 matrices $V(\theta_L)$ and $V(\theta_R)$ of the fundamental representation of the $SU(2)$ group: $V(\theta) = \exp(-i\theta^a T^a)$ (here $T^a = \frac{1}{2}\,\tau^a$ and τ^a, $a =$

$1, 2, 3$ are three Pauli matrices). Parametrizing $\Sigma(x)$ with three pion fields $\pi^a(x)$ and the constant f,

$$\Sigma(x) = \exp\left(-\frac{i}{f}\,\pi^a(x)\tau^a\right),$$

find explicitly, up to terms quadratic in fields π^a, the conserved Noether currents $(j_R)^a_\mu$ and $(j_L)^a_\mu$ associated with the $SU(2)_L \times SU(2)_R$ symmetry of the Lagrangian.

Using the method of Problem III.11 derive also general formulae for the $SU(N)_L \times SU(N)_R$ currents in terms of Σ, without resorting to any explicit parametrization of Σ. Check that for $N = 2$ and the exponential parametrization of Σ the result is as previously.

Problem III.23

Find the classical equations of motion of the matrix field Σ following from the Lagrangian density

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{tr}\left\{\partial_\mu\Sigma(x)\partial^\mu\Sigma^\dagger(x) + \Sigma(x)\,\chi^\dagger(x) + \chi(x)\,\Sigma^\dagger(x)\right\},$$

in which Σ and χ are unitary $N \times N$ matrices. $\chi(x)$ plays the role of an external field.

Problem III.24

Show that if in a relativistic quantum theory model there exists a Lorentz-covariant (four-vector) current operator j^μ, then in the spectrum there cannot be massless particles of spin $s > \frac{1}{2}$ carrying a nonzero quantum number of the charge operator $\hat{Q} = \int d^3\mathbf{x}\, j_0(t, \mathbf{x})$. Similarly, show that if there exists a Lorentz-covariant energy-momentum tensor operator $T^{\mu\nu}$ such that the energy-momentum four-vector $\hat{P}^\mu = \int d^3\mathbf{x}\, T^{0\mu}(t, \mathbf{x})$, then in the spectrum there cannot be massless particles of spin $s > 1$.[4]

[4] S. Weinberg, E. Witten *Phys. Lett.* **B96** (1980), 59.