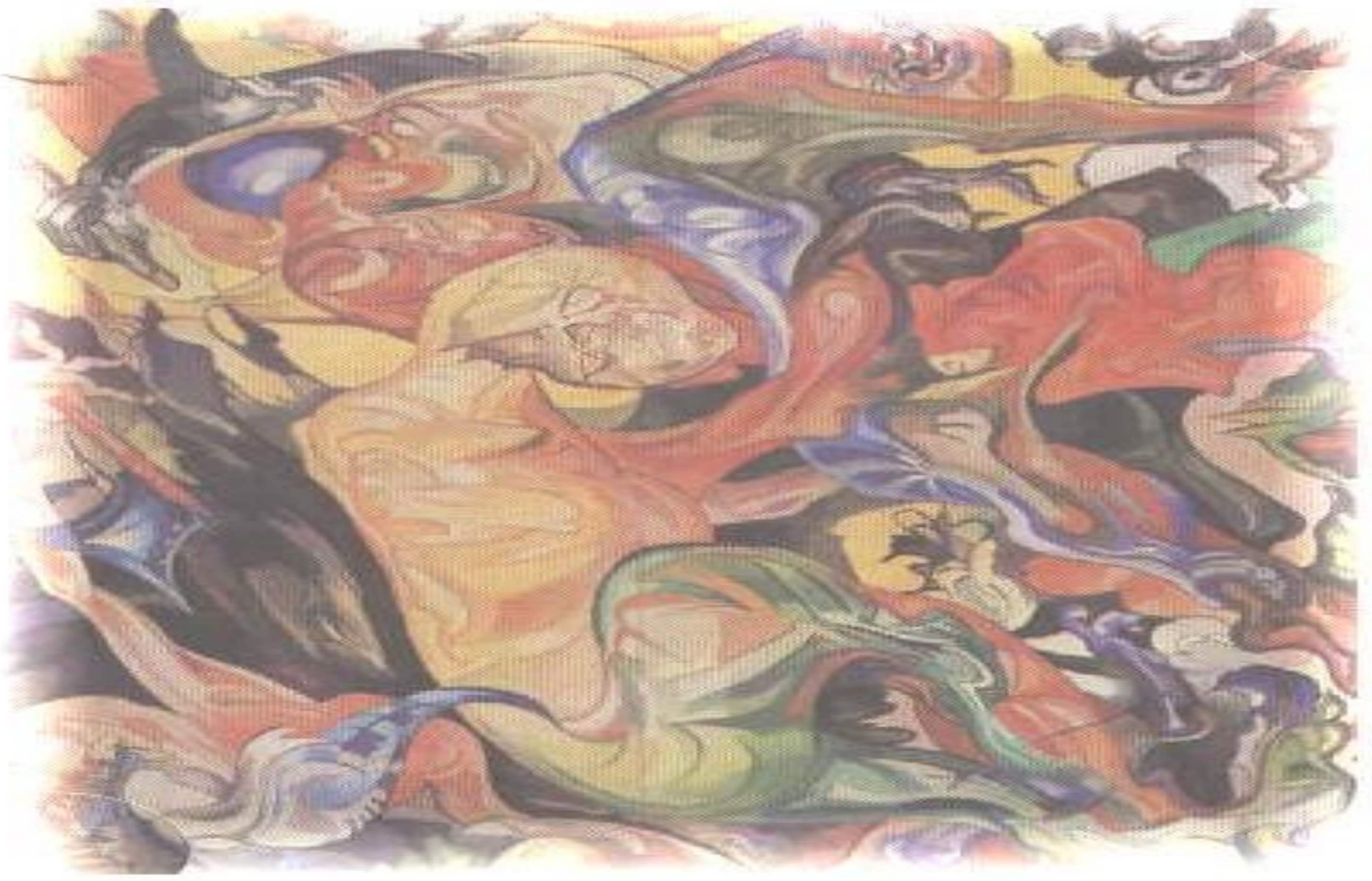
Entangling power of chaotic quantum systems – A case study



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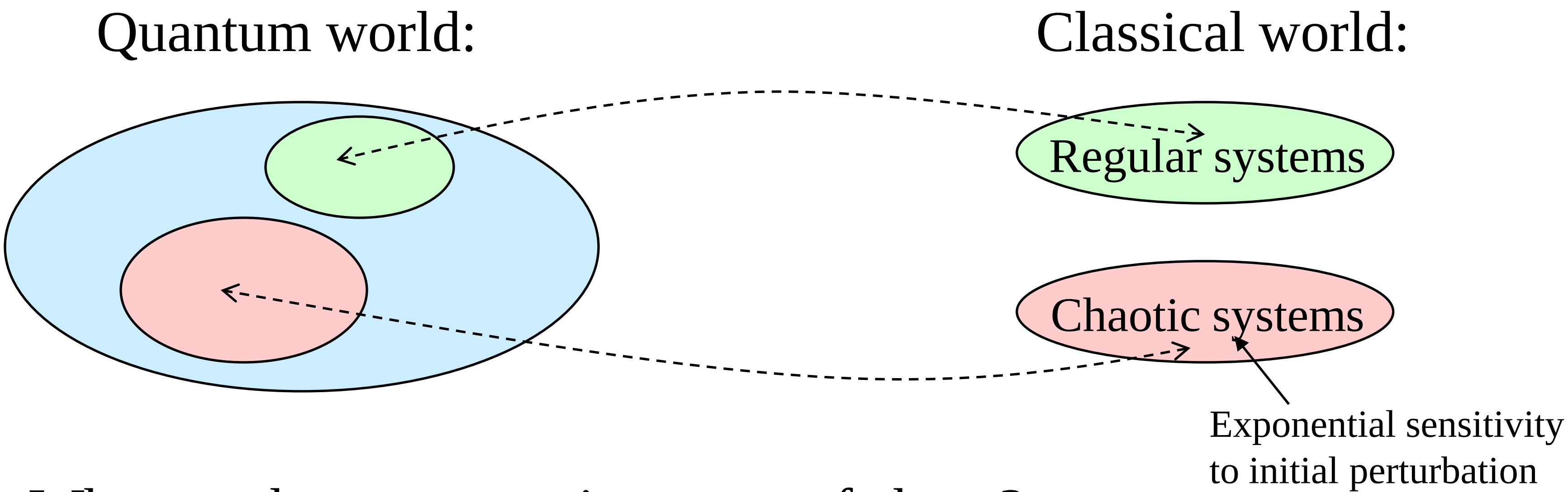
Quantum Chaos

- Do quantum systems which in the classical limit are chaotic differ form the ones which are classically regular?



- What are the quantum signatures of chaos?
  - Distribution of energy levels
  - Stability of quantum motion under perturbed Hamiltonian
  - Entanglement ???

Measure of entanglement

- Bipartite quantum system:

H_1 H_2

$$\left|\Psi\right\rangle = \sum_{i=1,\,j=1}^{d_1,d_2} c_{ij} \left|i\right\rangle \otimes \left|j\right\rangle$$

$$\boldsymbol{H} = \boldsymbol{H}_1 \otimes \boldsymbol{H}_2$$

$$d = \min(d_1, d_2)$$

- Mixedness of the reduced density matrix (linear entropy):

$$\rho_1 = Tr_2 \left|\Psi\right\rangle\left\langle\Psi\right|$$

$$E(\Psi) = 1 - Tr(\rho_1{}^2)$$

$$0 \le E(\Psi) \le 1 - \frac{1}{d}$$

product state

$$\left|\Psi\right\rangle = \left|\psi\right\rangle \otimes \left|\varphi\right\rangle$$

maximally entangled state

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \left|\psi_i\right\rangle \otimes \left|\varphi_i\right\rangle$$

Entangling properties of quantum evolutions

- How an initially product state is being entangled?

U – unitary operator acting in $\boldsymbol{H} = \boldsymbol{H}_1 \otimes \boldsymbol{H}_2$

$|\varphi\rangle \otimes |\psi\rangle \xrightarrow{U} |\Psi\rangle$ $\qquad E(\Psi)$ - entanglement produced

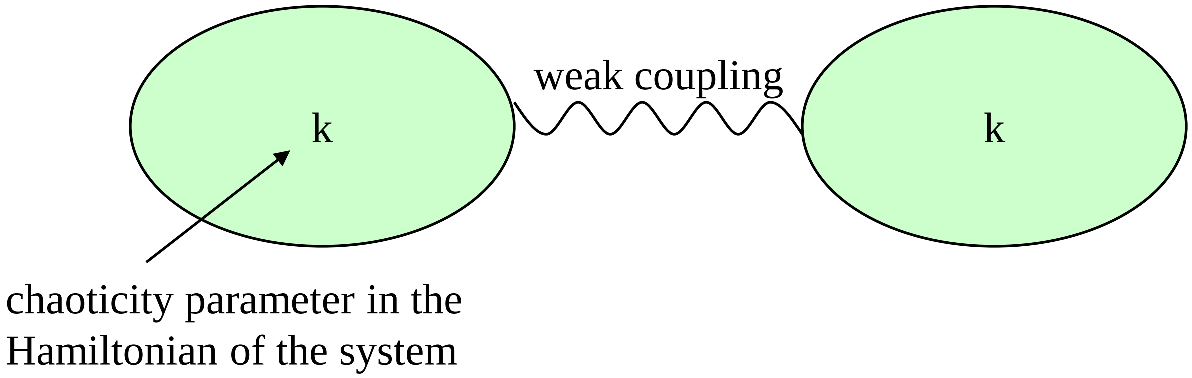
- Entangling power of U

$$E_U = \int \underbrace{d\mu(\psi) d\mu(\varphi)}_{} \, E\Big(U\big(|\psi\rangle \otimes |\varphi\rangle\big)\Big)$$

Invariant measure under SU(d_1)×SU(d_1)

- How chaos influences entangling properties?

Two weakly coupled chaotic systems



• How entanglement production changes when varying k, under fixed coupling?

- initial entanglement growth rate
- long time behaviour of entanglement

• How it depends on the type of initial product state chosen?

Does chaos help entanglement?

- Miller & Sarkar PRE 60, 1542 (1998)

  - initial entanglement growth rate proportional to Lyapunov exponent

- Tanaka, Fujisaki, Miyadera PRE 66, 045201 (2002)

  - stronger chaos does not mean higher entanglement growth rate

- Bandyopadhyay, Lakshmirayan PRE 69, 016201 (2004)

  - asymptotic value of entanglement is very high in chaotic regime

Chaos and Entanglement: friends or enemies?

Kicked Top

j – total spin

$$d = \dim \boldsymbol{H} = 2j+1 \qquad |-j\rangle, \ldots, |j\rangle \in \boldsymbol{H} \quad \text{- basis}$$

- Hamiltonian:

$$\mathrm{H(t)} = pJ_y + \tfrac{k}{2j} J_z{}^2 \sum_{n=-\infty}^{+\infty} \delta(n-t) \qquad p = \tfrac{\pi}{2}$$

- One period evolution:

$$U = e^{-i\frac{k}{2j}J_z{}^2} e^{-i\frac{\pi}{2}J_y}$$

Kicked Top

$$U = e^{-i\frac{k}{2j}J_z{}^2} e^{-i\frac{\pi}{2}J_y}$$

- Heisenberg picture:

$$\begin{cases} \tilde{J}_x = U^+ J_x U = \frac{1}{2}(J_z + iJ_y) e^{-i\frac{k}{j}(J_x - \frac{1}{2})} + h.c. \\ \tilde{J}_y = U^+ J_y U = \frac{i}{2}(J_z - iJ_y) e^{-i\frac{k}{j}(J_x - \frac{1}{2})} + h.c. \\ \tilde{J}_z = U^+ J_z U = -J_x \end{cases}$$

- Direction operators:

$$X = \frac{J_x}{j} \qquad Y = \frac{J_y}{j} \qquad Z = \frac{J_z}{j}$$

For large j direction operators commute: $[X, Y] = \frac{iZ}{j} \approx 0$

Classical limit for the kicked top

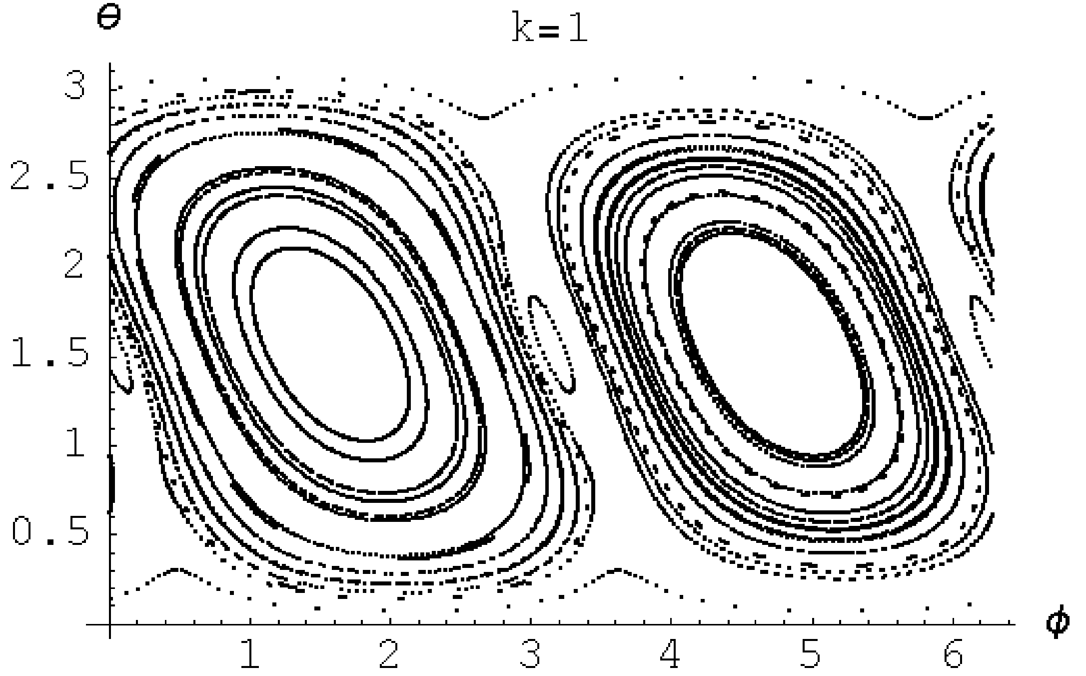
- $j \to \infty$

$$\begin{cases} \tilde{X} = Z\cos(kX) + Y\sin(kX) \\ \tilde{Y} = -Z\sin(kX) + Y\cos(kX) \\ \tilde{Z} = -X \end{cases}$$

$$X^2 + Y^2 + Z^2 = 1$$

discreet dynamics on a sphere

$$X = \sin\theta\cos\varphi \quad Y = \sin\theta\sin\varphi \quad Z = \cos\theta$$



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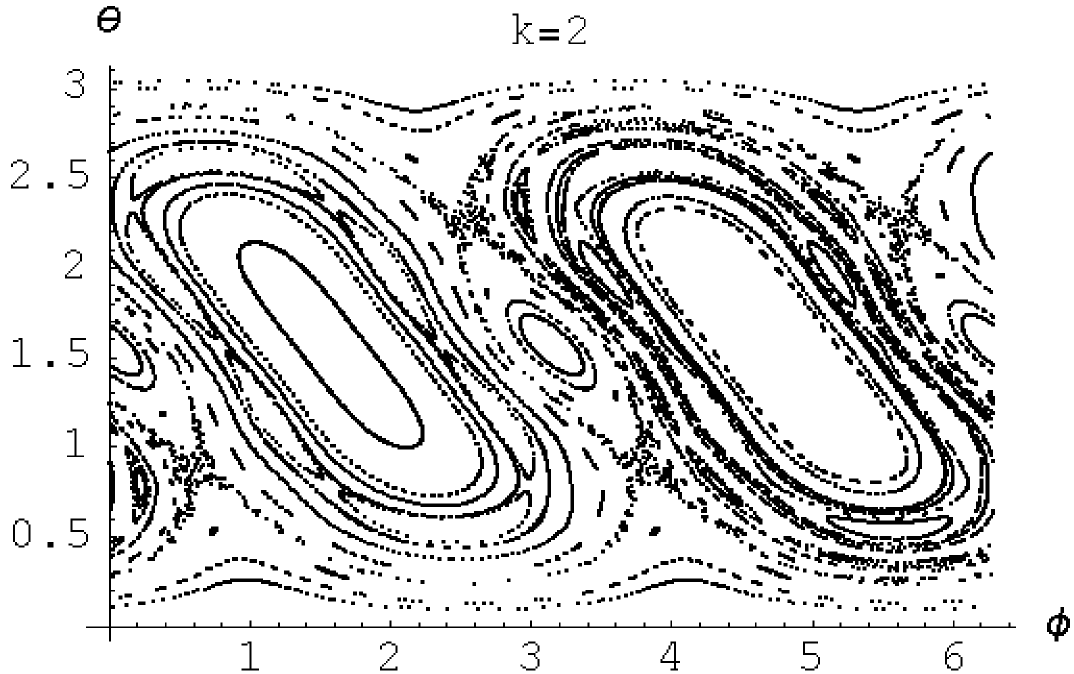

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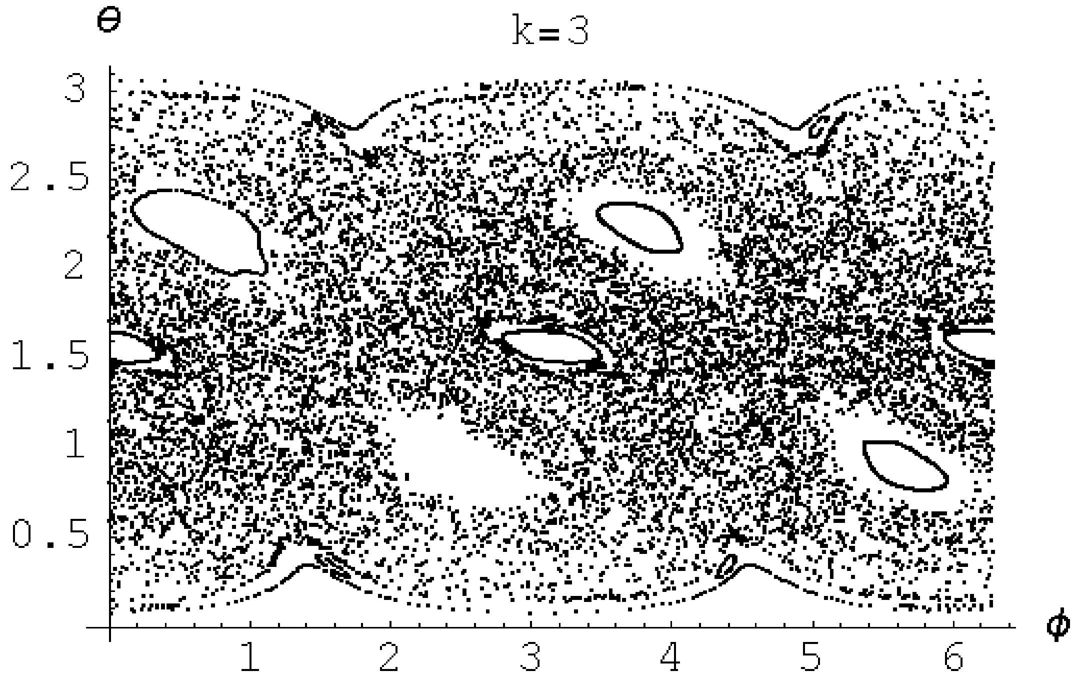

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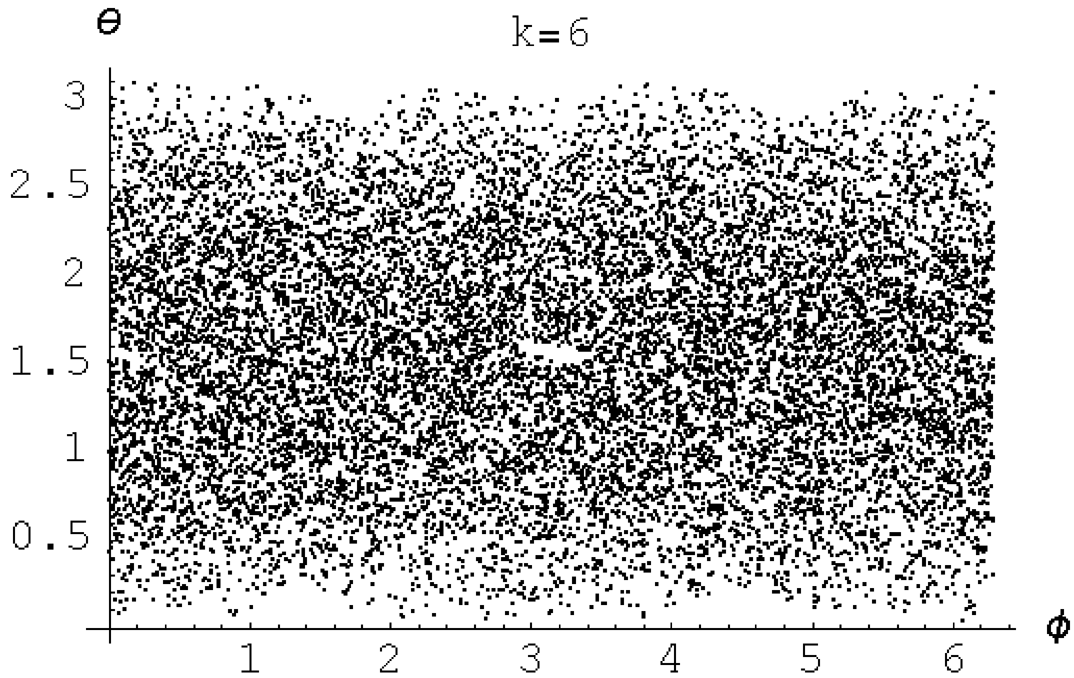


Most classical quantum states

- Spin-coherent states:

$$|\theta,\varphi\rangle = R(\theta,\varphi)|\,j\rangle \qquad R(\theta,\varphi)=\exp\left(-i\theta\left(J_x\sin\varphi - J_y\cos\varphi\right)\right)$$

  - minimal uncertainty with respect to angular momentum components

$$\Delta J^2 = \Delta J_x^{\,2} + \Delta J_y^{\,2} + \Delta J_z^{\,2}$$

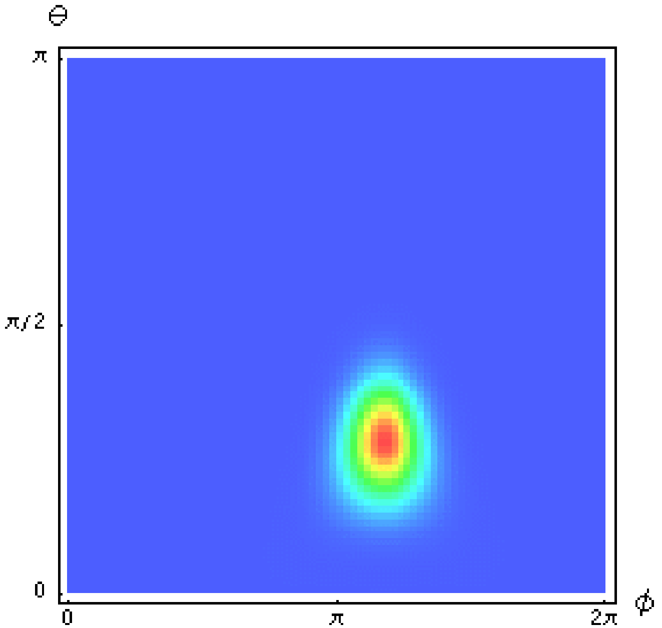
  - overcomplete set

- Phase space picture of spin states

Hussimi function:

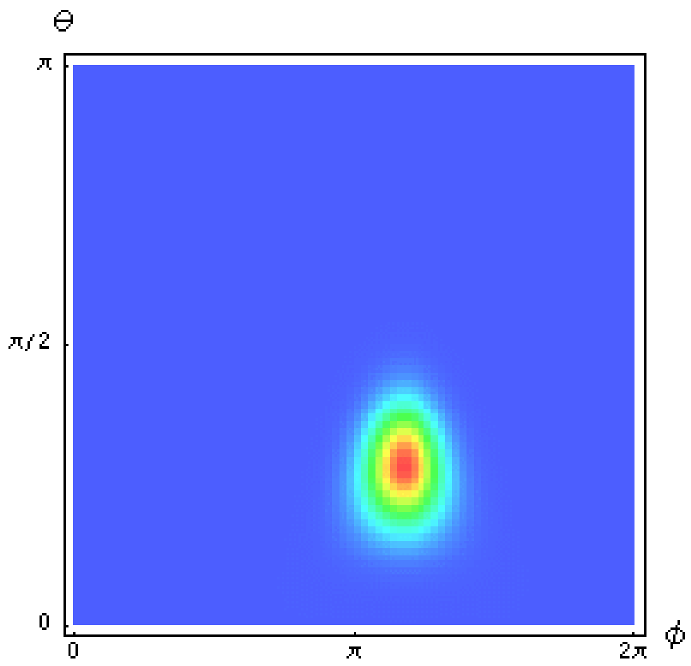
$$Q(\theta,\varphi) = \frac{2j+1}{4\pi}\langle\theta,\varphi|\rho|\theta,\varphi\rangle$$

Hussimi function of
a spin-coherent state (*j=20*):

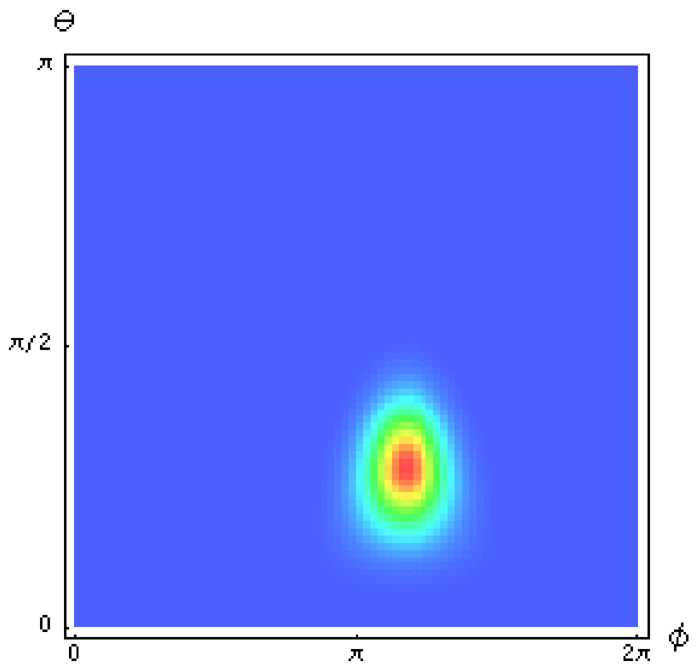


Evolution of a spin-coherent state

- *j=20* $\left|\theta = 0.89, \varphi = 3.77\right\rangle$

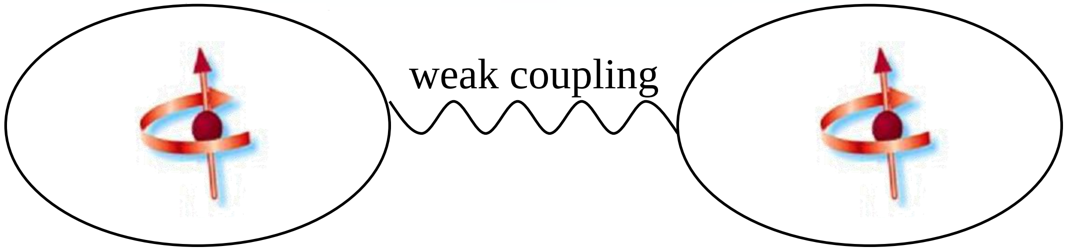


k=1



k=6

Coupled kicked tops



- Hamiltonian:

$$\mathrm{H} = \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_{\mathrm{int}}$$

$$\mathrm{H}_1(\mathrm{t}) = \tfrac{\pi}{2} J_{y_1} + \tfrac{k}{2j} J_{z_1}{}^2 \sum_{n=-\infty}^{+\infty} \delta(n-t)$$

$$\mathrm{H}_2(\mathrm{t}) = \tfrac{\pi}{2} J_{y_2} + \tfrac{k}{2j} J_{z_2}{}^2 \sum_{n=-\infty}^{+\infty} \delta(n-t)$$

$$\mathrm{H}_{\mathrm{int}}(\mathrm{t}) = \tfrac{\varepsilon}{j} J_{z_1} J_{z_2} \sum_{n=-\infty}^{+\infty} \delta(n-t)$$

- One period evolution operator:

$$U = U_1 \otimes U_2 \, U_{\mathrm{int}}$$

$$U_1 = e^{-i\frac{k}{2j}J_{z_1}{}^2} e^{-i\frac{\pi}{2}J_{y_1}}$$

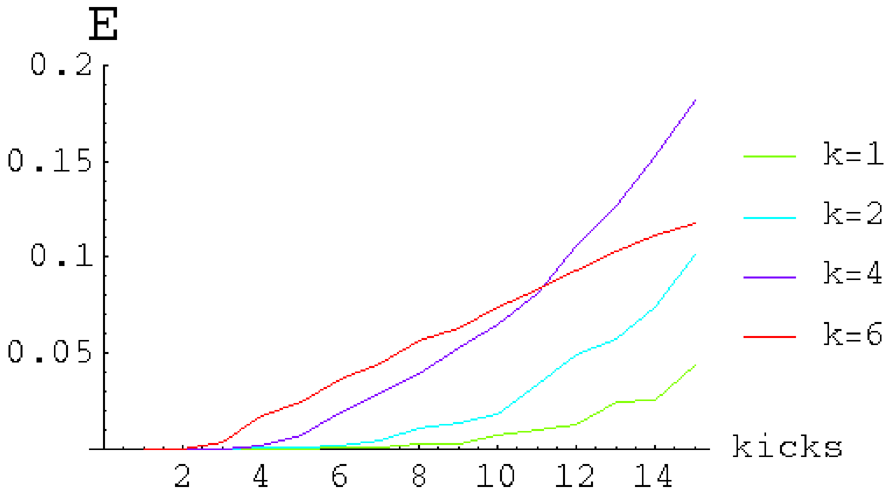
$$U_2 = e^{-i\frac{k}{2j}J_{z_2}{}^2} e^{-i\frac{\pi}{2}J_{y_2}}$$

$$U_{\mathrm{int}} = e^{-i\frac{\varepsilon}{j}J_{z_1}J_{z_2}}$$

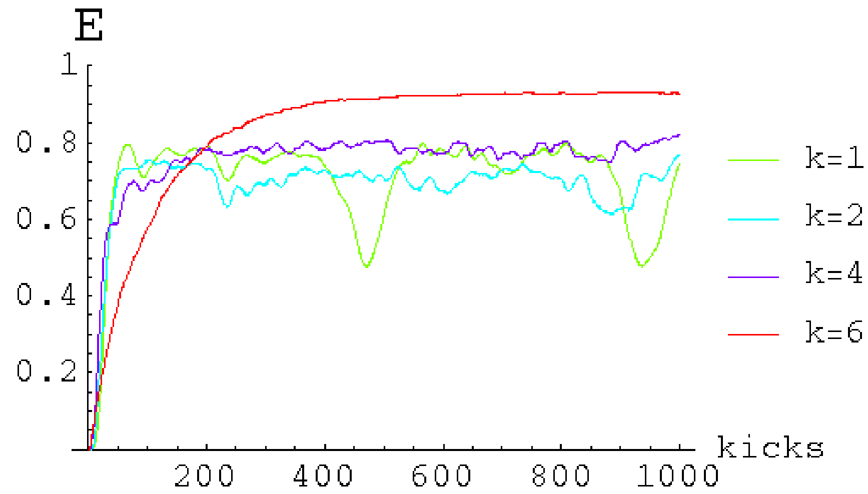
Entanglement growth for initial product state of spin-coherent states

- Initial state

$$|\psi\rangle = |\theta,\varphi\rangle \otimes |\theta,\varphi\rangle \qquad \theta=0.89 \quad \varphi=3.77 \quad j=20 \quad \varepsilon=0.01$$



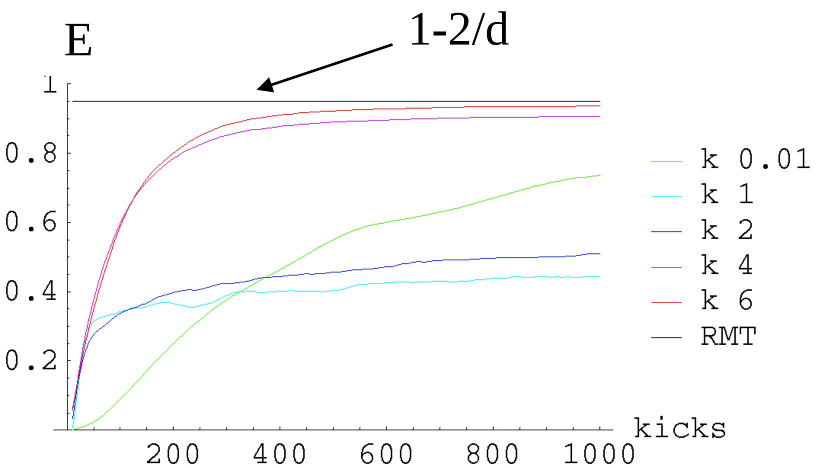
Chaos enhances initial production rate!?



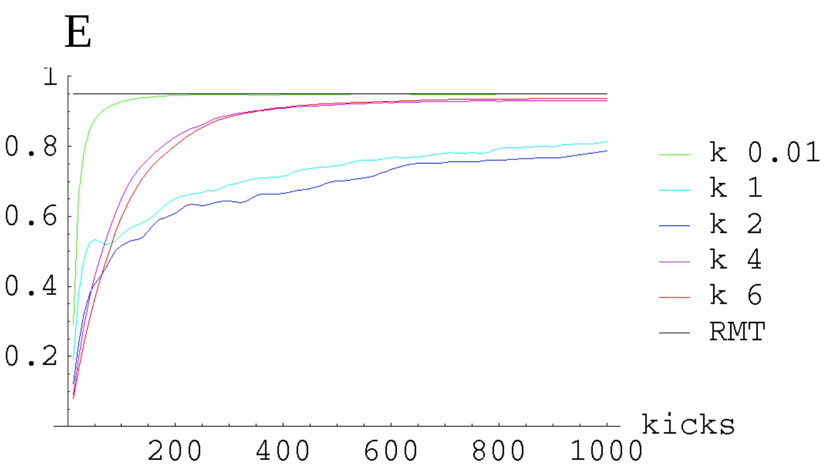
Chaos helps achieving high asymptotic entanglement

Averaging over initial product states

• spin-coherent states



• random states



- chaos helps in achieving high asymptotic entanglement

- very regular dynamics has extremely low initial entanglement growth rate.

- asymptotic entanglement is very high both for regular and chaotic dynamics.

- initial entanglement growth is the highest for very regular dynamics.

Short time behaviour of entanglement

- Perturbative formula (Tanaka et al. 2002)

$$U = \underbrace{U_1 \otimes U_2}_{U_0} U_{\text{int}}$$

Interaction picture:

$$\hat{A}(t) = \left(U_0^{+}\right)^t \hat{A} \left(U_0\right)^t$$

$$\left|\tilde{\Psi}(t)\right\rangle = \left(U_0^{+}\right)^t \left|\Psi(t)\right\rangle$$

$$U_{\text{int}} = \exp\left(-i \tfrac{\varepsilon}{j} J_{z_1} J_{z_2}\right)$$

$$\left|\tilde{\Psi}(t)\right\rangle = U_{\text{int}}(t) \left|\tilde{\Psi}(t-1)\right\rangle$$

- Entanglement of $\left|\Psi(t)\right\rangle$ to the second order in ε:

$$E(t) = 1 - Tr(\rho_1^{\ 2}) = 2\varepsilon^2 j^2 \sum_{t_1=1}^{t} \sum_{t_2=1}^{t} C_1(t_1, t_2) C_2(t_1, t_2)$$

$$C_1(t_1, t_2) = \frac{1}{j^2} \left(\left\langle J_{z_1}(t_1) J_{z_1}(t_2) \right\rangle - \left\langle J_{z_1}(t_1) \right\rangle \left\langle J_{z_1}(t_2) \right\rangle \right)$$

time correlation function

Chaos and time correlation function

$$E(t) = 2\varepsilon^2 j^2 \sum_{t_1=1}^{t} \sum_{t_2=1}^{t} C^2(t_1, t_2)$$

- $t_1 = t_2$

$$C(t,t) = \frac{1}{j^2}\left(\left\langle J_z{}^2(t)\right\rangle - \left\langle J_z(t)\right\rangle^2\right)$$

very low for coherent state: $\propto \frac{1}{j}$

for random state: $\propto 1$

Chaos increases C(t,t) !

- $t_1 \neq t_2$

but kills correlations for $t_1 \neq t_2$

Chaotic motion:

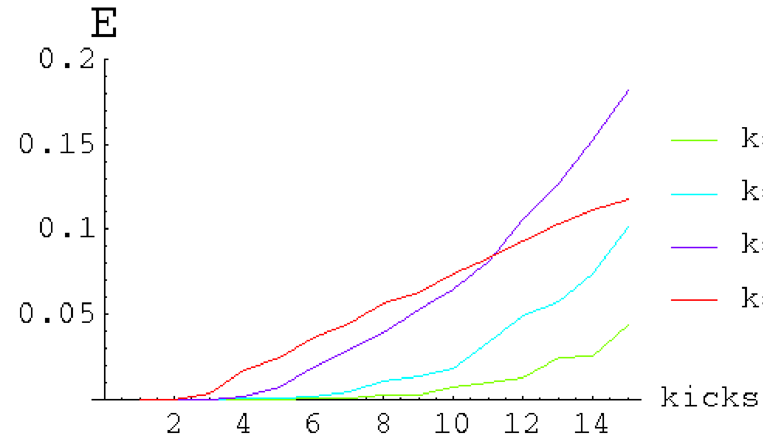
$$C(t, t+\Delta t) \xrightarrow{\Delta t \to \infty} 0$$

Regular motion:

$$C(t, t+\Delta t) \xrightarrow{\Delta t \to \infty} \overline{C}$$

Chaos induces initial linear entanglement increase

$$E(t) = 2\varepsilon^2 j^2 \sum_{t_1=1}^{t} \sum_{t_2=1}^{t} C^2(t_1, t_2)$$



- Chaotic regime:

$$E(t) = 2\varepsilon^2 j^2 \frac{t}{\alpha} = \frac{t}{\tau_c}$$

linear increase

- Regular regime:

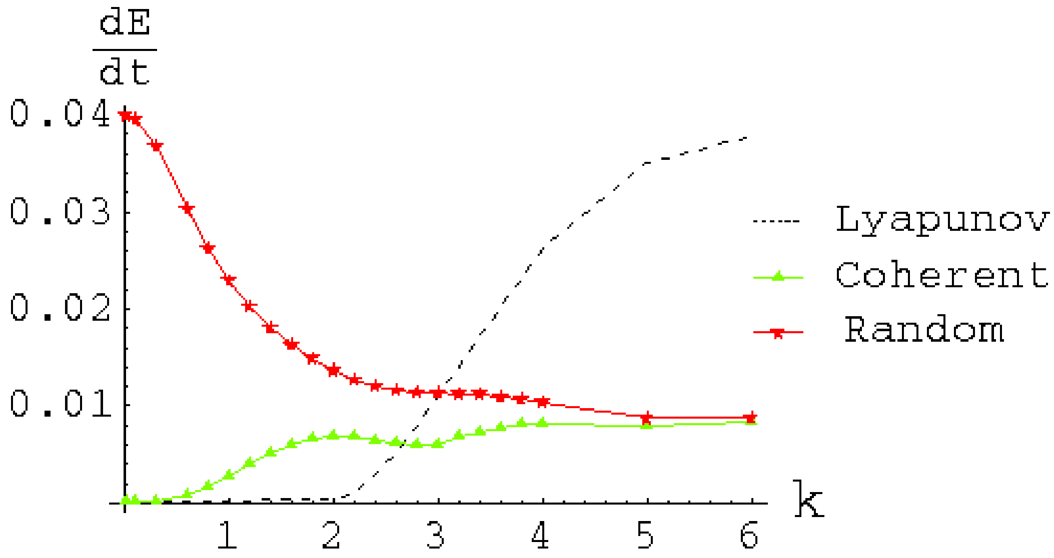
$$E(t) = 2\varepsilon^2 j^2 \frac{t^2}{\beta} = \frac{t^2}{\tau_r^{\,2}}$$

quadratic increase

$\tau_c \sim \frac{1}{\varepsilon^2 j^2}$ ⟵ characteristic times ⟶ $\tau_r \sim \frac{1}{\varepsilon j}$

Initial entangling power for the coupled kicked tops

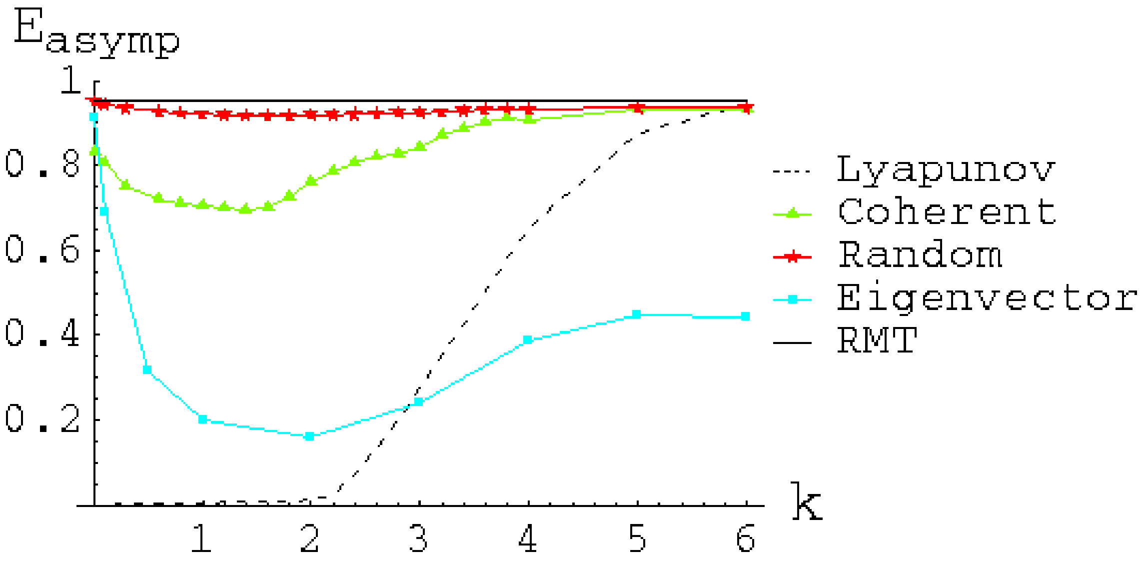
• Initial entanglement growth rate, averaged over either random or coherent states:



Chaos always diminishes initial entangling power!

Averaged asymptotic behaviour and eigenvectors entanglement

- Asymptotic entanglement, averaged over either random or coherent states:



- Averaged asymptotic entanglement and eigenvectors entanglement

$$\overline{E}_{asymp} \geq 2\overline{E}_{eigen} - 1$$

Conclusions

Chaos and entanglement are....

- Friends:

  a) Chaos drives low-uncertainty states into highly smeared states and thus increases initial entanglement growth rate

  b) Chaos assures high asymptotic entanglement

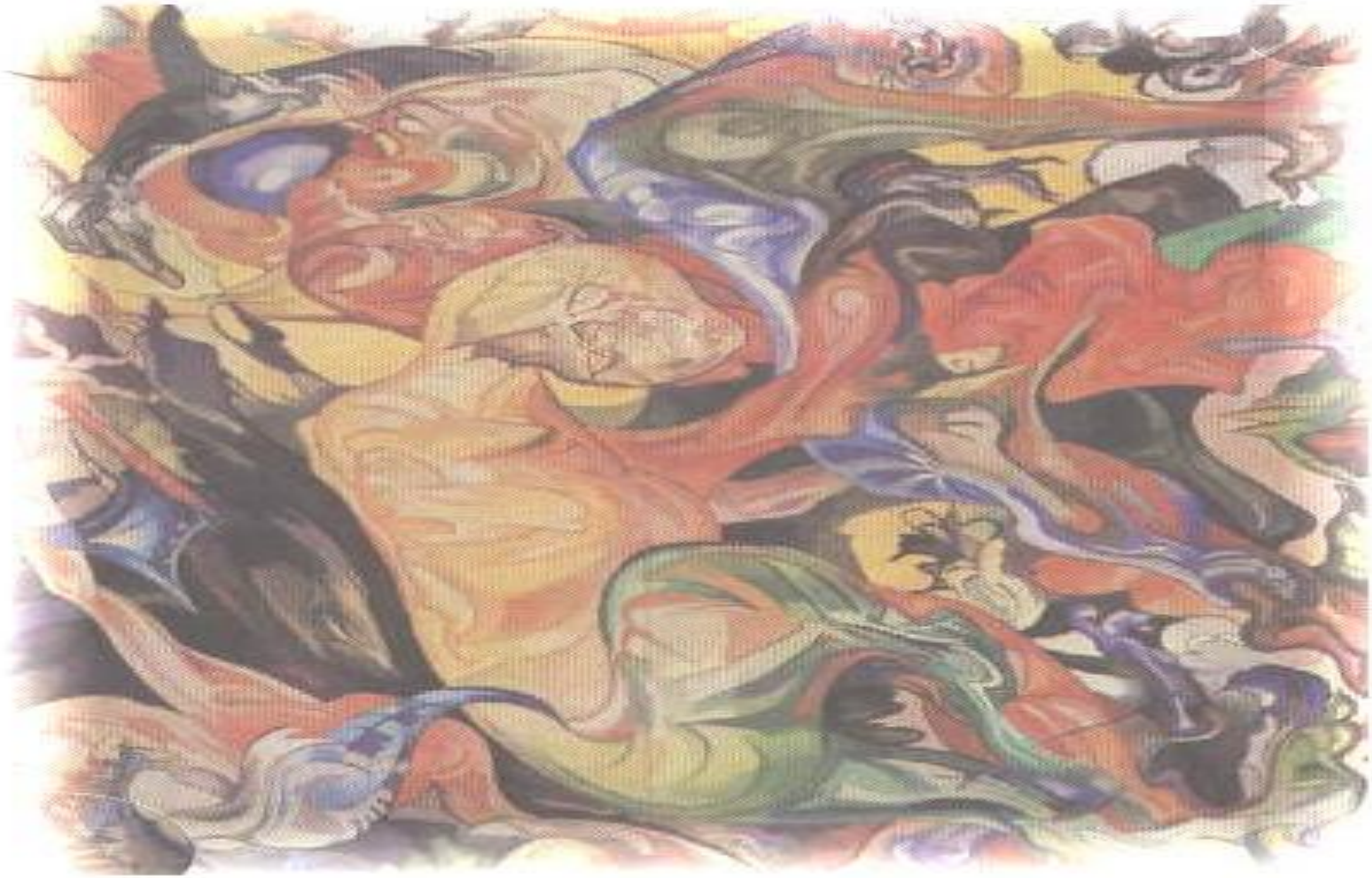
- Enemies:

  a) For certain choices of parameters (j, ε), regular dynamics, thanks to non-vanishing time correlations, outperforms chaotic dynamics in terms of initial entanglement production.

  b) For weakly coupled systems initial entangling power is always worst in chaotic case

  c) In the case of coupled kicked tops, very regular dynamics has equally high (even a little bit higher) asymptotic entanglement than chaotic cases.

Entangling power of chaotic quantum systems – A case study



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