Decorrelation of quantum states

a first step towards the quantum cocktail party

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Decorrelation

how to remove correlations while preserving local properties?

$$\Lambda: \ \rho_{AB} \longrightarrow \rho_A \otimes \rho_B$$

Decorrelating operation $\qquad \mathrm{Tr}_B \rho_{AB} \qquad \mathrm{Tr}_A \rho_{AB}$

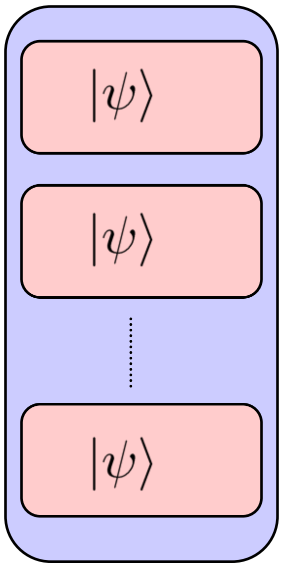
Is decorrelation possible for a given set of states?

$$\exists_\Lambda \forall_{\rho_{AB} \in \mathcal{S}}$$

Personal motivation

cloning, estimation and the role of correlations

Universal cloning

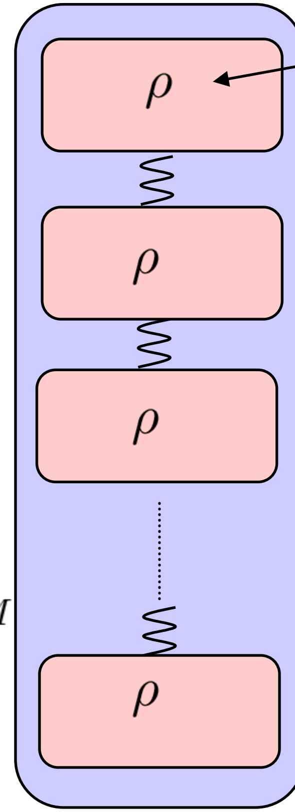


N uncorrelated copies

$$\mathcal{E} : |\psi\rangle^{\otimes N} \mapsto \rho_M$$

Cloning operation should work for every $|\psi\rangle$

$$\mathcal{E} : (U|\psi\rangle)^{\otimes N} = U^{\otimes M} \rho_M U^{\dagger \otimes M}$$



M correlated approximate clones $\rho_M \neq \rho^{\otimes M}$

single clone density matrix

$$\rho = \mathrm{Tr}_{2\ldots M}\, \rho_M =$$

$$= \eta |\psi\rangle\langle\psi| + \frac{1-\eta}{d} \mathbb{1}$$

noise

$$F = \langle\psi|\rho|\psi\rangle$$

$$= \frac{N(d+M)+M-N}{(d+N)M}$$

fidelity

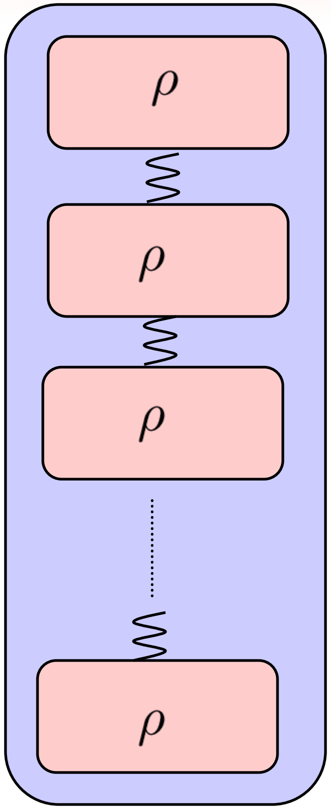
Asymptotic cloning is state estimation

D. Bruss, A. Ekert, C. Macchiavello, PRL 81, 2598 (1997)

J. Bae, A. Acin, quant-ph/0603078 (2006)

$$\lim_{M\to\infty} F = \frac{N+1}{N+d}$$

Can clones be decorrelated?

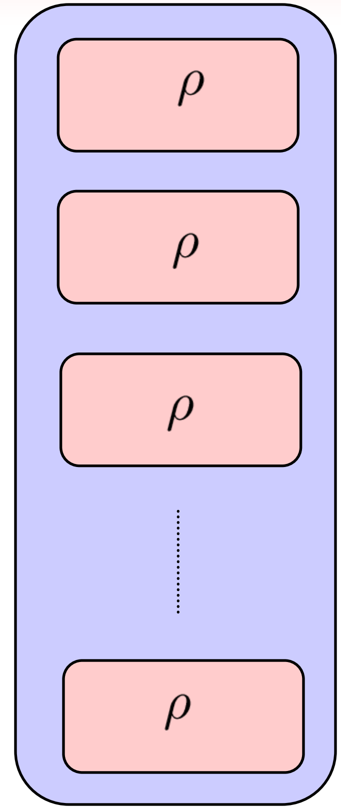


M correlated approximate clones

$$\Lambda : \rho_M \mapsto \rho^{\otimes M}$$

Decorrelating operation

$$\Lambda(U^{\otimes M} \rho_M U^{\dagger \otimes M}) = (U \rho U^{\dagger})^{\otimes M}$$



M uncorrelated approximate clones

Correlations influence estimation fidelitites → Constraints on decorrelation (not tight)

RDD, PRA 71, 062321 (2005)

Official motivation

the quantum cocktail party

Classical cocktail party

$\varphi(t)$
signal 1

$\psi(t)$
signal 2

$x(t)$
mixed signals

$y(t)$
mixed signals

$$x(t) = C_{11}\varphi(t) + C_{12}\psi(t)$$

$$y(t) = C_{21}\varphi(t) + C_{22}\psi(t)$$

How to decorrelate signals without knowing C_{ij} ?

e.g. Independent component analysis (ICA)

What can be decorrelated?

No-decorrelation theorem

There is no operation that decorrelates all states

D. R. Terno PRA 59, 3320 (1999)

Let ρ'_{AB}, ρ''_{AB} be bipartite states such that:

$$\underbrace{\mathrm{Tr}_B\rho'_{AB}}_{\rho'_A} \neq \underbrace{\mathrm{Tr}_B\rho''_{AB}}_{\rho''_A}, \quad \underbrace{\mathrm{Tr}_A\rho'_{AB}}_{\rho'_B} \neq \underbrace{\mathrm{Tr}_A\rho''_{AB}}_{\rho''_B}.$$

Let us assume that Λ decorrelates both states:

$$\Lambda(\rho'_{AB}) = \rho'_A \otimes \rho'_B, \quad \Lambda(\rho''_{AB}) = \rho''_A \otimes \rho''_B.$$

However, it will not decorrelate their convex combination

$$\rho_{AB} = p\rho'_{AB} + (1-p)\rho''_{AB},$$

since from linearity of Λ we get

$$\Lambda(\rho_{AB}) = p\Lambda(\rho'_{AB}) + (1-p)\Lambda(\rho''_{AB}) = \underbrace{p\rho'_A \otimes \rho'_B + (1-p)\rho''_A \otimes \rho''_B}_{\text{This is not a product state!}}$$

Yes-decorrelation theorem

There is an operation that decorrelates a given state

$$\Lambda(\rho_{AB}) = \rho_A \otimes \rho_B$$

Discard the state ρ_{AB} and prepare the state $\rho_A \otimes \rho_B$

Interesting cases

sets of density matrices, where no element is a convex combination of the others, e.g. orbits of unitary representations

Different signals ("quantum cocktail party")

$$\Lambda(\underbrace{U_1 \otimes \cdots \otimes U_M \rho_M U_1^\dagger \otimes \cdots \otimes U_M^\dagger}_{\text{signals encoded on correlated systems}}) = \underbrace{U_1 \rho U_1^\dagger \otimes \cdots \otimes U_M \rho U_M^\dagger}_{\text{decorrelated signals}}$$

Identical signals (decorrelating clones)

$$\Lambda(\underbrace{U^{\otimes M} \rho_M U^{\dagger \otimes M}}_{\text{a signal encoded on correlated systems}}) = \underbrace{(U \rho U^\dagger)^{\otimes M}}_{\text{a signal encoded on uncorrelated systems}}$$

Covariance condition

$$\Lambda(U_1 \otimes \cdots \otimes U_M \rho_M U_1^\dagger \otimes \cdots \otimes U_M^\dagger) = U_1 \otimes \cdots \otimes U_M \Lambda(\rho_M) U_1^\dagger \otimes \cdots \otimes U_M^\dagger$$

Covariant operations

Choi-Jamiołkowski isomorphism

$$\Lambda : \mathcal{L}(\mathcal{H}^{\text{in}}) \mapsto \mathcal{L}(\mathcal{H}^{\text{out}}) \text{ - arbitrary CP map}$$

$$\updownarrow$$

$$R_\Lambda \in \mathcal{L}\left(\mathcal{H}^{\text{out}} \otimes \mathcal{H}^{\text{in}}\right) \text{ - positive operator}$$

1 to 1 relation

$$R_\Lambda = \Lambda \otimes \mathcal{I}\left(|\Psi\rangle\langle\Psi|\right), \text{where } |\Psi\rangle = \frac{1}{\sqrt{\dim \mathcal{H}^{\text{in}}}} \sum_i |i\rangle \otimes |i\rangle$$

$$\Lambda(\rho) = \mathrm{Tr}_{\mathcal{H}^{\text{in}}}\left(\mathbb{1}_{\mathcal{H}^{\text{out}}} \otimes \rho^T \; R_\Lambda\right)$$

Trace preservation condition

$$\mathrm{Tr}_{\mathcal{H}^{\text{out}}}\left(R_\Lambda\right) = \mathbb{1}_{\mathcal{H}^{\text{in}}}.$$

Covariance condition

$$\forall_{g\in G} \Lambda\left(V_g \rho V_g^\dagger\right) = W_g \Lambda(\rho) W_g^\dagger \quad \longleftrightarrow \quad \forall_{g\in G} [R_\Lambda, W_g \otimes V_g^*] = 0$$

Two qubits

Permutational invariant state of two qubits

singlet subspace

$$\rho_{AB} = \begin{pmatrix} \boxed{\rho_{00}} & 0 & 0 & 0 \\ 0 & \rho_{11} & \rho_{12}^* & \rho_{13}^* \\ 0 & \rho_{12} & \rho_{22} & \rho_{23}^* \\ 0 & \rho_{13} & \rho_{23} & \rho_{33} \end{pmatrix}$$

triplet subspace

Two qubits

Permutational invariant state of two qubits

singlet subspace

$$\rho_{AB} = \begin{pmatrix} \rho_{00} & 0 & 0 & 0 \\ 0 & \rho_{11} & 0 & 0 \\ 0 & 0 & \rho_{22} & 0 \\ 0 & 0 & 0 & \rho_{33} \end{pmatrix}$$

triplet subspace

$$\rho_A = \rho_B = \tfrac{1}{2}\left(\mathbb{1} + r\sigma_z\right), \quad r = \rho_{33} - \rho_{11}$$

Decorrelation condition

allow additional noise after decorrelation

$$\Lambda(\rho_{AB}) = \frac{1}{2}(\mathbb{1} + \check{r}\sigma_z) \otimes \frac{1}{2}(\mathbb{1} + \check{r}\sigma_z) =$$

$$= \begin{pmatrix} 1/4(1-\check{r}^2) & 0 & 0 & 0 \\ 0 & 1/4(1-\check{r})^2 & 0 & 0 \\ 0 & 0 & 1/4(1-\check{r}^2) & 0 \\ 0 & 0 & 0 & 1/4(1+\check{r})^2 \end{pmatrix}.$$

Two qubits

different signals

Covariance condition

$$\Lambda(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = U_A \otimes U_B \Lambda(\rho_{AB}) U_A^\dagger \otimes U_B^\dagger$$

$$\Lambda : \mathcal{L}(\mathcal{H}_A^{\text{in}} \otimes H_B^{\text{in}}) \mapsto \mathcal{L}(\mathcal{H}_A^{\text{out}} \otimes H_B^{\text{out}})$$

$$[R_\Lambda, \underbrace{U_A \otimes U_B}_{\mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}}} \otimes \underbrace{U_A^* \otimes U_B^*}_{\mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{in}}}] = 0 \qquad R_\Lambda \in \mathcal{L}(\mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}} \otimes \mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{in}})$$

Thanks to the equivalence of conjugated representation of SU(2)

$$\tilde{R}_\Lambda = \mathbb{1}_{\mathcal{H}^{\text{out}}} \otimes C \; R_\Lambda \; \mathbb{1}_{\mathcal{H}^{\text{out}}} \otimes C^\dagger, \text{ where } C = (i\sigma_y)^{\otimes 2}$$

$$[\tilde{R}_\Lambda, U_A \otimes U_B \otimes U_A \otimes U_B] = 0$$

$$\Lambda(\rho_{AB}) = \mathrm{Tr}_{\mathcal{H}^{\text{in}}}(\mathbb{1}_{\mathcal{H}^{\text{out}}} \otimes (C\rho_{AB}^T C^\dagger) \; \tilde{R}_\Lambda)$$

Two qubits

different signals

Structure of the decorrelating operation

$$[\tilde{R}_\Lambda, U_A \otimes U_B \otimes U_A \otimes U_B] = 0$$

after changing the order of subspaces

$$\mathcal{H}_A^{\mathrm{out}} \otimes \mathcal{H}_B^{\mathrm{out}} \otimes \mathcal{H}_A^{\mathrm{in}} \otimes \mathcal{H}_B^{\mathrm{out}} \mapsto \mathcal{H}_A^{\mathrm{out}} \otimes \mathcal{H}_A^{\mathrm{in}} \otimes \mathcal{H}_B^{\mathrm{out}} \otimes \mathcal{H}_B^{\mathrm{in}}.$$

$$[\tilde{R}_\Lambda, U_A \otimes U_A \otimes U_B \otimes U_B] = 0$$

$$\downarrow$$

$$\tilde{R}_\Lambda = \sum_{i,j=0}^{1} a_{i,j} P_A^{(i)} \otimes P_B^{(j)}$$

positive coefficients

$$a_{ij} = a_{ji}$$

projection on the singlet (i=0), triplet (i=1) subspace of $\mathcal{H}_A^{\mathrm{out}} \otimes \mathcal{H}_A^{\mathrm{in}}$

trace preservation condition

$$a_{00} + 9a_{11} + 6a_{01} = 4$$

2 parameters

Two qubits

different signals

Solution for the decorrelation problem

$$\Lambda(\rho_{AB}) = \frac{1}{2}(\mathbb{1} + \check{r}\sigma_z) \otimes \frac{1}{2}(\mathbb{1} + \check{r}\sigma_z)$$

trivial solution (complete mixing) always exists $\check{r} = 0$

non-trivial solutions exists only for states with $\rho_{00} = \rho_{22}$

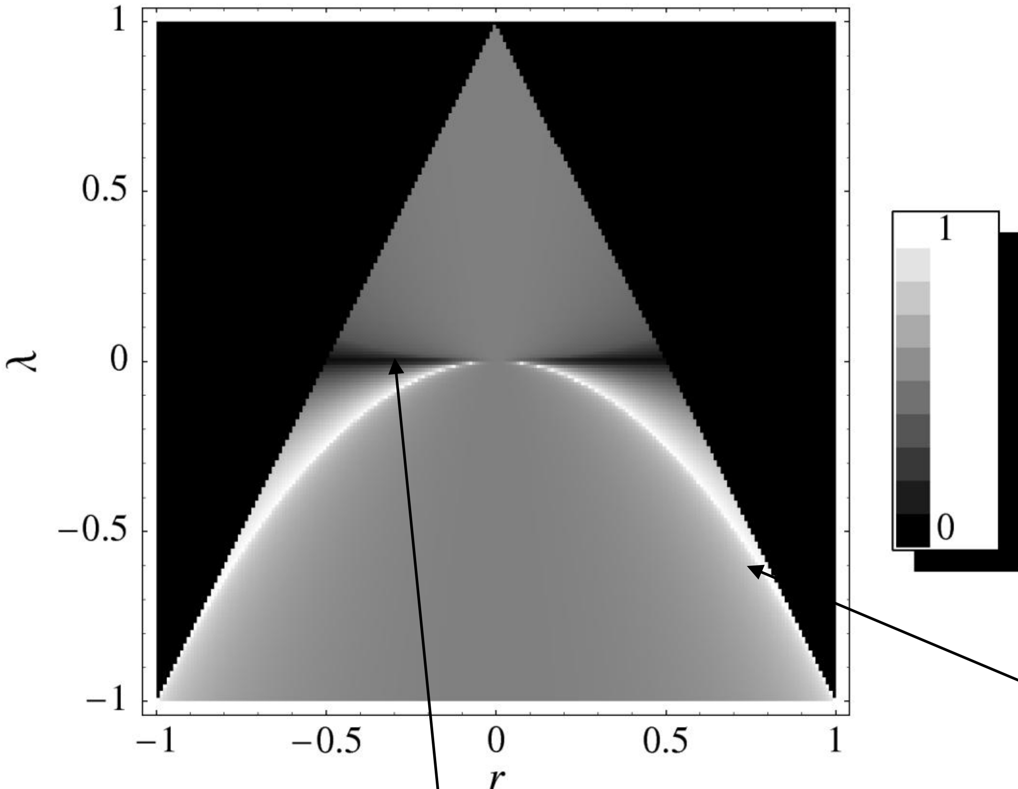
$$\rho_{AB} = \tfrac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + r(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - \lambda\sigma_z \otimes \sigma_z\right)$$

condition that $a_{ij} \geq 0$**, puts constraint on maximall achievable** $\check{r}$

Decorrelable states of two qubits

different signals

$$\rho_{AB} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + r(\sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z) - \lambda\sigma_z\otimes\sigma_z\right)$$



Maximall achievable ratio $\check{r}/r$

parabolla $\lambda = -r^2$ coresponds to product states

States with $\lambda = 0$ are not decorrelable

Two qubits

identical signals

Covariance condition (weaker)

$$\Lambda(U \otimes U \rho_{AB} U^\dagger \otimes U^\dagger) = U \otimes U \Lambda(\rho_{AB}) U^\dagger \otimes U^\dagger$$

$$[\tilde{R}_\Lambda, \underbrace{U \otimes U}_{\mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}}} \otimes \underbrace{U \otimes U}_{\mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{in}}}] = 0$$

additionally permutation covariance

$$\mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}} \otimes \mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{out}} = \left(\bigoplus_{j=0}^{1} \mathcal{H}_j^{\text{out}} \right) \otimes \left(\bigoplus_{l=0}^{1} \mathcal{H}_l^{\text{in}} \right) = \sum_{j,l=0}^{1} \bigoplus_{J=|j-l|}^{j+l} \mathcal{H}_{j,l}^J$$

$$\tilde{R}_\Lambda = \sum_{j,l=0}^{1} \bigoplus_{J=|j-l|}^{j+l} s_{j,l}^J P_{j,l}^J$$

positive coefficients **projection on** $\mathcal{H}_{j,l}^J$

trace preservation condition

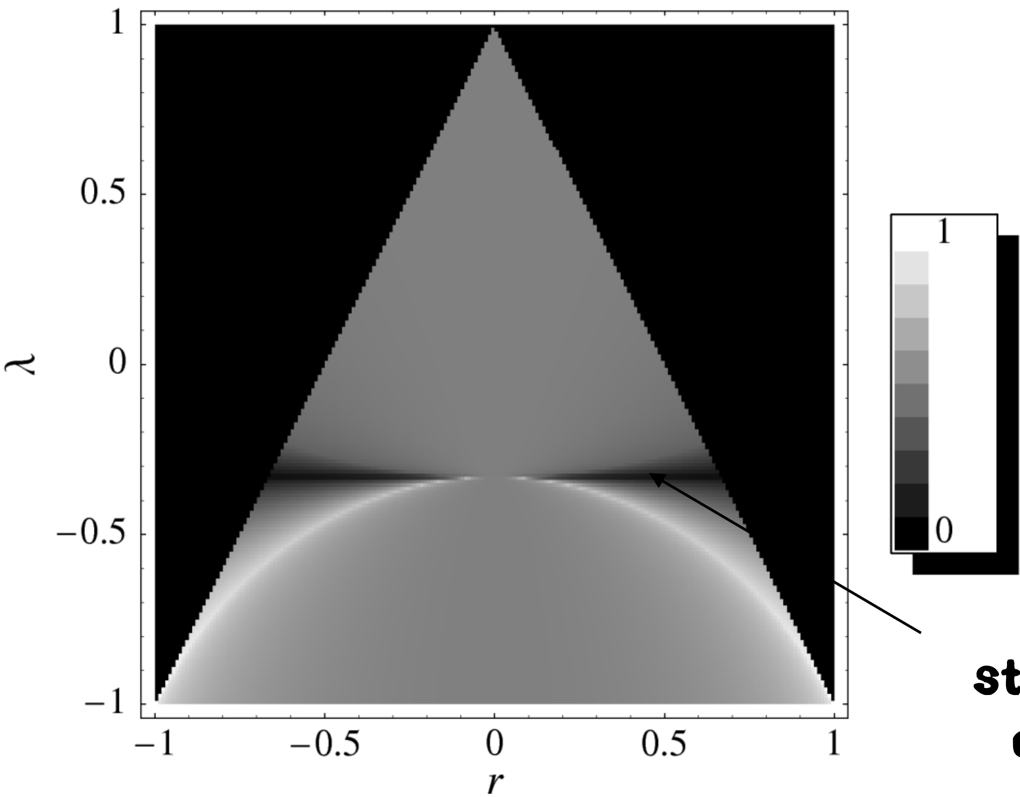
$$\sum_{j=0}^{1} \sum_{J=|j-l|}^{j+l} s_{j,l}^J \frac{2J+1}{2l+1} = 1, \quad \text{for} \quad l = 0, 1$$

4 parameters

almost all states are decorelable

e.g. State from symetric subspace

$$\rho_{AB}^{\mathrm{sym}} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + r(\sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z) + \frac{1+\lambda}{2}(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y) - \lambda\sigma_z\otimes\sigma_z\right)$$



Maximall achievable ratio $\check{r}/r$

states obtainable by cloning of one qubit - not decorrelable (generalizes to N qubits)

arbitrary state: $\rho_{AB} = p|\psi^-\rangle\langle\psi^-| + (1-p)\rho_{AB}^{\mathrm{sym}}$

N qubits

efficient numerical procedure

$$\Lambda(\rho_{AB}) = \frac{1}{2}(\mathbb{1} + \check{r}\sigma_z) \otimes \frac{1}{2}(\mathbb{1} + \check{r}\sigma_z)$$

Different signals

For given $\check{r}$:
linear equations for
O(N) positive parameters

Identical signals

For given $\check{r}$:
linear equations for
O(N^3) positive parameters

Linear programming

Two-mode gaussian states

Zero mean gaussian states

$$\rho_{AB} = \frac{1}{\pi^4}\int d^4x e^{-\frac{1}{2}x^T M x} D(x), \quad D(x) = D_A(x_1 + ix_2) \otimes D_B(x_3 + ix_4)$$

correlation matrix

displacement operators

Signals = displacements

$$D_A(\alpha) \otimes D_B(\beta)\rho_{AB} D_A^\dagger(\alpha) \otimes D_B^\dagger(\beta),$$

Covariance condition

$$\Lambda\left[D_A(\alpha) \otimes D_B(\beta)\rho_{AB} D_A^\dagger(\alpha) \otimes D_B^\dagger(\beta)\right] =$$

$$= D_A(\alpha) \otimes D_B(\beta)\Lambda(\rho_{AB}) D_A^\dagger(\alpha) \otimes D_B^\dagger(\beta)$$

in short

$$\Lambda\left[D(y)\rho_{AB} D^\dagger(y)\right] = D(y)\Lambda(\rho_{AB}) D^\dagger(y)$$

Decorrelation is easy

Covariant gaussian channel

$$\mathcal{G}(\rho_{AB}) = \frac{\sqrt{\det G}}{(2\pi)^2} \int \mathrm{d}^4 z \, e^{-\frac{1}{2} z^T G z} D(z) \rho D^\dagger(z)$$

postive definite matrix

Output state is a gaussian

$$M' = M + \Sigma G^{-1} \Sigma$$

new correlation matrix

old correlation matrix

$$\Sigma = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Decorrelation is easy

Covariant gaussian channel

$$\mathcal{G}(\rho_{AB}) = \frac{\sqrt{\det G}}{(2\pi)^2} \int \mathrm{d}^4 z\, e^{-\frac{1}{2} z^T G z} D(z) \rho D^\dagger(z)$$

postive definite matrix

Output state is a gaussian

$$M' = M + \Sigma G^{-1} \Sigma$$

$$\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \qquad \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \qquad \begin{pmatrix} W & V \\ V^T & Z \end{pmatrix}$$

to decorrelate, take $V = -\sigma_y C \sigma_y$ **, and** W, Z **high enough to make** $G^{-1} > 0$

Decorrelation easy - orbits of covariance group small

Example: Mixed EPR states

$$M' = M + \Sigma G^{-1} \Sigma$$

$$M = \begin{pmatrix} 1+2n & 0 & 2m & 0 \\ 0 & 1+2n & 0 & -2m \\ 2m & 0 & 1+2n & 0 \\ 0 & -2m & 0 & 1+2n \end{pmatrix}$$

Single mode states are termal states with n photons

$$G^{-1} = \begin{pmatrix} 2m+\epsilon & 0 & 2m & 0 \\ 0 & 2m+\epsilon & 0 & -2m \\ 2m & 0 & 2m+\epsilon & 0 \\ 0 & -2m & 0 & 2m+\epsilon \end{pmatrix}$$

$$\downarrow$$

$$M' = (1 + 2n + 2m + \epsilon)\mathbb{1}$$

Single mode states are termal states with n+m photons

Decorrelation vs. Cocktail party

signals are encoded on correlated state

$$U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger$$

signals get correlated via unknown interaction

$$\mathcal{E}\left(U_A |0\rangle\langle 0| U_A^\dagger \otimes U_B |0\rangle\langle 0| U_B^\dagger \right)$$

obtain uncorrelated signals

Reference:
G. M. D'Ariano, RDD, P. Perinotti, M. Sacchi, quant-ph/0609020 (2006)