Znaleźć objętość $|T_n|$ n-wymiarowego sympleksu

$$T_n := \{x \in \mathbb{R}^n,\ x_1 \geq 0, \dots, x_n \geq 0,\ x_1 + x_2 + \dots + x_n \leq h,\ h > 0\}.$$

Widać, że

$$|T_n| = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \int_0^{h-x_1-x_2} dx_3 \dots \int_0^{h-x_1-\dots-x_{n-2}} dx_{n-1} \int_0^{h-x_1-\dots-x_{n-1}} dx_n$$

Więc,

$$|T_n| = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \dots \int_0^{h-x_1-\dots-x_{n-2}} dx_{n-1} \left[\overset{x_n}{\cancel{h-x_1-\dots-x_{n-1}-x_n}}\right]_0^{h-x_1-\dots-x_{n-1}}$$

$$|T_n| = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \dots \int_0^{h-x_1-\dots-x_{n-3}} dx_{n-2} \left[\frac{(h-x_1-\dots-x_{n-1})^2}{-2}\right]_0^{x_{n-1}=h-x_1-\dots-x_{n-2}}$$

$$|T_n| = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \dots \int_0^{h-x_1-\dots-x_{n-4}} dx_{n-3} \frac{(h-x_1-\dots-x_{n-2})^3}{3!}$$

$$\Rightarrow \boxed{|T_n| = \frac{h^n}{n!}}$$

$n=1$ $|T_1| = h$

$|T_2| = h^2/2$

$|T_3| = h^3/6$

Odwracając kolejność całkowania wykazać, że

$$J = \int_a^b dx_n \int_a^{x_n} dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \dots \int_a^{x_3} dx_2 \int_a^{x_2} f(x_1)\,dx_1 = \int_a^b dx\, f(x) \frac{(b-x)^{n-1}}{(n-1)!}$$

Dowód: Widać, że

$a \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq \dots \leq x_n = b$ $(x_1, \dots, x_n) \Rightarrow$

$$J = \int_a^b dx_1 f(x_1) \int_{x_1}^b dx_2 \dots \int_{x_{n-1}}^b dx_n = \int_a^b dx_1 f(x_1) \dots \int_{x_{n-2}}^b dx_{n-1} (b - x_{n-1})\Big]_{x_{n-1}}^{b}$$

$=$

$$I = \int_a^b dx_1\, f(x_1) \int_{x_1}^b dx_2 \dots \int_{x_{n-2}}^b dx_{n-2} \frac{(b-x_{n-2})^2}{2}$$

$$= \int_a^b dx_1\, f(x_1) \frac{(b-x_1)^{n-1}}{(n-1)!}$$

Obliczyć pole ograniczone krzywą $(x^2+y^2)^2 = 2a^2(x^2-y^2)$

$$S = \iint_D dx\,dy$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & +r\cos\varphi \end{vmatrix}$$

$$dx\,dy = |J|\,dr\,d\varphi$$

$$dx\,dy = r\,dr\,d\varphi$$

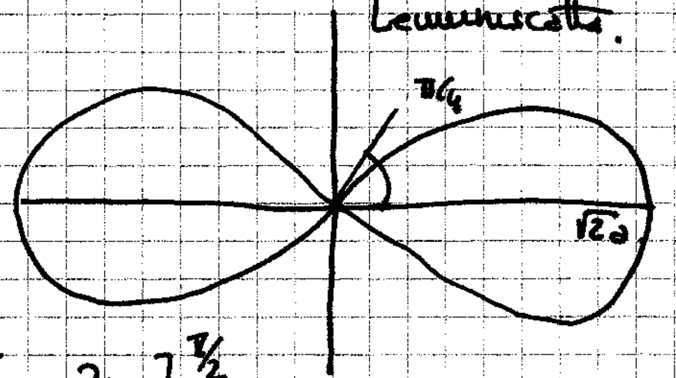
$$= r$$

$$(x^2+y^2)^2 = 2a^2(x^2-y^2) \Rightarrow r^4 = 2a^2 r^2 (\cos^2\varphi - \sin^2\varphi)$$

$$\Rightarrow r^2 = 2a^2 \cos 2\varphi .$$

$$S = \iint_D r\,dr\,d\varphi$$

Lemniscata.



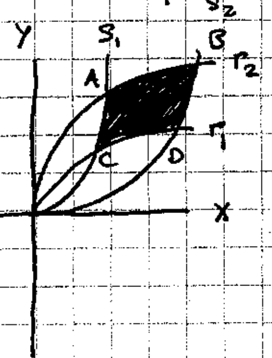
$$S = 4 \int_0^{\pi/2} \int_0^{\sqrt{2}a\sqrt{\cos 2\varphi}} r\,dr\,d\varphi$$

$$= 2 \int_0^{\pi/2} \int_0^{2a^2\cos 2\varphi} dr^2\,d\varphi$$

$$= 2 \int_0^{\pi/2} 2a^2 \cos 2\varphi\, d\varphi = 4a^2 \left[\frac{\sin 2\varphi}{2}\right]_0^{\pi/2}$$

$$= 2a^2 .$$

Znaleźć pole figury ograniczonej parabolami:



$y^2 = px \equiv r_1$ $\qquad 0 < p < q$

$y^2 = qx \equiv r_2$ $\qquad 0 < a < b$

$x^2 = ay \equiv s_1$

$x^2 = by \equiv s_2$

Obliczmy punkty A, B, C i D

W tym przypadku to nie potrzebny

→ w podobnej sytuacji wszystkie mogą być potrzebne

A to rozwiązanie równania $\left.\begin{matrix} y^2 = qx \\ x^2 = ay \end{matrix}\right\} \Rightarrow y^4 = q^2 x^2 = q^2 a y \Rightarrow$

$y^3 = q^2 a \Rightarrow y = q^{2/3} a^{1/3}$

$\Rightarrow x = q^{4/3} a^{2/3} / q = q^{1/3} a^{2/3}.$

B to rozwiązanie równania $\left.\begin{matrix} y^2 = qx \\ x^2 = by \end{matrix}\right\} \Rightarrow \begin{cases} y = q^{2/3} b^{1/3} \\ x = q^{1/3} b^{2/3} \end{cases}$

C to rozwiązanie równania $\begin{matrix} y^2 = px \\ x^2 = ay \end{matrix} \Rightarrow \begin{cases} y = p^{2/3} a^{1/3} \\ x = p^{1/3} a^{2/3}. \end{cases}$

D to rozwiązanie równania $\begin{matrix} y^2 = px \\ x^2 = by \end{matrix} \Rightarrow \begin{cases} y = p^{2/3} b^{1/3} \\ x = p^{1/3} b^{2/3} \end{cases}$

Teraz

$$\mu(D) = \iint_D d\overset{dxdy}{\overset{\shortparallel}{S}} = \iint_D |J| du_1 \, du_2$$

$$u_1 = \frac{y^2}{x} \quad u_2 = \frac{x^2}{y} \qquad |J_{(u_1 u_2) \to (x,y)}| = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{vmatrix}$$

$\Rightarrow u_1 x = y^2 \wedge u_2^2 y^2 = x^4 \Rightarrow x^4 = u_2^2 u_1 x \Rightarrow$

$\Rightarrow x^3 = u_2^2 u_1 \Rightarrow x = u_2^{2/3} u_1^{1/3}$

$$y = \frac{x^2}{u_2} = \frac{u_2^{4/3} u_1^{2/3}}{u_2} = u_2^{1/3} u_1^{2/3}.$$

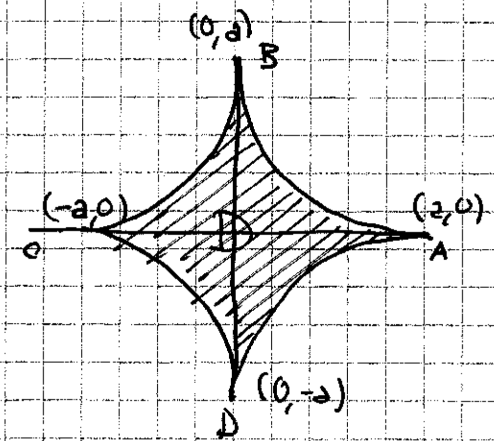
$$\frac{\partial x}{\partial u_1} = \frac{1}{3} u_2^{2/3} u_1^{-2/3} \qquad \frac{\partial x}{\partial u_2} = \frac{2}{3} u_2^{-1/3} u_1^{1/3}$$

$$\frac{\partial y}{\partial u_1} = \frac{2}{3} u_1^{-1/3} u_2^{1/3} \qquad \frac{\partial y}{\partial u_2} = \frac{1}{3} u_2^{-2/3} u_1^{2/3}$$

$$\Rightarrow \quad |J| = \begin{vmatrix} u_2^{2/3} u_1^{-2/3} & 2 u_2^{-1/3} u_1^{1/3} \\ 2 u_1^{-1/3} u_2^{1/3} & u_2^{-2/3} u_1^{2/3} \end{vmatrix} \frac{1}{9} = -\frac{1}{3}$$

$$\mu(D) = \iint_D dS = \iint_D dx\,dy = \frac{1}{3} \int_a^b \int_p^q du_1\,du_2 = \left| \frac{(q-p)(b-a)}{3} \right|.$$

Znaleźć pole asteroidy $x^{2/3} + y^{2/3} = a^{\frac{2}{3}}$



Wprowadzamy zamianę zmiennych.

$x = t\cos^3\varphi$

$y = t\sin^3\varphi$

Z tego

$$dS = dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \varphi} \end{vmatrix} dt\,d\varphi = \begin{vmatrix} \cos^3\varphi & -3t\cos^2\varphi\sin\varphi \\ \sin^3\varphi & +3t\sin^2\varphi\cos\varphi \end{vmatrix} dt\,d\varphi$$

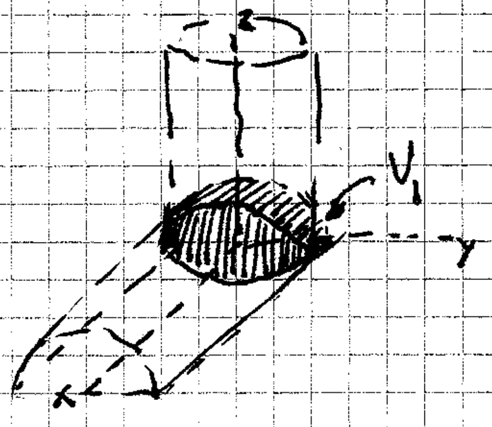
$$= +3t(\cos^4\varphi\sin^2\varphi + \cos^2\varphi\sin^4\varphi) = -3t(\cos^2\varphi\sin^2\varphi)$$

$$= +\frac{3}{4} t \sin^2 2\varphi \Rightarrow$$

$$\mu(D) = \iint_D dS = \iint_D dx\,dy = \int_0^{2\pi}\int_0^a +\frac{3}{4} t \sin^2 2\varphi \, dt\,d\varphi.$$

$$= \frac{3}{8} a^2 \int_0^{2\pi} \sin^2 2\varphi \, d\varphi = = \frac{3\pi a^2}{8}.$$

Dwa walce o jednakowym promieniu $a>0$ (i nieskończonej objętości) przenikają się wzajemnie tak, że ich osie symetrii obrotowej przecinają się pod kątem $\varphi = \pi/2$. Oblicz objętość wspólnej części obu walców.



$$V_1 = \iiint_D dV = 8\int_0^{\pi/2}\int_0^{a}\int_0^{\sqrt{a^2-r^2\sin^2\varphi}} r\,dz\,dr\,d\varphi$$

$$= 8\int_0^{\pi/2}\int_0^{a} r\left[\sqrt{a^2-r^2\sin^2\varphi}\right] dr\,d\varphi$$

$$= 8\int_0^{\pi/2}\left[-\frac{(a^2-r^2\sin^2\varphi)^{\frac{3}{2}}}{3\sin^2\varphi}\right]_0^{a} d\varphi$$

$$= \frac{8}{3}\int_0^{\pi/2}\left[-\frac{a^3\cos^3\varphi}{\sin^2\varphi}+\frac{a^3}{\sin^2\varphi}\right] d\varphi$$

$$= \frac{8}{3}a^3\int_0^{\pi/2}\frac{1-\cos^3\varphi}{\sin^2\varphi}\,d\varphi = \frac{16}{3}a^3$$

$\varphi \in [0, 2\pi]$

$r \in [0, a]$

$z \in [0, \sqrt{a^2-y_1^2}\,]$

$r^2\sin^2\varphi$