Decorrelation of quantum states
a first step towards the quantum cocktail party

Giacomo Mauro D'Ariano
Massimiliano Sacchi
Paolo Perinotti

Rafał Demkowicz-Dobrzański
Center for Theoretical Physics of the Polish Academy of Sciences, Warsaw, Poland
Decorrelation

how to remove correlations while preserving local properties?

\[ \Lambda : \rho_{AB} \rightarrow \rho_A \otimes \rho_B \]

Decorrelating operation

\( \text{Tr}_B \rho_{AB} \)
\( \text{Tr}_A \rho_{AB} \)

Is decorrelation possible for a given set of states?

\[ \exists \Lambda \forall \rho_{AB} \in S \]
Personal motivation
cloning, estimation and the role of correlations
Universal cloning

\[ E : |\psi\rangle^\otimes N \mapsto \rho_M \]

Cloning operation should work for every \(|\psi\rangle\)

\[ E : (U |\psi\rangle)^\otimes N = U^\otimes M \rho_M U^{\dagger \otimes M} \]

\[ \rho = \text{Tr}_{2\ldots M} \rho_M = \eta |\psi\rangle \langle \psi | + \frac{1-\eta}{d} \mathbb{1} \]

Asymptotic cloning is state estimation

\[ D.\text{Bruss}, A.\text{Ekert}, C.\text{Macchiavello}, \text{PRL} 81,2598 (1997) \]

\[ J.\text{Bae}, A.\text{Acin}, \text{quant-ph/0603078} (2006) \]

\[ \lim_{M \to \infty} F = \frac{N+1}{N+d} \]
Can clones be decorrelated?

\[ \Lambda : \rho_M \leftrightarrow \rho^\otimes M \]

Decorrelating operation

\[ \Lambda(U^\otimes M \rho_M U^{\dag \otimes M}) = (U \rho U^{\dag})^\otimes M \]

M correlated approximate clones

M uncorrelated approximate clones

Correlations influence estimation fidelities

Constraints on decorrelation (not tight)
Official motivation
the quantum cocktail party
Classical cocktail party

\[ x(t) = C_{11} \varphi(t) + C_{12} \psi(t) \]
\[ y(t) = C_{21} \varphi(t) + C_{22} \psi(t) \]

How to decorrelate signals without knowing \( C_{ij} \)?

e.g. Independent component analysis (ICA)
What can be decorrelated?
No-decorrelation theorem

There is no operation that decorrelates all states

D. R. Terno PRA 59, 3320 (1999)

Let $\rho'_{AB}, \rho''_{AB}$ be bipartite states such that:

$$\begin{align*}
\text{Tr}_B \rho'_{AB} &\neq \text{Tr}_B \rho''_{AB}, \\
\rho'_A &\neq \rho''_A \\
\text{Tr}_A \rho'_{AB} &\neq \text{Tr}_A \rho''_{AB}, \\
\rho'_B &\neq \rho''_B.
\end{align*}$$

Let us assume that $\Lambda$ decorrelates both states:

$$\begin{align*}
\Lambda(\rho'_{AB}) &= \rho'_A \otimes \rho'_B, \\
\Lambda(\rho''_{AB}) &= \rho''_A \otimes \rho''_B.
\end{align*}$$

However, it will not decorrelate their convex combination

$$\rho_{AB} = p \rho'_{AB} + (1 - p) \rho''_{AB},$$

since from linearity of $\Lambda$ we get

$$\begin{align*}
\Lambda(\rho_{AB}) &= p \Lambda(\rho'_{AB}) + (1 - p) \Lambda(\rho''_{AB}) \\
&= p \rho'_A \otimes \rho'_B + (1 - p) \rho''_A \otimes \rho''_B
\end{align*}$$

This is not a product state!
Yes-decorrelation theorem

There is an operation that decorrelates a given state

\[ \Lambda(\rho_{AB}) = \rho_A \otimes \rho_B \]

Discard the state \( \rho_{AB} \) and prepare the state \( \rho_A \otimes \rho_B \)
Interesting cases

sets of density matrices, where no element is a convex combination of the others, e.g. orbits of unitary representations

**Different signals** ("quantum cocktail party")

\[ \Lambda(U_1 \otimes \cdots \otimes U_M \rho_M U_1^\dagger \otimes \cdots \otimes U_M^\dagger) = U_1 \rho U_1^\dagger \otimes \cdots \otimes U_M \rho U_M^\dagger \]

signals encoded on correlated systems  
decorrelated signals

**Identical signals** (decorrelating clones)

\[ \Lambda(U^\otimes M \rho_M U^\dagger \otimes M) = (U \rho U^\dagger)^\otimes M \]

a signal encoded on correlated systems  
a signal encoded on uncorrelated systems

**Covariance condition**

\[ \Lambda(U_1 \otimes \cdots \otimes U_M \rho_M U_1^\dagger \otimes \cdots \otimes U_M^\dagger) = U_1 \otimes \cdots \otimes U_M \Lambda(\rho_M) U_1^\dagger \otimes \cdots \otimes U_M^\dagger \]
Covariant operations

Choi-Jamiołkowski isomorphism

\[ \Lambda : \mathcal{L}(\mathcal{H}^{\text{in}}) \rightarrow \mathcal{L}(\mathcal{H}^{\text{out}}) - \text{arbitrary CP map} \]

\[ R_\Lambda \in \mathcal{L}(\mathcal{H}^{\text{out}} \otimes \mathcal{H}^{\text{in}}) - \text{positive operator} \]

1 to 1 relation

\[ R_\Lambda = \Lambda \otimes \mathcal{I} (|\Psi\rangle \langle \Psi|), \text{where } |\Psi\rangle = \frac{1}{\sqrt{\dim \mathcal{H}^{\text{in}}}} \sum_i |i\rangle \otimes |i\rangle \]

\[ \Lambda(\rho) = \text{Tr}_{\mathcal{H}^{\text{in}}} \left( \mathbb{1}_{\mathcal{H}^{\text{out}}} \otimes \rho^T R_\Lambda \right) \]

Trace preservation condition

\[ \text{Tr}_{\mathcal{H}^{\text{out}}} (R_\Lambda) = \mathbb{1}_{\mathcal{H}^{\text{in}}}. \]

Covariance condition

\[ \forall g \in G \Lambda \left( V_g \rho V_g^\dagger \right) = W_g \Lambda(\rho) W_g^\dagger \quad \iff \quad \forall g \in G [R_\Lambda, W_g \otimes V_g^*] = 0 \]
Two qubits

Permutational invariant state of two qubits

\[ \rho_{AB} = \begin{pmatrix} \rho_{00} & 0 & 0 & 0 \\ 0 & \rho_{11} & \rho_{12}^* & \rho_{13}^* \\ 0 & \rho_{12} & \rho_{22} & \rho_{23}^* \\ 0 & \rho_{13} & \rho_{23} & \rho_{33} \end{pmatrix} \]

singlet subspace

triplet subspace
Two qubits

Permutational invariant state of two qubits

\[ \rho_{AB} = \begin{pmatrix} \rho_{00} & 0 & 0 & 0 \\ 0 & \rho_{11} & 0 & 0 \\ 0 & 0 & \rho_{22} & 0 \\ 0 & 0 & 0 & \rho_{33} \end{pmatrix} \]

singlet subspace  \quad \text{triplet subspace}

\[ \rho_A = \rho_B = \frac{1}{2} (\mathbb{1} + r\sigma_z), \quad r = \rho_{33} - \rho_{11} \]

Decorrelation condition

\[ \Lambda(\rho_{AB}) = \frac{1}{2} (\mathbb{1} + \tilde{r}\sigma_z) \otimes \frac{1}{2} (\mathbb{1} + \tilde{r}\sigma_z) = \]

\[ = \begin{pmatrix} 1/4(1 - \tilde{r}^2) & 0 & 0 & 0 \\ 0 & 1/4(1 - \tilde{r}^2) & 0 & 0 \\ 0 & 0 & 1/4(1 - \tilde{r}^2) & 0 \\ 0 & 0 & 0 & 1/4(1 + \tilde{r}^2) \end{pmatrix}. \]
Two qubits

Different signals

Covariance condition

$$\Lambda(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = U_A \otimes U_B \Lambda(\rho_{AB}) U_A^\dagger \otimes U_B^\dagger$$

$$\Lambda : \mathcal{L}(\mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{in}}) \leftrightarrow \mathcal{L}(\mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}})$$

$$[R_\Lambda, U_A \otimes U_B \otimes U_A^* \otimes U_B^*] = 0$$

$$R_\Lambda \in \mathcal{L}(\mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}} \otimes \mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{in}})$$

Thanks to the equivalence of conjugated representation of SU(2)

$$\tilde{R}_\Lambda = 1_{\mathcal{H}_A^{\text{out}}} \otimes C \ R_\Lambda \ 1_{\mathcal{H}_B^{\text{out}}} \otimes C^\dagger,$$ where \( C = (i\sigma_y)^{\otimes 2} \)

$$[\tilde{R}_\Lambda, U_A \otimes U_B \otimes U_A \otimes U_B] = 0$$

$$\Lambda(\rho_{AB}) = \text{Tr}_{\mathcal{H}_A^{\text{in}}}(1_{\mathcal{H}_A^{\text{out}}} \otimes (C \rho_{AB}^T C^\dagger) \ \tilde{R}_\Lambda)$$
Two qubits

different signals

Structure of the decorrelating operation

\[ [\tilde{R}_A, U_A \otimes U_B \otimes U_A \otimes U_B] = 0 \]

after changing the order of subspaces

\[ \mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_B^{\text{out}} \otimes \mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{out}} \leftrightarrow \mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_A^{\text{in}} \otimes \mathcal{H}_B^{\text{out}} \otimes \mathcal{H}_B^{\text{in}}. \]

\[ [\tilde{R}_A, U_A \otimes U_A \otimes U_B \otimes U_B] = 0 \]

\[ \tilde{R}_A = \sum_{i,j=0}^{1} a_{ij} P_A^{(i)} \otimes P_B^{(j)} \]

positive coefficients

\[ a_{ij} = a_{ji} \]

projection on the singlet (i=0), triplet (i=1) subspace of \( \mathcal{H}_A^{\text{out}} \otimes \mathcal{H}_A^{\text{in}} \)

trace preservation condition

\[ a_{00} + 9a_{11} + 6a_{01} = 4 \]
Two qubits
different signals

Solution for the decorrelation problem

\[ \Lambda(\rho_{AB}) = \frac{1}{2}(1 + \tilde{r}\sigma_z) \otimes \frac{1}{2}(1 + \tilde{r}\sigma_z) \]

trivial solution (complete mixing) always exists \( \tilde{r} = 0 \)

non-trivial solutions exists only for states with \( \rho_{00} = \rho_{22} \)

\[ \rho_{AB} = \frac{1}{4} (1 \otimes 1 + r(\sigma_z \otimes 1 + 1 \otimes \sigma_z) - \lambda \sigma_z \otimes \sigma_z) \]

condition that \( a_{ij} \geq 0 \), puts constraint on maximall achievable \( \tilde{r} \)
Decorrelable states of two qubits

different signals

$$\rho_{AB} = \frac{1}{4} (1 \otimes 1 + r(\sigma_z \otimes 1 + 1 \otimes \sigma_z) - \lambda \sigma_z \otimes \sigma_z)$$

Maximall achievable ratio $\tilde{r}/r$

parabolla $\lambda = -r^2$

corresponds to product states

States with $\lambda = 0$ are not decorrelable
Two qubits

**identical signals**

**Covariance condition (weaker)**

\[
\Lambda(U \otimes U \rho_{AB} U^\dagger \otimes U^\dagger) = U \otimes U \Lambda(\rho_{AB}) U^\dagger \otimes U^\dagger
\]

\[
[\tilde{R}_\Lambda, \underbrace{U \otimes U}_{\mathcal{H}^{{\text{out}}} _A \otimes \mathcal{H}^{{\text{out}}} _B} \otimes \underbrace{U \otimes U}_{\mathcal{H}^{{\text{in}}} _A \otimes \mathcal{H}^{{\text{in}}} _B}] = 0
\]

**additionally permutation covariance**

\[
\mathcal{H}^{{\text{out}}} _A \otimes \mathcal{H}^{{\text{out}}} _B \otimes \mathcal{H}^{{\text{in}}} _A \otimes \mathcal{H}^{{\text{out}}} _B = \left( \bigoplus_{j=0}^{1} \mathcal{H}^{{\text{out}}} _j \right) \otimes \left( \bigoplus_{l=0}^{1} \mathcal{H}^{{\text{in}}} _l \right) = \sum_{j,l=0}^{1} \bigoplus_{J=|j-l|}^{j+l} \mathcal{H}^J _{j,l}
\]

\[
\tilde{R}_\Lambda = \sum_{j,l=0}^{1} \bigoplus_{J=|j-l|}^{j+l} s^{J}_{j,l} P^{J}_{j,l}
\]

**positive coefficients**

**projection on** \(\mathcal{H}^J _{j,l}\)

**trace preservation condition**

\[
\frac{1}{2} \sum_{j=0}^{1} \sum_{J=|j-l|}^{j+l} s^{J}_{j,l} \frac{2J+1}{2l+1} = 1, \quad \text{for} \quad l = 0, 1
\]

**4 parameters**

**almost all states are decorrelable**
e.g. State from symetric subspace

\[
\rho_{AB}^{\text{sym}} = \frac{1}{4} \left( \mathbb{1} \otimes \mathbb{1} + r (\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \frac{1 + \lambda}{2} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) - \lambda \sigma_z \otimes \sigma_z \right)
\]

Maximally achievable ratio \( \tilde{\gamma} / \gamma \)

States obtainable by cloning of one qubit - not decorrelable (generalizes to N qubits)

**arbitrary state:** \( \rho_{AB} = p |\psi^-\rangle \langle \psi^-| + (1 - p) \rho_{AB}^{\text{sym}} \)
\[ \Lambda(\rho_{AB}) = \frac{1}{2}(1 + \tilde{r}\sigma_z) \otimes \frac{1}{2}(1 + \tilde{r}\sigma_z) \]

**Different signals**

For given \( \tilde{r} \):
- \text{linear equations for } \mathcal{O}(N) \text{ positive parameters}

**Identical signals**

For given \( \tilde{r} \):
- \text{linear equations for } \mathcal{O}(N^3) \text{ positive parameters}

**Linear programming**
Two-mode gaussian states

Zero mean gaussian states

$$\rho_{AB} = \frac{1}{\pi^4} \int d^4 x e^{-\frac{1}{2} x^T M x} D(x), \quad D(x) = D_A(x_1 + i x_2) \otimes D_B(x_3 + i x_4)$$

correlation matrix

displacement operators

Signals = displacements

$$D_A(\alpha) \otimes D_B(\beta) \rho_{AB} D_A^\dagger(\alpha) \otimes D_B^\dagger(\beta),$$

Covariance condition

$$\Lambda \left[ D_A(\alpha) \otimes D_B(\beta) \rho_{AB} D_A^\dagger(\alpha) \otimes D_B^\dagger(\beta) \right] =$$

$$= D_A(\alpha) \otimes D_B(\beta) \Lambda(\rho_{AB}) D_A^\dagger(\alpha) \otimes D_B^\dagger(\beta)$$

in short

$$\Lambda \left[ D(y) \rho_{AB} D^\dagger(y) \right] = D(y) \Lambda(\rho_{AB}) D^\dagger(y)$$
Decorrelation is easy

**Covariant gaussian channel**

\[
G(\rho_{AB}) = \frac{\sqrt{\det G}}{(2\pi)^2} \int d^4 z \ e^{-\frac{1}{2} z^T G z} D(z) \rho D^\dagger(z)
\]

positive definite matrix

**Output state is a gaussian**

\[
M' = M + \Sigma G^{-1} \Sigma
\]

new correlation matrix  
old correlation matrix

\[
\Sigma = \begin{pmatrix}
\sigma_y & 0 \\
0 & -\sigma_y \\
\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}
0 & -i \\
i & 0 \\
\end{pmatrix}.
\]
Decorrelation is easy

Covariant gaussian channel

\[ G(\rho_{AB}) = \frac{\sqrt{\det G}}{(2\pi)^2} \int d^4 z \ e^{-\frac{1}{2} z^T G z} D(z) \rho D^\dagger(z) \]

postive definite matrix

Output state is a gaussian

To decorrelate, take \( V = -\sigma_y C \sigma_y \), and \( W, Z \) high enough to make \( G^{-1} > 0 \)

Decorrelation easy - orbits of covariance group small
Example: Mixed EPR states

\[ M' = M + \sum G^{-1} \Sigma \]

\[ M = \begin{pmatrix}
1 + 2n & 0 & 2m & 0 \\
0 & 1 + 2n & 0 & -2m \\
2m & 0 & 1 + 2n & 0 \\
0 & -2m & 0 & 1 + 2n
\end{pmatrix} \]

Singel mode states are termal states with \( n \) photons

\[ G^{-1} = \begin{pmatrix}
2m + \epsilon & 0 & 2m & 0 \\
0 & 2m + \epsilon & 0 & -2m \\
2m & 0 & 2m + \epsilon & 0 \\
0 & -2m & 0 & 2m + \epsilon
\end{pmatrix} \]

Singel mode states are termal states with \( n + m \) photons

\[ M' = (1 + 2n + 2m + \epsilon) \mathbb{1} \]
Decorrelation vs. Cocktail party

Signals are encoded on correlated state

\[ U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger \]

Signals get correlated via unknown interaction

\[ \mathcal{E} \left( U_A |0\rangle \langle 0| U_A^\dagger \otimes U_B |0\rangle \langle 0| U_B^\dagger \right) \]

Obtain uncorrelated signals

Reference: