
**Problem 1.** Find the expression for the time derivative of the chronological product:
\[ \partial_{\mu} \delta \left[ A(x) B(y) C(z) \right] , \]
of three arbitrary bosonic (i.e. transforming under rotations as a sum of integer spin terms) operators \( A(x), B(y) \) and \( C(z) \).

**Problem 2.** Consider the \( \varphi^4 \) theory. Justify the operator relation
\[
(\partial_x^2 + M_B^2) T \left\{ \varphi_B(x) \varphi_B(y_1) \ldots \varphi_B(y_n) \right\} + \frac{\lambda_B}{3!} T \left\{ \varphi_B^3(x) \varphi_B(y_1) \ldots \varphi_B(y_n) \right\} \]
\[ = -i \sum_{k=1}^n \delta^{(4)}(x - y_k) T \left\{ \varphi_B(y_1) \ldots \left[ \text{without } \varphi_B(y_k) \right] \ldots \varphi_B(y_n) \right\} , \]
where \( \varphi_B \equiv \varphi_H \) is the bare canonical Heisenberg picture field operator. Check this relation through order \( \lambda_B \) in the perturbative expansion for \( n = 1 \) computing its matrix element between two one-particle states.

**Hint:** Computation of the matrix elements is most straightforward using the physically renormalized field operator \( \varphi_{ph} = Z^{-1/2} \varphi_B \equiv Z^{-1/2}_{(OS)} \varphi_B \).

**Problem 3.** Assuming validity of the perturbative expansion investigate the operator \( \varphi^3_B(x) \) as the interpolating field in the \( \varphi^4 \) theory. Reduce first the matrix element
\[
\langle (p_1, p_2) - | \varphi^3_B(x) | (k_1)_+ \rangle ,
\]
computed to order \( \lambda_B^2 \) and show that the \( S \) matrix element \( \langle (p_1, p_2) - | \varphi^3_B(x) | (k_1, k_2)_+ \rangle \) can be obtained from it with the help of the LSZ prescription. Try to generalize the proof to the case of more \( \varphi^3_B \) operators used as the interpolating fields for the remaining final/initial state particles. Can the operator \( \varphi^2_B(x) \) be used in the perturbation expansion as the interpolating field?

**Problem 4.** Check (extending the analysis to the one loop order) that the equation of motion of the \( \varphi^4 \) theory
\[
(\partial_x^2 + M_{ph}^2) \varphi_B(x) = -\frac{\lambda}{3!} \varphi^3_B(x) - (M_B^2 - M_{ph}^2) \varphi_B(x) ,
\]
applied to the operator \( \varphi_B(x) \) in the LSZ formula
\[
i Z^{-1/2}_\varphi \lim_{k^2 \to M_{ph}^2} \int d^4 x e^{-ik^2 x} (\partial_x^2 + M_{ph}^2) \langle (p_1, p_2) - | \varphi_B(x) | (k_1)_+ \rangle ,
\]
leads to the same (connected part of the) $S$ matrix element $\langle(p_1, p_2)\ | (k_1, k_2)_+ \rangle$ as the standard LSZ prescription.

Try to apply the equation of motion once and then twice to transform the formula

$$(i)^2 Z^{-1}_\varphi \lim_{k_2^2 \to M^2_{\text{ph}}} \lim_{k_1^2 \to M^2_{\text{ph}}} \int d^4x \int d^4y e^{-ik_1 \cdot x} e^{-ik_2 \cdot y} (\partial_x^2 + M^2_{\text{ph}})(\partial_y^2 + M^2_{\text{ph}})$$

$$\times \langle(p_1, p_2)\ | T[\varphi_B(x)\varphi_B(y)] | \Omega_+ \rangle \, .$$

In both cases study to the one-loop order how the known form of the $S$ matrix element $\langle(p_1, p_2)\ | (k_2, k_1)_+ \rangle$ is recovered.

**Problem 5.** Suppose in a theory there are two scalar fields (complex or real) $\varphi_1$ and $\varphi_2$ which can mix, i.e. the two-point Green’s function $\langle \Omega | T[\varphi_{1B}(x_1)\varphi_{2B}(x_2)] | \Omega \rangle \neq 0$. Formulate the prescription for calculating $S$ matrix elements with particles created from the vacuum by the operators $\varphi_{1rmB}$ and $\varphi_{2B}$ in the initial and/or final state.

**Problem 6.** Consider the lowest order amplitude of the Compton scattering on spin 0 particles of electric charge $Q$. Using the appropriate equation of motion of the photon field operator in the LSZ reduction formula show that the amplitude can be obtained also from the vacuum Green’s function

$$\langle \Omega | T^*[J_{\text{EM}}(x_2) J_{\text{EM}}^\mu(x_1)\phi_B(y_2)\phi_B^\dagger(y_1)] | \Omega \rangle \, ,$$

where $J_{\text{EM}}^\mu(x) = Q \phi^\dagger(x) D^\mu_x \phi_B(x)$ is the electromagnetic current (Heisenberg picture) operator and $T^*$ is the covariant chronological product resulting from adding to the standard chronological product of the so-called “sea-gull” operator term $S^{\mu\nu}(x_2, x_1)$:

$$T^*[J_{\text{EM}}^\nu(x_2) J_{\text{EM}}^\mu(x_1) \ldots] \equiv T[J_{\text{EM}}^\nu(x_2) J_{\text{EM}}^\mu(x_1) \ldots] + T[S^{\mu\nu}(x_2, x_1) \ldots] \, .$$

Find the explicit form of $S^{\mu\nu}(x_2, x_1)$ in this case.

**Problem 7.** In the $\varphi^4$ theory in $d = 4$ dimensions construct (up to one-loop order) a renormalized operator $\frac{1}{3!} \varphi^3_R$ which has finite matrix elements between the $in$ and $out$ states. Show that if the counterterms to this operator are specified either by the MS (or MS) scheme or the by the requirement that for $q = 0$ (where $q$ is the momentum transferred through this operator) the Green’s function

$C^{(4)}_{\varphi^3}(p_1, p_2, p_3, q) = \int \frac{d^4q}{(2\pi)^4} e^{-i q \cdot y} \prod_{i=1}^3 \left( \int \frac{d^4p_i}{(2\pi)^4} e^{i p_i \cdot x_i} \right) \times \langle \Omega \ | T \{ \varphi_R(x_1) \varphi_R(x_2) \varphi_R(x_3) \left[ \frac{1}{3!} \varphi^3_R(y) \right] \} | \Omega_+ \rangle \, ,$
takes on the tree-level form, the renormalized equation of motion
\[
\left( \partial_x^2 + M_R^2 \right) \varphi_R(x) = -\lambda_R \left[ \frac{1}{3!} \varphi^3(x) \right]_R,
\]
in which \( \lambda_R \) is defined either in the \( \overline{\text{MS}} \) (or MS) scheme (in which case \( M_R^2 = \hat{M}^2 \) and \( \lambda_R = \hat{\lambda}_\mu^{2\epsilon} \)) or in the OS scheme with the zero momentum subtraction in the four-point 1PI vertex function (\( \lambda_R = \lambda_{\text{ph}} \)), is equivalent to the equation of motion for the bare canonical operator \( \varphi_B \)
\[
\left( \partial_x^2 + M_B^2 \right) \varphi_B(x) = -\frac{\lambda_B}{3!} \varphi^3(x)_B.
\]

**Problem 8.** Working in the \( \overline{\text{MS}} \) scheme with the \( \varphi^3 \) theory in \( d = 6 \) dimensions construct (up to one-loop order) renormalized operator \( \left[ \frac{1}{2} \varphi^2 \right]_R \). As in Problem 7 show that the renormalized operator equation of motion
\[
\left( \partial_x^2 + \hat{M}^2 \right) \varphi_R(x) = -\mu^{-\epsilon} \hat{g} \left[ \frac{1}{2} \varphi^2(x) \right]_R,
\]
is equivalent to the equation of motion for the bare operator \( \varphi_B \).

**Problem 9.** For the theory defined by the Lagrangian
\[
\mathcal{L} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} M^2 \varphi^2 + i \tilde{\psi} \gamma^\mu \psi - m \tilde{\psi} \psi - ig \tilde{\psi} \gamma^5 \psi \varphi - \frac{\lambda}{4!} \varphi^4
\]
in \( d = 4 \) dimensions construct (up to one-loop order) renormalized operators \( \left[ \frac{1}{3!} \varphi^3 \right]_R \), \( [\varphi \psi]_R \), \( [\bar{\psi} \gamma^5 \psi]_R \) and \( [\bar{\psi} \psi]_R \) which have finite matrix elements between the in and out states. As in the preceding Problems show that if the operator counterterms are fixed in the \( \overline{\text{MS}} \) scheme similarly as the counterterms in the interaction \( V_{\text{int}} \), the operator equations
\[
\left( \partial_x^2 + \hat{M}^2 \right) \varphi_R(x) = -\hat{\lambda} \mu^{-2\epsilon} \left[ \frac{1}{3!} \varphi^3 \right]_R (x) - i \hat{g} \mu^{-\epsilon} \left[ \bar{\psi} \gamma^5 \psi \right]_R (x),
\]
\[
(\hat{\sigma} - \hat{m}) \psi_R(x) = i \hat{g} \mu^{-\epsilon} \left[ \varphi \psi \right]_R (x),
\]
are equivalent to the equations satisfied by the bare operators \( \varphi_B(x) \) and \( \psi_B(x) \).

**Problem 10.** In \( \varphi^4 \) theory construct all possible renormalized operators of dimension 4 in the \( \overline{\text{MS}} \) (or MS) renormalization scheme.