Zbadać punkty krytyczne funkcji $f:(x,y)\in\mathbb{R}^2 \longrightarrow z\in\mathbb{R}$ zdefiniowanej niejawnie równaniem

a) $$\frac{1}{2}(x^2+y^2)z^3+xyz^2+z-2=0.$$

Zdefiniujemy funkcję $F:\mathbb{R}^3\longrightarrow\mathbb{R}$ postaci

$$F(x,y,z)=\frac{1}{2}(x^2+y^2)z^3+xyz^2+z-2.$$

Jeżeli $F(x,y,z)=0$ i $\left.\frac{\partial F}{\partial z}\right|_{p=(x_0,y_0,z_0)}\neq 0$ to ~~że~~ $F(x,y,z(x,y))=0$ dla pewnej funkcji $z(x,y)$ takiej, że $z(x_0,y_0)=z_0$ i z jest zdefiniowana w otoczeniu punktu (x_0,y_0). Teraz,

$$\frac{\partial F(x,y,z(x,y))}{\partial x}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \overset{\frac{\partial F}{\partial z}\neq 0}{\Longrightarrow} \frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial F(x,y,z(x,y))}{\partial y}=\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0 \overset{\frac{\partial F}{\partial z}\neq 0}{\Longrightarrow} \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial y}}$$

Więc ~~f~~ ma punkt krytyczny w punkcie p wtedy i tylko wtedy

$$\left.\frac{\partial F}{\partial x}\right|_p=0 \qquad \left.\frac{\partial F}{\partial y}\right|_p=0$$

W naszym przypadku

$$\frac{\partial F}{\partial x}=xz^3+yz^2, \qquad \frac{\partial F}{\partial y}=yz^3+xz^2.$$

W punktach krytycznych

$\underline{z\neq 0 \text{ bo } F(x,y,0)\neq 0}$

$$\left.\begin{array}{l} xz^3+yz^2=0 \Rightarrow (xz+y)z^2=0 \\ yz^3+xz^2=0 \Rightarrow (yz+x)z^2=0 \end{array}\right\} \quad \begin{array}{l} xz+y=0 \\ yz+x=0 \end{array} \Rightarrow \begin{array}{l} (x-y)z+(y-x)=0 \Rightarrow \\ (x-y)(z-1)=0\,. \end{array}$$

Jeżeli $x=y$, to $xz+y=0 \Rightarrow \begin{cases} x=0 \Rightarrow y=0 \quad F(0,0,z)=0 \Rightarrow z=2 \\ \underset{x\neq 0}{z=-1}. \quad Z \ F(x,x,-1)=0 \Rightarrow -x^2+x^2-1-2\neq 0 \text{ !!} \end{cases}$ nie wozw

$z=1$, to $xz+y=0 \Rightarrow x=-y \quad Z \ F(y,-x,1)=0 \Rightarrow x^2-x^2+1-2\neq 0$!! niewozw

Jedyne rozwiązanie, to $(0,0,2)$. Teraz,

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\to 0 \qquad 0 \text{ w punktach krytycznych}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right) = -\frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial z} - \cancel{\frac{\partial F}{\partial x}}\frac{\partial^2 F}{\partial x \partial z}}{\left(\frac{\partial F}{\partial z}\right)^2} = -\frac{\frac{\partial^2 F}{\partial x^2}}{\frac{\partial F}{\partial z}}$$

Podobnie,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\frac{\partial^2 F}{\partial x \partial y}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{\frac{\partial^2 F}{\partial y^2}}{\frac{\partial F}{\partial z}}.$$

Teraz

$$H = -\begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} \frac{1}{\frac{\partial F}{\partial z}} =$$

$$\frac{\partial^2 F}{\partial x^2} = z^3 \qquad \frac{\partial^2 F}{\partial y^2} = z^3 \qquad \frac{\partial^2 F}{\partial x \partial y} = z^2$$

$$\frac{\partial F}{\partial z} = \frac{3}{2}(x^2+y^2)z^2 + 2xyz + 1$$

Więc, w punktach krytycznych

$$H = -\frac{1}{1}\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} \Rightarrow$$ $(0,0,2)$ to maksimum!!

b) $\frac{1}{2}(x^2+y^2)z^3 + xyz^2+1=0$

Zdefiniujemy funkcję $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$ postaci

$$F(x,y,z) = \frac{1}{2}(x^2+y^2)z^3 + xyz^2+1$$

Punkty krytyczne funkcji $z(x,y)$ są rozwiązaniami układu

$$\frac{\partial F}{\partial x} = 0 = \quad xz^3+yz^2=0 \Rightarrow \quad z^2(xz+y)=0.$$

$$\frac{\partial F}{\partial y} = 0 = \quad yz^3+xz^2=0 \Rightarrow \quad z^2(x+yz)=0.$$

Można zauważyć, że nie istnieje punkt $(x,y,0)$ taki, że $F(x,y,0)=0$, więc takie punkty nie należą do dziedziny $z(x,y)$. Więc

$$\left.\begin{aligned} z^2(xz+y)=0 &\Rightarrow \quad xz+y=0 \\ z^2(x+yz)=0 &\Rightarrow \quad x+yz=0 \end{aligned}\right\} \Rightarrow (x-y)z-(y-x)=0$$

$$(x-y)(z-1)=0.$$

Jeżeli $x=y\neq 0$, to $\begin{cases} F(x,x,-1) \overset{\Uparrow}{=} -x^2 + x^2 + 1 = 1 > 0 \\ xz+x=0 \Rightarrow z=-1 \end{cases}$

Nie istnieje takie rozwiązanie.

$x=y=0$ to $\Big\}\ F(0,0,z)=0 \Rightarrow 1=0$!! nie istnieje rozwiązanie.

$z=1 \Rightarrow \quad \overset{xz+y=0}{x \overset{\Downarrow}{=} -y}. \quad \Rightarrow \quad F(x,-x,1)=1\neq 0$

nie ma rozwiązania.

IV

Zbadać punkty krytyczne funkcji $f: (x,y)\in\mathbb{R}^2_+ \longrightarrow z\in\mathbb{R}$

zdefiniowanej niejawnie równaniem

$$F = (x+z)(y+z)\left(1+\frac{z}{xy}\right)-8$$

Punkty krytyczne tej f są rozwiązaniami układu

$$\frac{\partial F}{\partial x}=0 \qquad \frac{\partial F}{\partial y}=0$$

Teraz

$$\frac{\partial F}{\partial x} = (y+z)\left(1+\frac{z}{xy}\right) + (x+z)(y+z)\left(-\frac{z}{x^2y}\right)$$

$$= (y+z)\left[1+\cancel{\frac{z}{xy}}-\cancel{\frac{z}{xy}}-\frac{z^2}{x^2y}\right] = \frac{(y+z)(x^2y-z^2)}{x^2y}$$

symetria $y\leftrightarrow x$

$$\frac{\partial F}{\partial y}=\frac{(x+z)(y^2x-z^2)}{y^2x}$$

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=0 \Rightarrow \left\{\begin{array}{l} x^2y-z^2=0 \\ y^2x-z^2=0 \end{array}\right\} \Rightarrow x^2y=yx^2 \Rightarrow \boxed{y=x}$$

$$\Rightarrow z^2=x^3 \Rightarrow \boxed{z=\pm x^{3/2}}$$

Łatwo zauważyć, że

$y+z\neq 0$ i $x+z\neq 0$.

Inaczej, $F(x,y,z)\neq 0$.

Pozostało, $F(x,x,\pm x^{3/2})=0 \Rightarrow \quad (x\pm x^{3/2})(x\pm x^{3/2})\left(1\pm\frac{1}{x^{1/2}}\right)-8=0$

$t=x^{1/2} \Rightarrow \quad (t^2\pm t^3)(t^2\pm t^3)\left(1\pm\frac{1}{t}\right)-8=0 \rightarrow$

$$t^4(1\pm t)^2\left(1\pm\frac{1}{t}\right)=t^3(1\pm t)^2(t\pm 1)=8$$

Przypadek a) $t^3(1+t)^3=8 \Rightarrow t(1+t)=2 \Rightarrow t+t^2=2 \Rightarrow$

$t=\frac{-1\pm\sqrt{1+4\cdot 2}}{2}$ $\nearrow \boxed{t_+=1} \Leftarrow$ skoro wybraliśmy $t>0$. $\searrow t_-=-2$

Więc, $x=1 \Rightarrow y=1$ Punkt krytyczny $(1,1,1)$

$$H=-\frac{1}{\frac{\partial F}{\partial z}}\begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial y\partial x} \\ \frac{\partial^2 F}{\partial x\partial y} & \frac{\partial^2 F}{\partial y^2}\end{pmatrix}$$

$$\begin{cases} \frac{\partial^2 F}{\partial x^2}=\frac{2z^2}{x^3y}(y+z) \\ \frac{\partial^2 F}{\partial y^2}=\frac{2z^2}{y^3x}(x+z) \\ \frac{\partial^2 F}{\partial x\partial y}=z\left(1-\frac{z^2}{y^2x}\right)+(x+z)\left(+\frac{2z}{y^2x^2}\right)\end{cases}$$

symetria $x\Leftrightarrow y$.

$$\frac{\partial F}{\partial z}=1+x+y+2\left(1+\frac{1}{y}+\frac{1}{x}\right)+\frac{3z^2}{xy}$$

W punkcie $(1,1,1)$

$$H=-\frac{1}{\frac{\partial F}{\partial z}}\begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}=-\frac{1}{12}\begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}$$

Maksymum
$D_1<0$
$D_2=\frac{1}{3}>0$

Przypadek b)

Dla $t^3(1-t)^3=8 \Rightarrow$ Punkt $(4,4,-8)$ i

$$H=-\frac{1}{2}\begin{pmatrix}1 & 3/2\\ 3/2 & 1\end{pmatrix} \Rightarrow \begin{matrix} D_1<0 \\ D_2>0\end{matrix}$$ maksymum.

Znaleźć pochodną odwzorowania $(u(x,y),v(x,y))$ w punkcie $(x,y,u,v)=(1,2,3,4)$ niejawnie danego równaniami $F:\mathbb{R}^4\to\mathbb{R}^2$ $F(x,y,u,v)=0$, gdzie

$$F(x,y,u,v)=(x+y+u+v-10,\ x^2+y^2+u^2+v^2-30)$$

Wiemy, że $F=0$ określa odwzorowanie $(u(x,y),v(x,y))$ gdy

$$\begin{vmatrix}\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v}\\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}\end{vmatrix}\neq 0$$

Τελος

$$F(x,y,u(x,y),v(x,y))=0 \Rightarrow$$

$$\frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F_1}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F_2}{\partial v}\frac{\partial v}{\partial x}=0$$

Μητρικώς:

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix}=-\begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix}=-\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 2u & 2v \end{pmatrix} \rightarrow \frac{1}{2(v-u)}\begin{pmatrix} 2v & -1 \\ -2u & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix}=\frac{-1}{2(v-u)}\begin{pmatrix} 2v & -1 \\ -2u & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2x \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{\cancel{2}(v-x)}{v-u} \\ -\frac{x-u}{v-u} \end{pmatrix}$$

Podobnie:

$$\begin{pmatrix} \frac{\partial u}{\partial y} \\[2ex] \frac{\partial v}{\partial y} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\[2ex] \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial y} \\[2ex] \frac{\partial F_2}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial y} \\[2ex] \frac{\partial v}{\partial y} \end{pmatrix} = -\frac{1}{2(v-u)} \begin{pmatrix} 2v & -1 \\[2ex] -2u & 1 \end{pmatrix} \begin{pmatrix} 1 \\[2ex] 2y \end{pmatrix} = \begin{pmatrix} \frac{v-y}{u-v} \\[2ex] \frac{y-u}{u-v} \end{pmatrix}.$$