Quantum Field Theory of Fundamental Interactions. Problems set 7.

Problem 1. Find the expression for the time derivative of the chronological product:

$$\partial_{x^0} T\left[A(x)B(y)C(z)\right] \ ,$$

of three arbitrary bosonic (i.e. transforming under rotations as a sum of integer spin terms) operators $A(x)$, $B(y)$ and $C(z)$.

Problem 2. Consider the φ^4 theory. Justify the operator relation

$$\begin{aligned}(\partial^2_{(x)} + M^2_{\rm B})\, T\left\{\varphi_{\rm B}(x)\varphi_{\rm B}(y_1)\dots\varphi_{\rm B}(y_n)\right\} + \frac{\lambda_{\rm B}}{3!}\, T\left\{\varphi^3_{\rm B}(x)\varphi_{\rm B}(y_1)\dots\varphi_{\rm B}(y_n)\right\} \\ = -i\sum_{k=1}^{n}\delta^{(4)}(x-y_k)\, T\left\{\varphi_{\rm B}(y_1)\dots[\text{without } \varphi_{\rm B}(y_k)]\dots\varphi_{\rm B}(y_n)\right\} \ ,\end{aligned}$$

where $\varphi_{\rm B} \equiv \varphi_H$ is the bare canonical Heisenberg picture) field operator. Check this relation through order $\lambda_{\rm B}$ in the perturbative expansion for $n = 1$ computing its matrix element between two one-particle states.
Hint: Computation of the matrix elements is most straightforward using the physically renormalized field operator $\varphi_{\rm ph} = \mathcal{Z}^{-1/2}\varphi_{\rm B} \equiv Z^{-1/2}_{\rm (OS)}\varphi_{\rm B}$.

Problem 3. Assuming validity of the perturbative expansion investigate the operator $\varphi^3_{\rm B}(x)$ as the interpolating field in the φ^4 theory. Reduce first the matrix element

$$\langle(\mathbf{p}_1,\mathbf{p}_2)_-|\varphi^3_{\rm B}(x)|(\mathbf{k}_1)_+\rangle \ ,$$

computed to order $\lambda^2_{\rm B}$ and show that the S matrix element $\langle(\mathbf{p}_1,\mathbf{p}_2)_-|(\mathbf{k}_1,\mathbf{k}_2)_+\rangle$ can be obtained from it with the help of the LSZ prescription. Try to generalize the proof to the case of more $\varphi^3_{\rm B}$ operators used as the interpolating fields for the remaining final/initial state particles. Can the operator $\varphi^2_{\rm B}(x)$ be used in the perturbation expansion as the interpolating field?

Problem 4. Check (extending the analysis to the one loop order) that the equation of motion of the φ^4 theory

$$(\partial^2_x + M^2_{\rm ph})\varphi_{\rm B}(x) = -\frac{\lambda}{3!}\varphi^3_{\rm B}(x) - (M^2_{\rm B} - M^2_{\rm ph})\varphi_{\rm B}(x) \ ,$$

applied to the operator $\varphi_{\rm B}(x)$ in the LSZ formula

$$i\mathcal{Z}^{-1/2}_{\varphi}\lim_{k^2_2\to M^2_{\rm ph}}\int d^4x\, e^{-ik_2\cdot x}\,(\partial^2_x + M^2_{\rm ph})\,\langle(\mathbf{p}_1,\mathbf{p}_2)_-|\varphi_{\rm B}(x)|(\mathbf{k}_1)_+\rangle \ ,$$

leads to the same (connected part of the) S matrix element $\langle(\mathbf{p}_1,\mathbf{p}_2)_-|(\mathbf{k}_1,\mathbf{k}_2)_+\rangle$ as the standard LSZ prescription.

Try to apply the equation of motion once and then twice to transform the formula

$$(i)^2\mathcal{Z}_\varphi^{-1}\lim_{k_2^2\to M^2_{\rm ph}}\lim_{k_1^2\to M^2_{\rm ph}}\int d^4x\int d^4y\, e^{-ik_1\cdot x}\, e^{-ik_2\cdot y}\,(\partial_x^2+M^2_{\rm ph})(\partial_y^2+M^2_{\rm ph})$$
$$\times\langle(\mathbf{p}_1,\mathbf{p}_2)_-|T[\varphi_{\rm B}(x)\varphi_{\rm B}(y)]|\Omega_+\rangle\ .$$

In both cases study to the one-loop order how the known form of the S matrix element $\langle(\mathbf{p}_1,\mathbf{p}_2)_-|(\mathbf{k}_2,\mathbf{k}_1)_+\rangle$ is recovered.

Problem 5. Suppose in a theory there are two scalar fields (complex or real) φ_1 and φ_2 which can mix, i.e. the two-point Green's function $\langle\Omega|T[\varphi_{1\rm B}(x_1)\varphi_{2\rm B}(x_2)]|\Omega\rangle\neq 0$. Formulate the prescription for calculating S matrix elements with particles crated from the vacuum by the operators φ_{1rmB} and $\varphi_{2\rm B}$ in the initial and/or final state.

Problem 6. Consider the lowest order amplitude of the Compton scattering on spin 0 particles of electric charge Q. Using the appropriate equation of motion of the photon field operator in the LSZ reduction formula show that the amplitude can be obtained also from the vacuum Green's function

$$\langle\Omega|T^*[J^\nu_{\rm EM}(x_2)J^\mu_{\rm EM}(x_1)\phi_{\rm B}(y_2)\phi^\dagger_{\rm B}(y_1)]|\Omega\rangle\ ,$$

where $J^\mu_{\rm EM}(x)=Q\,i\phi^\dagger_{\rm B}(x)\overleftrightarrow{\partial}^\mu_x\phi_{\rm B}(x)$ is the electromagnetic current (Heisenberg picture) operator and T^* is the covariant chronological product resulting from adding to the standard chronological product of the so-called "sea-gull" operator term $S^{\nu\mu}(x_2,x_1)$:

$$T^*[J^\nu_{\rm EM}(x_2)J^\mu_{\rm EM}(x_1)\dots]\equiv T[J^\nu_{\rm EM}(x_2)J^\mu_{\rm EM}(x_1)\dots]+T[S^{\nu\mu}(x_2,x_1)\dots]\ .$$

Find the explicit form of $S^{\nu\mu}(x_2,x_1)$ in this case.

Problem 7. In the φ^4 theory in $d=4$ dimensions construct (up to one-loop order) a renormalized operator $\left[\frac{1}{3!}\varphi^3\right]_R$ which has finite matrix elements between the *in* and *out* states. Show that if the counterterms to this operator are specified either by the $\overline{\rm MS}$ (or MS) scheme or the by the requirement that for $q=0$ (where q is the momentum transfered through this operator) the Green's function

$$G^{(4)}_{\varphi^3}(p_1,p_1,p_3,q)=\int\frac{d^4q}{(2\pi)^4}\,e^{-iq\cdot y}\prod_{i=1}^3\left(\int\frac{d^4p_i}{(2\pi)^4}\,e^{ip_i\cdot x_i}\right)$$
$$\times\langle\Omega_-|T\left\{\varphi_{\rm R}(x_1)\varphi_{\rm R}(x_2)\varphi_{\rm R}(x_3)\left[\frac{1}{3!}\varphi^3(y)\right]_{\rm R}\right\}|\Omega_+\rangle\ ,$$

takes on the tree-level form, the renormalized equation of motion

$$\left(\partial_x^2 + M_{\rm R}^2\right)\varphi_R(x) = -\lambda_R\left[\frac{1}{3!}\varphi^3(x)\right]_{\rm R} ,$$

in which λ_R is defined either in the $\overline{\rm MS}$ (or MS) scheme (in which case $M_{\rm R}^2 = \hat{M}^2$ and $\lambda_R = \hat{\lambda}\mu^{-2\epsilon}$) or in the OS scheme with the zero momentum subtraction in the four-point 1PI vertex function ($\lambda_R = \lambda_{\rm ph}$), is equivalent to the equation of motion for the bare canonical operator $\varphi_{\rm B}$

$$\left(\partial_x^2 + M_{\rm B}^2\right)\varphi_B(x) = -\frac{\lambda_B}{3!}\varphi^3(x)_{\rm B} .$$

Problem 8. Working in the $\overline{\rm MS}$ scheme with the φ^3 theory in $d = 6$ dimensions construct (up to one-loop order) renormalized operator $\left[\frac{1}{2}\varphi^2\right]_R$. As in Problem 7 show that the renormalized operator equation of motion

$$\left(\partial_x^2 + \hat{M}^2\right)\varphi_R(x) = -\mu^{-\epsilon}\hat{g}\left[\frac{1}{2}\varphi^2(x)\right]_{\rm R} ,$$

is equivalent to the equation of motion for the bare operator $\varphi_{\rm B}$.

Problem 9. For the theory defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}M^2\varphi^2 + i\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi - ig\bar{\psi}\gamma^5\psi\varphi - \frac{\lambda}{4!}\varphi^4$$

in $d = 4$ dimensions construct (up to one-loop order) renormalized operators $\left[\frac{1}{3!}\varphi^3\right]_R$, $[\varphi\psi]_R$, $\left[\bar{\psi}\gamma^5\psi\right]_R$ and $\left[\bar{\psi}\psi\right]_R$ which have finite matrix elements between the *in* and *out* states. As in the preceding Problems show that if the operator counterterms are fixed in the $\overline{\rm MS}$ scheme similarly as the counterterms in the interaction $V_{\rm int}$, the operator equations

$$\begin{aligned}\left(\partial_x^2 + \hat{M}^2\right)\varphi_R(x) &= -\hat{\lambda}\mu^{-2\epsilon}\left[\frac{1}{3!}\varphi_{\rm R}^3\right]_R(x) - i\hat{g}\mu^{-\epsilon}\left[\bar{\psi}\gamma^5\psi\right]_R(x) ,\\ \left(\partial\!\!\!/ - \hat{m}\right)\psi_{\rm R}(x) &= i\hat{g}\mu^{-\epsilon}\left[\varphi\psi\right]_R(x) ,\end{aligned}$$

are equivalent to the equations sitisfied by the bare operators $\varphi_{\rm B}(x)$ and $\psi_{\rm B}(x)$.

Problem 10. In φ^4 theory construct all possible renormalized operators of dimension 4 in the $\overline{\rm MS}$ (or MS) renormalization scheme.