

# Quantum state estimation on co copies

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## One copy of the "unknown"

What can we learn about an unknown quantum state when we have one copy of it?

Polarization of a single photon

$$|\psi\rangle = \cos(\theta/2)|\leftrightarrow\rangle + \sin(\theta/2)e^{i\phi}|\updownarrow\rangle$$

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## Infinite number of copies

Infinite number of copies allows to determine exactly:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2} (\mathbb{I} + \vec{n} \cdot \vec{\sigma}), \quad n_i = \text{Tr}(\rho \sigma_i)$$

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + \langle\sigma_z\rangle & \langle\sigma_x\rangle - i\langle\sigma_y\rangle \\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & 1 - \langle\sigma_z\rangle \end{bmatrix}$$

Three types of measurements are necessary:

$$\langle\sigma_z\rangle : |\leftrightarrow\rangle, |\updownarrow\rangle$$

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The more copies we have, the easier it is to  
state. Different states become more orthog

$$\lim_{N \rightarrow \infty} \langle \psi |^{\otimes N} | \phi \rangle^{\otimes N} = \lim_{N \rightarrow \infty} \langle \psi | \phi \rangle^N =$$

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# Generalized measurement

Standard measurement:

$$A = \sum_i \lambda_i P_i \quad P_i P_j = \delta_{ij} P_i \quad \sum_i P_i = I$$

observable    measurement outcomes    projection op

$$p_i = \text{Tr}(\rho P_i) \quad \text{probability of the res}$$

## Generalized measurement

evolution and standard measurement on *system + ancilla*

$$p_\mu = \text{Tr}(\rho P_\mu) \quad P_\mu > 0 \quad \sum_\mu P_\mu = I$$

$P_\mu$  form a Positive Operator Valued Measure (POVM)

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## State estimation strategy

N copies of an unknown state:  $\rho^{\otimes N}$ ,  $\rho = |\psi\rangle\langle\psi|$

Choose a measurement – set of  $P_\mu$

Choose a guessing strategy:  $\mu \rightarrow \rho_\mu$

Quality (fidelity) of a guess:  $F_\mu = \langle\psi|\rho_\mu|\psi\rangle$

Quality of estimation

$$\begin{aligned} F &= \sum_{\mu} p_{\mu} F_{\mu} = \sum_{\mu} \text{Tr}(\rho^{\otimes N} P_{\mu}) \langle\psi|\rho_{\mu}|\psi\rangle = \\ &= \langle\psi|\sum_{\mu} \text{Tr}(\rho^{\otimes N} P_{\mu}) \rho_{\mu}|\psi\rangle \quad \text{should not de} \end{aligned}$$

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*Massar, Popescu PRL, 74, 1259 (1995)*

one copy:  $F = \frac{2}{3}$

$N$  copies:  $F = \frac{N+1}{N+2}$

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*Bruss, Machiavello, PLA, 253, 249 (1999)*

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## Cloning and Estimation – the Moti

$N \rightarrow M$  cloning

$$|\psi\rangle^{\otimes N} \otimes |0\rangle^{M-N} \otimes |A\rangle \quad |\Phi\rangle \in \mathcal{H}^{\otimes M}$$

input states    blank states    ancilla

$$\tilde{\rho} = \text{Tr}_A |\Phi\rangle\langle\Phi| \quad \text{the state of } M \text{ clones}$$

$$\rho_1 = \text{Tr}_{2,\dots,M} \tilde{\rho} \quad \text{the state of the first clone}$$

$$\rho_1 = \rho_2 = \dots = \rho_M =: \rho \quad \text{clones should be identical}$$

optimal fidelity:

$$F = \langle\psi|\rho|\psi\rangle = \frac{M(N+1)+N(d-1)}{M(N+d)}$$

Clones are correlated!:  $\tilde{\rho} \neq \rho^{\otimes M}$

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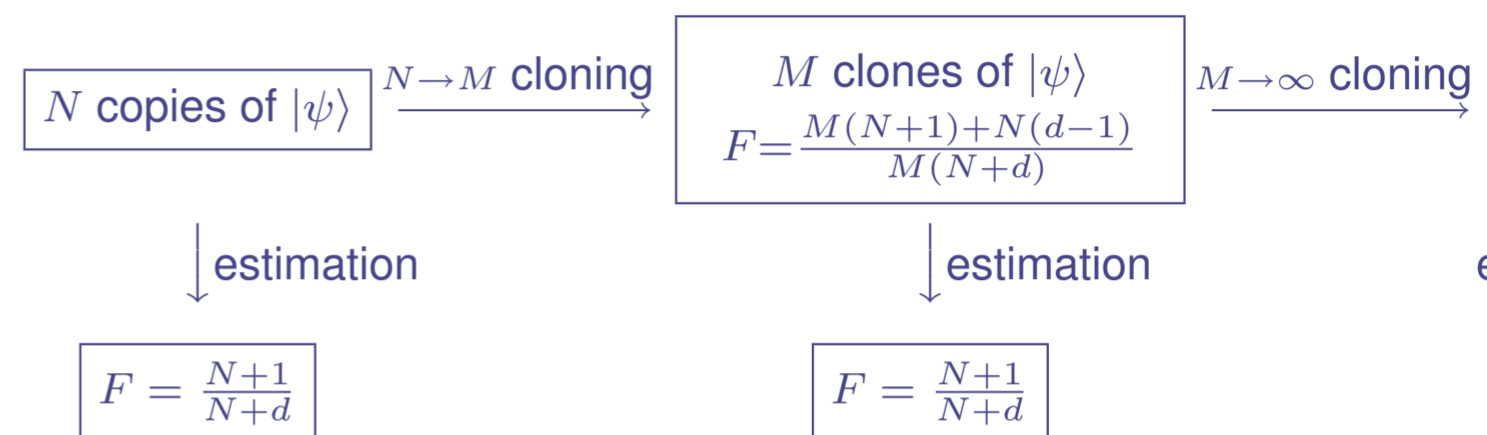
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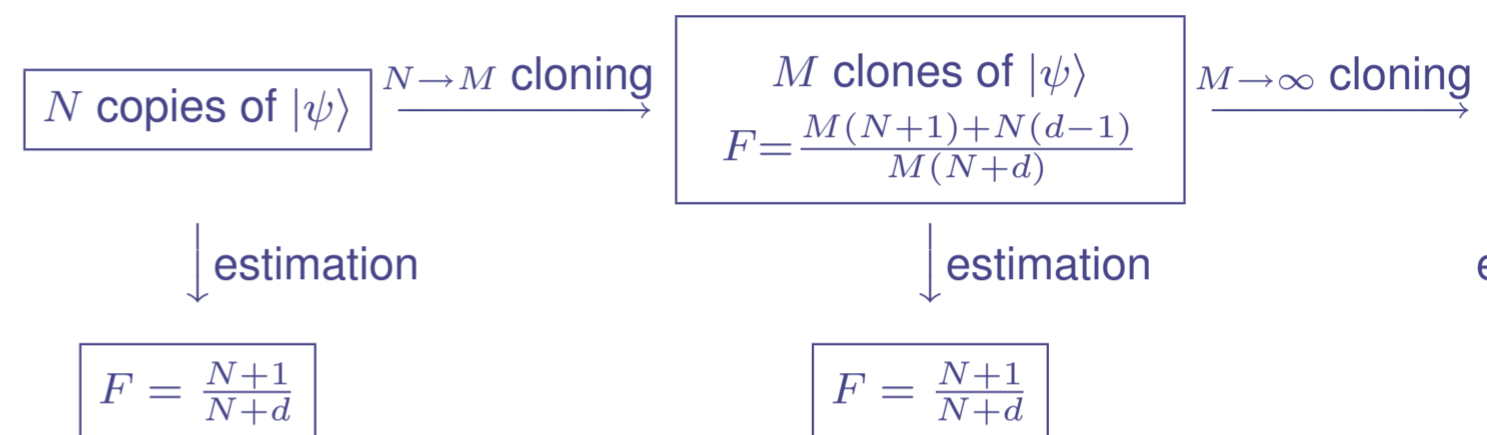


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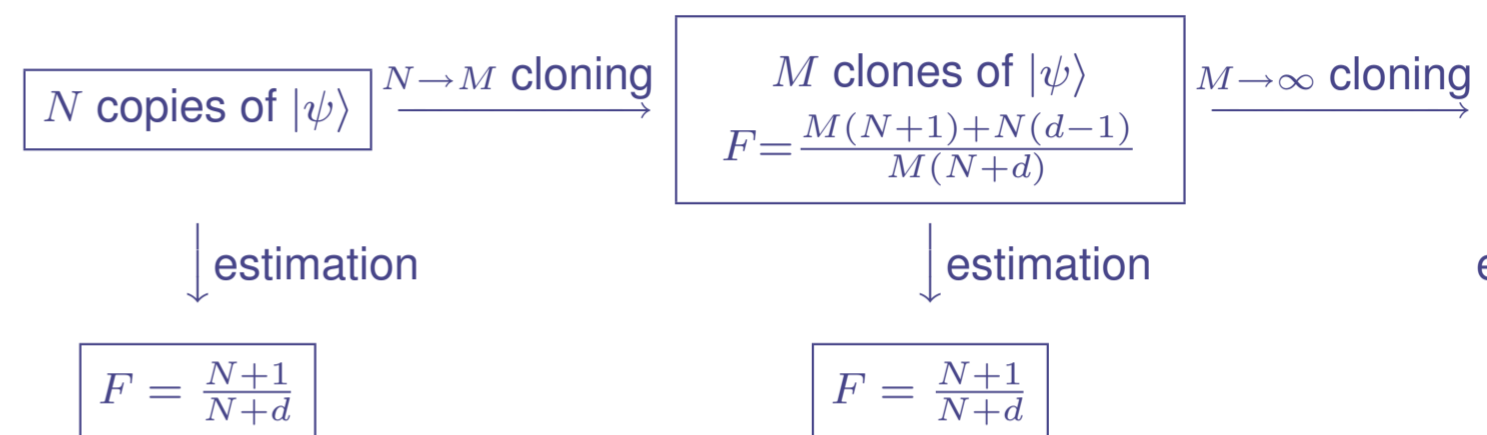


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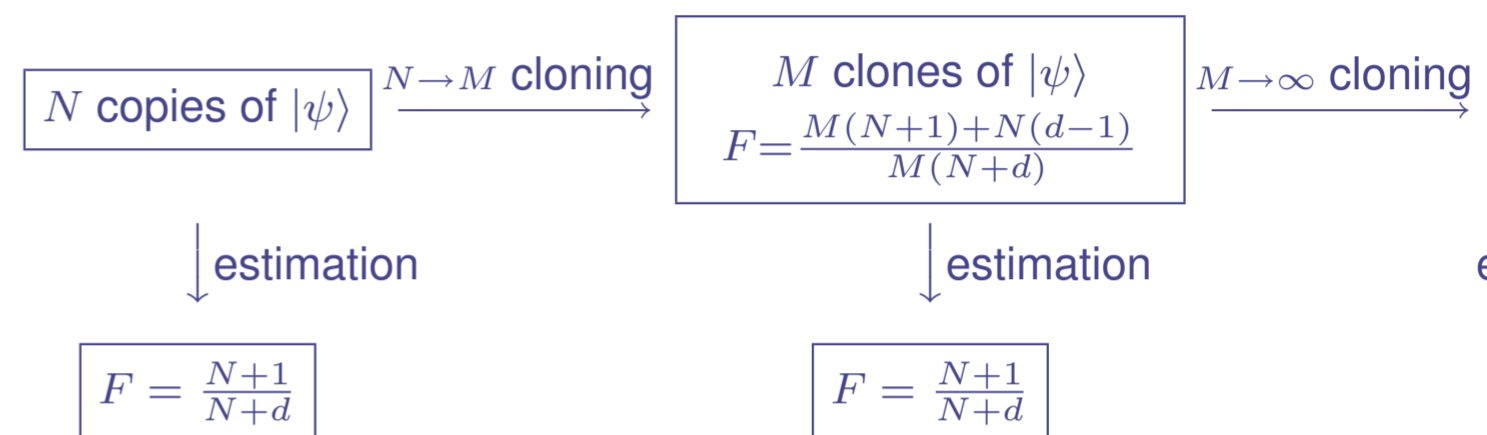
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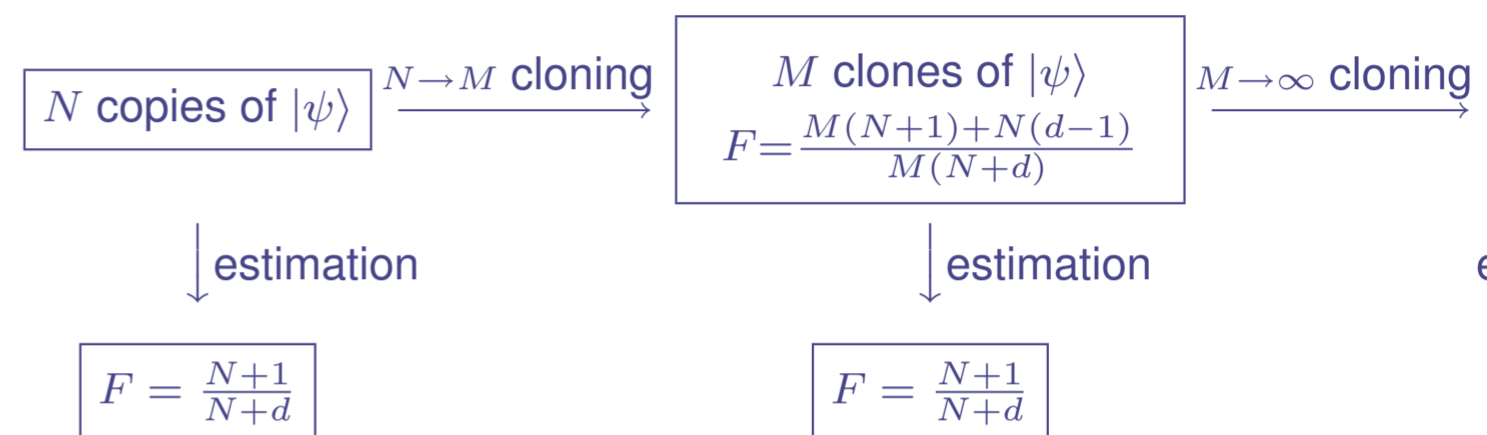


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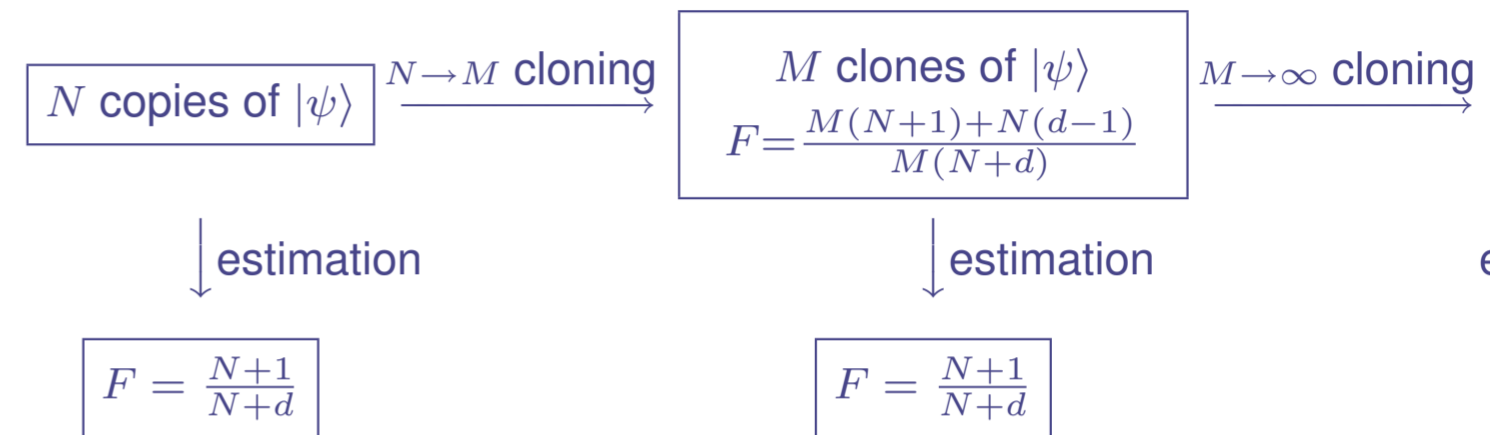


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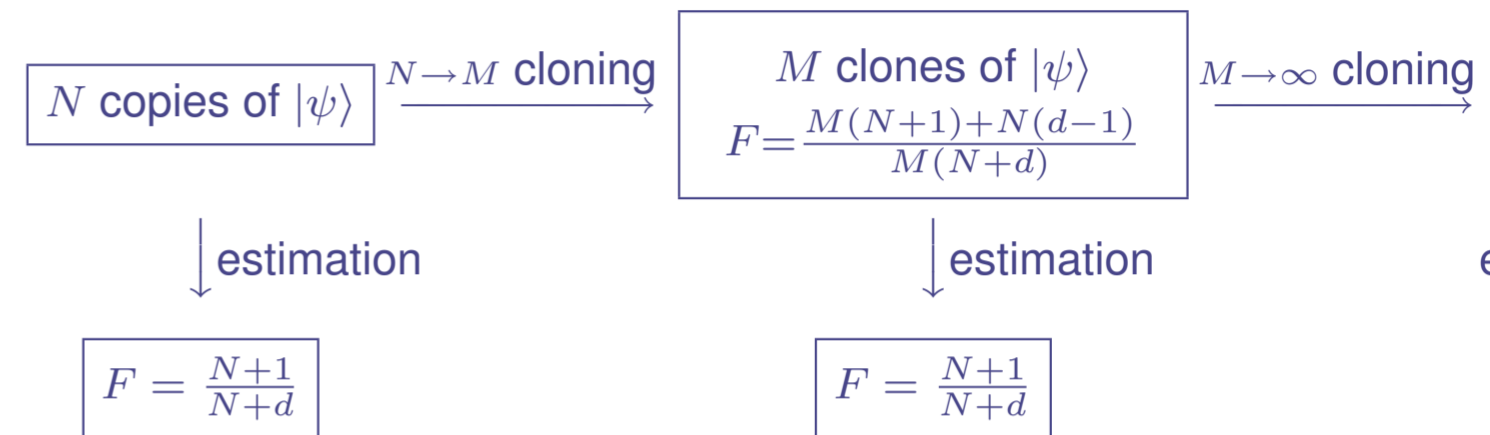


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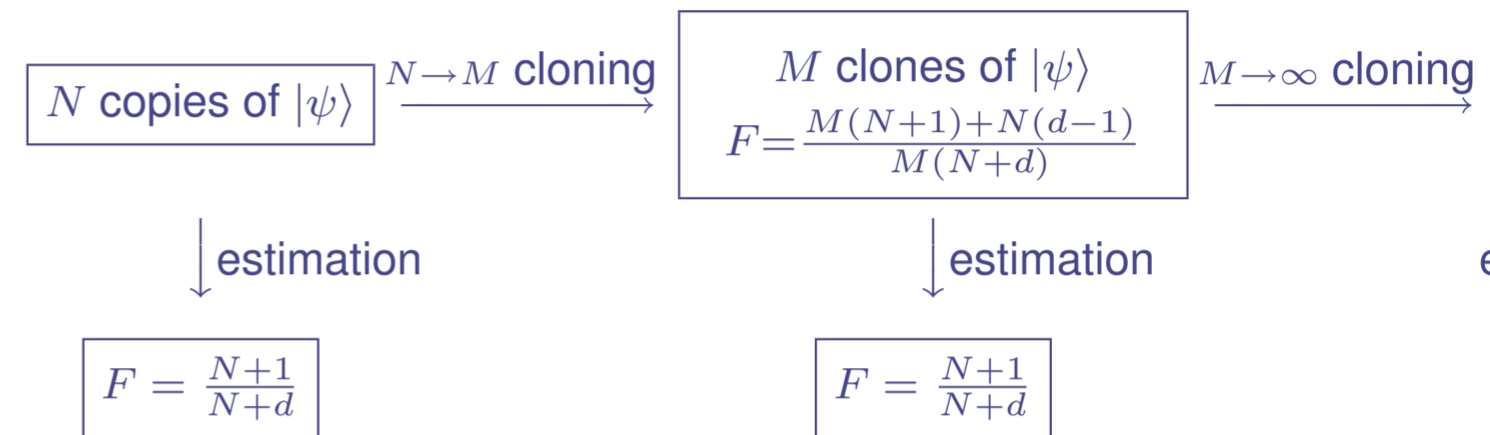


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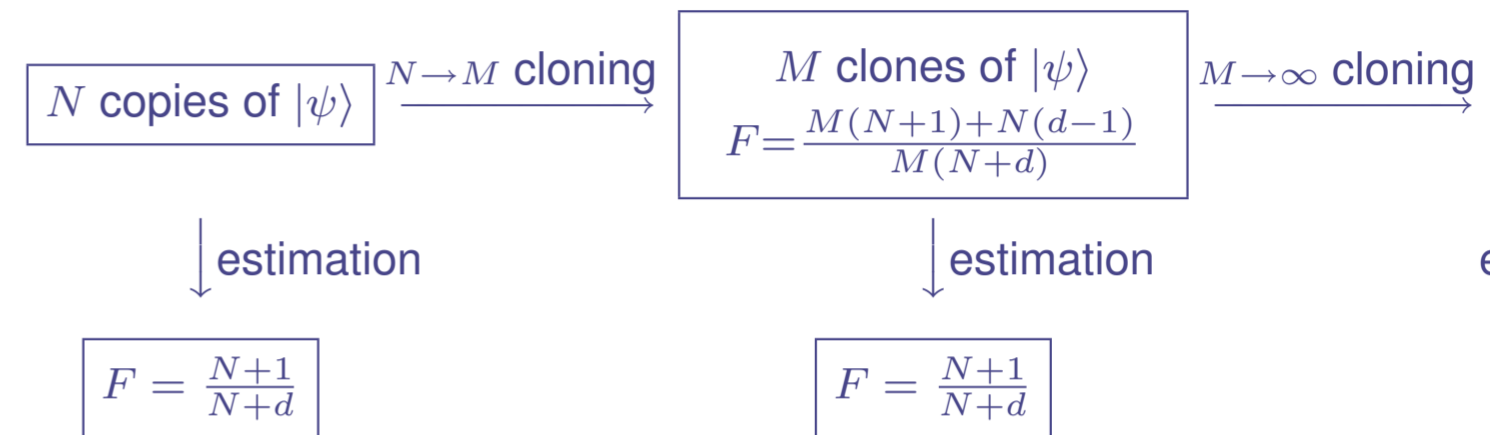


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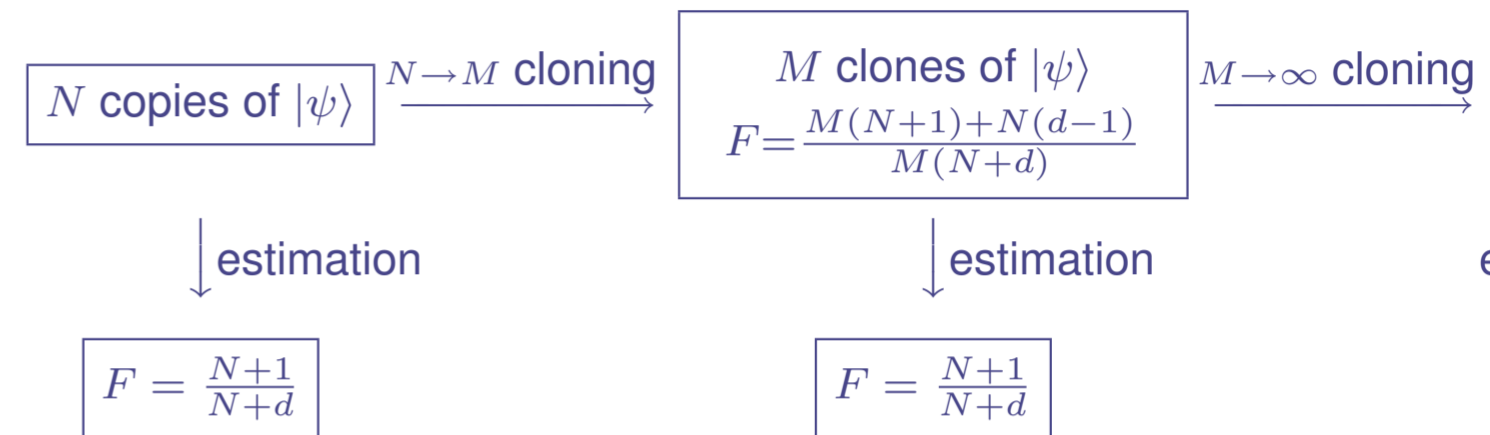


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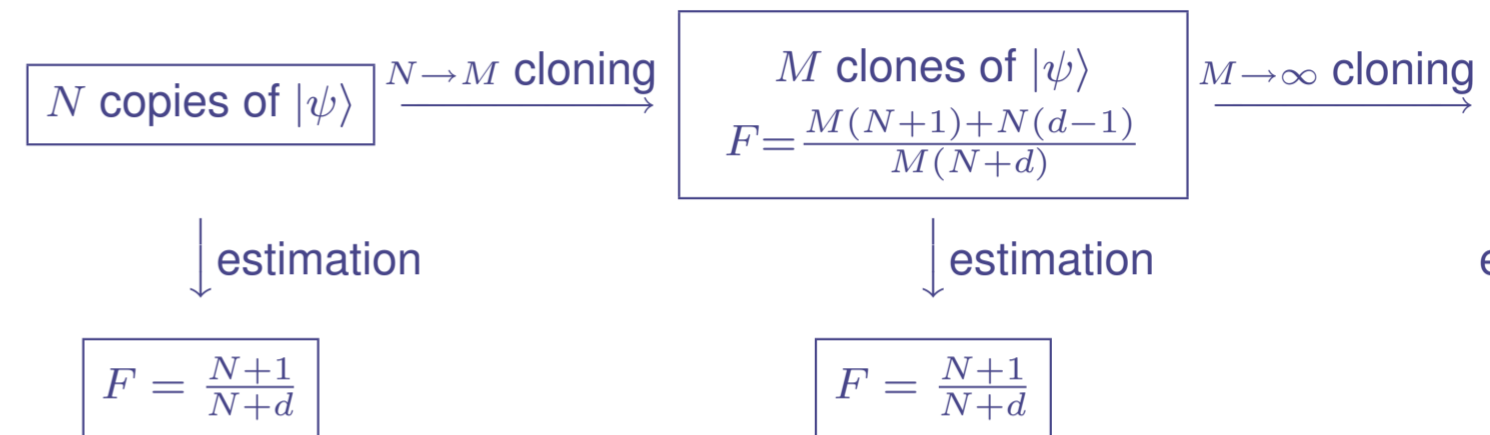


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How correlations between copies influence estimation quality (RDD *Phys. Rev. A* 71, 062321)

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## Permutation invariance

$$\Pi \tilde{\rho} \Pi^\dagger = \tilde{\rho}, \quad \forall \Pi \in \text{repr}(S_N)$$

$N$  qubits ( $N$  spin  $1/2$ )

a convenient basis, under action of  $SU(2)^{\otimes N}$  :

$$|j, m, \alpha\rangle, \quad j = 0, \dots, N/2; \quad m = -j, \dots, j; \quad \alpha$$

total angular momentum    projection on the "z" axis    equivalent r

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## Permutation invariant states

Example: 4 qubits

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## Fidelity of estimation

Arbitrary permutationally invariant state:

$$\tilde{\rho} = \sum_{j=0(\frac{1}{2})}^{N/2} \frac{p_j}{d_j} \tilde{\rho}_j \otimes \mathbf{1}_{d_j} \quad d_j - \text{multiplicity of rep}$$

$p_j \geq 0$ ,  $\sum_j p_j = 1$ ,  $\tilde{\rho}_j$  - are density matrices on  $2j + 1$  dim

Optimal estimation fidelity

$$F = \frac{1}{2} \left( 1 + \sum_{j=0(\frac{1}{2})}^{N/2} \frac{p_j \text{Tr}(J_z \tilde{\rho}_j)}{j + 1} \right)$$

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The best state  $\tilde{\rho}$  for estimation

Subspaces with high  $j$  are bad for the state

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Fidelity dependence on  $\eta$ :

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## Product state

Product state:  $\tilde{\rho} = \rho^{\otimes 4}$

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RDD *Phys. Rev. A* 71, 062321 (2005)

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