Problems in Quantum Field Theory of Fundamental Interactions IV.

Problem IV.1
Derive the vacuum functional $\Psi_{\Omega_0}[\varphi] \equiv \langle\varphi(\mathbf{x})|\Omega_0\rangle$ of the free real scalar field φ of mass M described by the action

$$I[\varphi] = \int dt \int d^3\mathbf{x} \left(\frac{c^2}{2}\,\partial_\mu\varphi\partial^\mu\varphi - \frac{M^2c^4}{2\hbar^2}\,\varphi^2\right),$$

quantized in the three-dimensional spatial box of volume V with periodic boundary conditions. Estimate vacuum fluctuations of each field Fourier mode. Take the continuum limit $V \to \infty$ of $\Psi_{\Omega_0}[\varphi]$. Show that the vacuum $|\Omega_0\rangle$ carries zero momentum.

Problem IV.2
Let φ be a free real scalar field of mass M quantized the in ordinary infinite flat Minkowski space-time and governed by the same action as in Problem IV.1. Consider a measurement of φ averaged over a cube of volume $V = L^3$, to which corresponds the operator

$$\varphi_V = \frac{1}{V}\int_{-L/2}^{+L/2} dx \int_{-L/2}^{+L/2} dy \int_{-L/2}^{+L/2} dz\, \varphi(\mathbf{x})\,.$$

Justify the order of magnitude estimate of vacuum fluctuations of the field φ_V

$$\Delta\varphi_V \equiv \sqrt{\langle\varphi_V^2\rangle} \sim \left((\Delta\varphi_{\mathbf{k}})^2\,|\mathbf{k}|^3\right)^{1/2}\,,$$

where $|\mathbf{k}| \sim L^{-1}$ and $\Delta\varphi_{\mathbf{k}} \sim E_{\mathbf{k}}^{-1/2}$ ($E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M^2}$). Generalize the result to the case of the field quantized in a nontrivial (but spatially flat) gravitational background with the line element $(ds)^2 = (dt)^2 - f(t)(d\mathbf{x})^2 = a(\eta)[(d\eta)^2 - (d\mathbf{x})^2]$, where η is the conformal time (see also Problem IV.4)
Hint: Use the decomposition of $\varphi(\mathbf{x})$ into the creation and annihilation operators to perform the integral over $d^3\mathbf{x}$ in the definition of φ_V. Then, approximate the integral over $d^3\mathbf{k}$ by the contribution of that domain of $\mathbf{k}$ which dominates it.

Problem IV.3
Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + \varphi J\,,$$

describing interaction of a real scalar field φ with an external source $J(x) \equiv J(t, \mathbf{x})$ which is assumed to vanish for $t \to \pm\infty$. Formulate the perturbative expansion of the S matrix along the standard lines (going over to the interaction picture and introducing the *in* and *out* states which for $t \to \pm\infty$ look the same as the free particle states) and compute the S matrix elements $\langle\Omega_{\text{out}}|\Omega_{\text{in}}\rangle$ and $\langle(\mathbf{k}_1, \dots, \mathbf{k}_n)_{\text{out}}|\Omega_{\text{in}}\rangle$ applying the Wick theorem. Check that the exact results (obtained by quantizing the field φ using the full set of solutions to the classical equation of motion) can be recovered after resummation of the perturbation series.

Problem IV.4
Consider a real scalar field χ whose classical dynamics is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2(\eta)\,\chi^2\,,$$

with the mass squared parameter $m^2(\eta)$ depending on time[1] η. Quantize the field $\chi(\eta, \mathbf{x})$ directly in the Heisenberg picture assuming that the complete set of real solutions $v_1(\eta, \mathbf{k})$, $v_2(\eta, \mathbf{k})$, of the equation $v'' + [\mathbf{k}^2 + m^2(\eta)]v = 0$ is known. Introduce the creation and annihilation operators $a^\dagger_v(\mathbf{k})$, $a_v(\mathbf{k})$ associated with the complex solutions $v(\eta, \mathbf{k})$ formed out of $v_1(\eta, \mathbf{k})$ and $v_2(\eta, \mathbf{k})$ and establish their commutation relations. Express the (conformal)time-dependent Hamiltonian

$$H = \frac{1}{2}\int d^3\mathbf{x}\left[\Pi^2 + (\boldsymbol{\nabla}\chi)^2 + m^2(\eta)\,\chi^2\right].$$

in terms of the operators $a_v(\mathbf{k})$ and $a^\dagger_v(\mathbf{k})$. Find the instantaneous values $v(\eta_0, \mathbf{k})$ and $v'(\eta_0, \mathbf{k})$ of the functions $v(\eta, \mathbf{k})$ and their (conformal)time derivatives which at η_0 minimize the expectation value $\langle 0_{(v)}|H(\eta_0)|0_{(v)}\rangle$ of the Hamiltonian in the state $|0_{(v)}\rangle$ annihilated by all $a_v(\mathbf{k})$.

Problem IV.5
Consider the same theory of the field χ as in Problem IV.4. Let $v(\eta, |\mathbf{k}|)$ and $u(\eta, |\mathbf{k}|)$ be two complete sets of complex solutions $v(\eta, |\mathbf{k}|)$ (for each value of $\mathbf{k}$) of the equation $v'' + [\mathbf{k}^2 + m^2(\eta)]v = 0$, normalized so that $v^*v' - v\,v^{*\prime} = i$ and $u^*u' - u\,u^{*\prime} = i$. Show that in the decomposition

$$v^*(\eta, \mathbf{k}) = \alpha(\mathbf{k})\,u^*(\eta, \mathbf{k}) + \beta(\mathbf{k})\,u(\eta, \mathbf{k})\,,$$

the coefficients $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ are η-independent. Give the explicit expressions for these coefficients and show that they satisfy the relation $|\alpha(\mathbf{k})|^2 - |\beta(\mathbf{k})|^2 = 1$. Find the relation between the creation and annihiation operators $a^\dagger_v(\mathbf{k})$, $a_v(\mathbf{k})$ associated with the v modes and $a^\dagger_u(\mathbf{k})$, $a_u(\mathbf{k})$ associated with the u modes.

Considering the χ field quantized in the spatial box L^3 with periodic boundary conditions express the "vacuum" state $|0_{(u)}\rangle$ annihilated by all operators $a_u(\mathbf{k})$ in terms of the states created by the operators $a^\dagger_v(\mathbf{k})$ from the "vacuum" $|0_{(v)}\rangle$ annihilated by all $a_v(\mathbf{k})$.

[1] Time is denoted η (instead of t) because such a Lagrangian density effectively describes the dynamics of a real scalar field $\varphi = \chi/a$ in the gravitational background with the line element $(ds)^2 = (dt)^2 - a^2(t)(d\mathbf{x})^2 \equiv a^2(\eta)[(d\eta)^2 - (d\mathbf{x})^2]$, conformally equivalent to the Minkowski space-time; η is then the conformal time.

Problem IV.6
Suppose the mode functions $v(\eta, \mathbf{k})$ of the field $\chi(\eta, \mathbf{x})$ considered in Problems IV.4 and IV.5 are such that at the moment η_1 the Hamiltonian takes the form

$$H(\eta_1) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, E(\eta_1, \mathbf{k}) \left[a_v^\dagger(\mathbf{k}) a_v(\mathbf{k}) + a_v(\mathbf{k}) a_v^\dagger(\mathbf{k})\right] ,$$

(see Problem IV.4). Suppose also that another set of mode functions, $u(\eta, \mathbf{k})$ satisfies the same condition at η_2. Assuming that the Bogolyubov coefficients relating $u(\eta, \mathbf{k})$ to $v(\eta, \mathbf{k})$ are known, compute $\langle 0_{(v)} | H(\eta_2) | 0_{(v)} \rangle$.

Problem IV.7
Check that the Schrödinger picture operators $V^i(\mathbf{x})$ and $\Pi_i(\mathbf{x})$ of the Proca field

$$V^i(\mathbf{x}) = \int d\Gamma_{\mathbf{k}} \sum_{\lambda=0,\pm 1} \left[a(\mathbf{k}, \lambda) \, \epsilon^i(\mathbf{k}, \lambda) \, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}, \lambda) \, \epsilon^{i*}(\mathbf{k}, \lambda) \, e^{-i\mathbf{k}\cdot\mathbf{x}}\right] ,$$

$$\Pi_i(\mathbf{x}) = \frac{1}{i} \int d\Gamma_{\mathbf{k}} \, E(\mathbf{k}) \sum_{\lambda=0,\pm 1} \left[a(\mathbf{k}, \lambda) \, \tilde{\epsilon}^i(\mathbf{k}, \lambda) \, e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}, \lambda) \, \tilde{\epsilon}^{i*}(\mathbf{k}, \lambda) \, e^{-i\mathbf{k}\cdot\mathbf{x}}\right] ,$$

where $\tilde{\epsilon}^i(\mathbf{k}, \lambda) = \epsilon^i(\mathbf{k}, \lambda) - (k^i/E)\epsilon^0(\mathbf{k}, \lambda)$, satisfy the canonical commutation relations $[V^i(\mathbf{x}), \Pi_j(\mathbf{y}) = \delta^{ij}\delta^{(3)}((\mathbf{x} - \mathbf{y})$, etc. provided the operators $a(\mathbf{k}, \lambda)$ and $a^\dagger(\mathbf{k}, \lambda)$ obey the standard rules

$$\left[a(\mathbf{k}, \lambda), \ a^\dagger(\mathbf{k}', \lambda')\right] = (2\pi)^3 2E(\mathbf{k}) \, \delta^{(3)}(\mathbf{k}' - \mathbf{k}) \, \delta_{\lambda'\lambda} \, ,$$

etc. Show also that the free part H_0 of the Hamiltonian,

$$H_0 = \frac{1}{2} \int d^3\mathbf{x} \left(\Pi_i \Pi_i + \frac{1}{M^2} \, (\partial_i \Pi_i)^2 + (\boldsymbol{\nabla} \times \mathbf{V})^2 + M^2 V^i V^i\right) ,$$

when expressed in terms of the $a(\mathbf{k}, \lambda)$ and $a^\dagger(\mathbf{k}, \lambda)$ operators takes the form

$$H_0 = \int d\Gamma_{\mathbf{k}} \, E(\mathbf{k}) \sum_{\lambda=0,\pm 1} a^\dagger(\mathbf{k}, \lambda) a(\mathbf{k}, \lambda) + \infty.$$

Problem IV.8
Consider a system the classical Lagrangian of which takes the form

$$L = \frac{1}{2} M_{ij} \, \dot{x}_i \dot{x}_j - \frac{1}{2} V_{ij} \, x_i x_j \, ,$$

in which

$$M = m \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} , \qquad V = m\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Identify all the constraints, check that that they are second class and construct the Hamiltonian H_{T} applying the Dirac procedure. Show that the resulting canonical equations give the same motion of the system as the Euler-Lagrange equations following from the above Lagrangian. Quantize the system and find the spectrum of its energies.

Problem IV.9
Analyze the constraints arising in the application of the Dirac quantization prescription to the Proca vector field V^μ with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} M^2 V_\mu V^\mu - V_\mu J^\mu + \mathcal{L}_{\mathrm{matter}} ,$$

in which $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and $\mathcal{L}_{\mathrm{matter}}$ is the Lagrangian density of other degrees of freedom with which the field V_μ interacts through the linear coupling to the current J^μ which (by assumption adopted for this problem) depends only on the variables of these other degrees of freedom.

Problem IV.10
Consider the theory defined by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\,\sigma^2 - g\,\sigma(\bar{\psi}\Gamma\psi) ,$$

in which ψ and $\psi^\dagger$ are Dirac spinors taking values in the Grassmann algebra and Γ is a matrix in the spinor space. Using the Dirac method perform its canonical quantization and show that it is equivalent to the theory of self-interacting Dirac fermions with the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - G(\bar{\psi}\Gamma\psi)^2 .$$

Relate G to g.

Problem IV.11
Consider a set of N classical relativistic charged particles (with electric charges $q_n e$ and masses m_n, $n = 1, \ldots, N$) interacting through the classical electromagnetic field. Write down the expression for the electromagnetic four-current $J^\mu(x)$ produced by these particles and check that it is conserved, $\partial_\mu J^\mu = 0$. Construct also the energy-momentum tensor $T^{\mu\nu}_{\mathrm{part}}$ of the particles. Show that it is symmetric. Find its four-divergence. Show by direct calculation that conserved is only the total energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{part}} + T^{\mu\nu}_{\mathrm{elmg}} ,$$

where $T^{\mu\nu}_{\mathrm{elmg}}$ is the symmetric Belinfante energy-momentum tensor of the electromagnetic field constructed in Problem III.14.

Problem IV.12

Consider the electromagnetic field interacting with some other (representing "matter") degrees of freedom through a linear coupling to the four-current $eJ^\mu = (c\rho, \mathbf{j})$; the classical Lagrangian density of the system (in the Heaviside-Lorentz system of units) reads

$$\mathcal{L} = -\frac{1}{4}\, f_{\mu\nu} f^{\mu\nu} - \frac{1}{c}\, eJ^\nu A_\nu + \mathcal{L}_{\text{matter}}\,.$$

In the Hamiltonian formulation the Gauss law $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho = eJ^0/c$, which is one of the Euler-Lagrange equations, arises as the secondary constraint $\Phi_2 = 0$, where

$$\Phi_2 \equiv c^2 \partial_i \Pi_i + eJ^0\,.$$

This follows from the requirement that the primary constraint, $\Phi_0 \equiv \Pi_0 = 0$, be compatible with the dynamics. Check that the Gauss law itself does not lead to any new constraints, i.e. that the condition $\{\Phi_2,\ H_{\rm T}\}_{\rm PB} \simeq 0$ ($H_{\rm T}$ is the total Hamiltonian which includes the primary constraints with arbitrary coefficients) is automatically satisfied if the current J^μ is conserved. Check explicitly that $\{\Phi_2,\ H_{\rm T}\}_{\rm PB} \simeq 0$ in the case of the electromagnetic field coupled to the dynamical set of nonrelativistic charged particles (the form of the current J^μ is established in Problem IV.11).

Analyse the full set of constraints arising in the Hamiltonian formulation when the Coulomb gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ is imposed as an additional primary constraints $\Phi_1 \equiv \partial_i A^i = 0$ in addition to the primary costraints $\Phi_0 \equiv \Pi_0 = 0$ following from the Lagrangian. Check also the commutators of the operator A^0 obtained by applying the Dirac quantization condition.

Problem IV.13

Express the Poincaré group generators

$$P^0 \equiv \int d^3\mathbf{x}\, T^{00}_{\text{symm}}\,, \qquad\qquad P^i \equiv \int d^3\mathbf{x}\, T^{0i}_{\text{symm}}\,,$$

and

$$J^{ij} \equiv \int d^3\mathbf{x}\, M^{0ij}\,, \qquad\qquad K^i \equiv \int d^3\mathbf{x}\, M^{00i}\,,$$

constructed out of the tensors $T^{\mu\nu}_{\text{symm}}$ and $M^{\mu\nu\lambda}$ of the free electromagnetic field found in Problem III.14 in terms of the canonical variables $A^i(\mathbf{x})$ and $\Pi_i(\mathbf{x})$ in the Coulomb gauge and check the Poincaré group algebra using the commutation rules satisfied by the operators $A^i(\mathbf{x})$ and $\Pi_i(\mathbf{x})$ (in the Schrödinger picture). Go next to the Heisenberg picture and argue that the Heisenberg picture generators are independent of time and satisfy the same algebra of commutators. Finally, check the transformation rules of the Heisenberg picture operators $A^i(x)$ and $\Pi_i(x)$.

Problem IV.14
Express the Poincaré group generators P^0, P^i, J^{ij} and K^i of the free electromagnetic field obtained in Problem IV.13 through the creation and annihilation operators and show that P^0 is just the Hamiltonian and P^i is the momentum operator

$$P^i = \int d\Gamma_{\mathbf{k}}\, k^i \sum_{\lambda=\pm 1} a^\dagger(\mathbf{k},\lambda) a(\mathbf{k},\lambda)\,.$$

Find the action of the Poincaré group generators on one-particle (one-photon) states.

Problem IV.15
The Canonical Ensemble statistical operator of the electromagnetic field in equilibrium with a heat bath of temperature T has the form

$$\hat{\rho} = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\left(e^{-\beta\hat{H}}\right)}\,, \qquad \text{where} \qquad \beta = 1/k_{\mathrm{B}}T\,.$$

Considering the electromagnetic field quantized in a box of volume $V = L^3$, find the mean number of photons with momentum $\mathbf{k}$ and polarization λ corresponding to temperature T. Find also the fluctuation of the number of photons. How many relic photons per cubic centimeter there are in the Universe at present, if $T_{\mathrm{relic}} = 2.73$ K?
Hint: Recall that the statistical average value of an observable represented by the operator $\hat{O}$ is given by $\langle\hat{O}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{O})$.

Problem IV.16 (Casimir force)
Quantize the electromagnetic field in the space between two large parallel conducting plates (perpendicular to the z axis) of area L^2 each, separated by the distance $d \ll L$. Impose periodic boundary conditions with the period L in the directions x and y. Find the Casimir force by which the plates attract each other, computing the difference of energies of zero point oscillations of the electromagnetic field quantized with the boundary conditions appropriate for conducting plates and of the field quantized with the periodic boundary conditions with the period L also in the z direction (but taking for the difference only the energy of the latter field contained in the box $L \times L \times d$).

Problem IV.17
Using the Dirac prescription for systems subject to constraints perform the canonical quantization in the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ of the spinor electrodynamics defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\,Q\bar{\psi}\gamma_\mu\psi A^\mu\,,$$

in which ψ and $\psi^\dagger$ ($\bar{\psi} \equiv \psi^\dagger\gamma^0$) are anticommuting (Grassmann algebra valued) “classical” fields of electric charge Q transforming as spinors under changes of the Lorentz frame.

Problem IV.18
Consider the electromagnetic field coupled to the complex scalar field ϕ. The Lagrangian density $\mathcal{L} = \mathcal{L}_{\rm EM} + \mathcal{L}_{\rm mat}$ of the system is

$$\mathcal{L} = -\frac{1}{4}\, f_{\mu\nu}f^{\mu\nu} + (\partial_\mu\phi^* - ie\,QA_\mu\phi^*)(\partial^\mu\phi + ie\,QA^\mu\phi) - M^2\phi^*\phi - \mathcal{H}_{\rm int}(\phi^*\phi)\,,$$

where Q is the electric charge of the field ϕ. Check that the generalized electromagnetic current J^μ defined by

$$-eJ^\mu(x) = \frac{\delta}{\delta A_\mu(x)}\int d^4y\,\mathcal{L}_{\rm mat}(y)\,,$$

is conserved and gauge invariant. Construct also the canonical energy-momentum tensor $T^{\mu\nu}$ of this theory and symmetrize it using the result of Problem III.14.

Problem IV.19
Perform the canonical quantization of the scalar electrodynamics (i.e. of the theory of the electromagnetic field coupled to one complex or two real scalar fields defined in Problem IV.18) in the Coulomb gauge using the Dirac method. Formulate also the perturbative expansion by going to the interaction picture. In particular, explain how the usual covariant Feynman rules (obtained in the path integral based quantization) are recovered in this framework.

Problem IV.20
Consider the Gupta-Bleuler quantization of the electromagnetic field coupled to a conserved current J_μ in the presence of the Nakanishi-Lautrup auxiliary field h. The system's Lagrangian density[2] is

$$\mathcal{L} = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \frac{\xi}{2}h^2 + \alpha\, h\,\partial_\mu A^\mu - (1-\alpha)A^\mu\,\partial_\mu h - eA^\mu J_\mu + \mathcal{L}_{\rm mat}\,,$$

in which $\mathcal{L}_{\rm mat}$ depends only (here by assumption) on the variables out of which the current J_μ is constructed. Show that by using the Dirac's quantization prescription one ends up with the same structure of the Schrödinger picture operators and of the Fock space of states $|\alpha_0\rangle$ as in the quantization which starts from the Lagrangian density obtained from the one given above by first eliminating the Nakanishi-Lautrup field h from the Lagrangian with the help of its classical equations of motion.

Problem IV.21
Consider the Gupta-Bleuler quantization of the electromagnetic field coupled to the complex scalar field ϕ. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}\, f_{\mu\nu}f^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (\partial_\mu\phi^* - ieA_\mu\phi^*)(\partial^\mu\phi + ieA^\mu\phi) - V(\phi^*\phi)\,.$$

[2] The Lagrangian density $\mathcal{L}$ is, up to a total derivative, independent of the arbitrary parameter α. Yet, the canonical momenta do depend on this parameter. It is the purpose of this exercise to convince onself that the resulting quantum theory is nevertheless independent of α.

Show that the Heisenberg picture operator $\Pi_0 \propto \partial_\mu A^\mu$ is free, that is, satisfies the equation

$$\partial_\mu \partial^\mu \Pi_0(x) = 0\,.$$

Formulate the perturbative expansion and show that one recovers in this framework the same results as with the covariant Feynman rules.

Problem IV.22
Write down the canonical equations satisfied by the field operators A^μ and Π_ν of the free electromagnetic field in the Gupta-Bleuler formalism (with the idefinite metric Hilbert space) for the general value of the parameter ξ in the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\,.$$

Show that they are equivalent to the Euler-Lagrange equations derived from $\mathcal{L}$. Postulating the general form of the field operators

$$A^\mu(t,\mathbf{x}) = \int d\Gamma_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, a^\mu(t,\mathbf{k})\,, \qquad \Pi_\mu(t,\mathbf{x}) = i\int d\Gamma_{\mathbf{k}}\, |\mathbf{k}|\, e^{i\mathbf{k}\cdot\mathbf{x}}\, b_\mu(t,\mathbf{k})\,,$$

solve the canonical equations for the time dependence of the operators $a^\mu(t,\mathbf{k})$ and $b_\mu(t,\mathbf{k})$ representing them as products of the creation and annihilation operators and the appropriate polarization vectors.

Problem IV.23
Using the form of the interaction picture operator $A^\mu(x)$ of in the Gupta-Bleuler approach to the electromagnetic field quantization (found in Problem IV.22) determine the corresponding free photon propagator.