Quantum enhanced metrology

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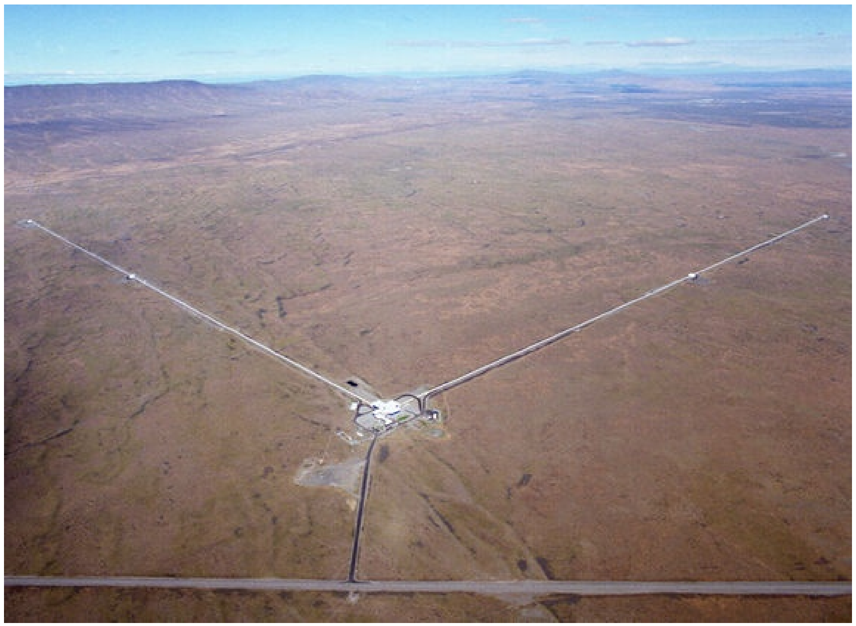
INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

FNP Foundation for Polish Science TEAM Programme

EUROPEAN UNION
EUROPEAN REGIONAL DEVELOPMENT FUND

Interferometry at its (classical) limits

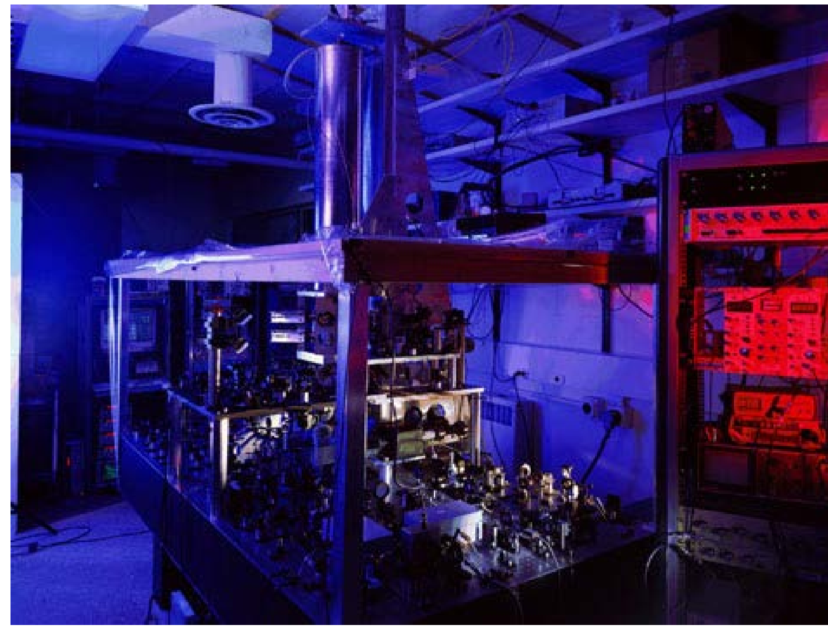
LIGO - gravitational wave detector



Michelson interferometer

$$\Delta L/L \approx 10^{-22}$$

NIST - Cs fountain atomic clock



Ramsey interferometry

$$\Delta t/t \approx 10^{-16}$$

Precision limited by:

$$\text{shot noise} \propto 1/\sqrt{N}$$

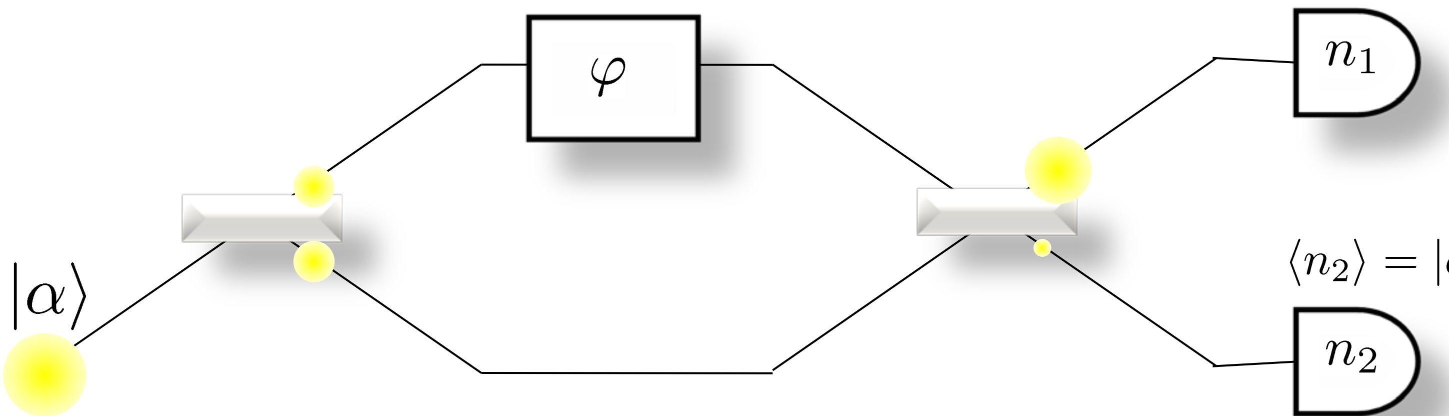
N - number of photons

$$\text{projection noise} \propto 1/\sqrt{N}$$

N - number of atoms

Classical phase estimation

$$\langle n_1 \rangle = |\alpha|^2(1+\cos\varphi)/2$$



$$\langle n_2 \rangle = |\alpha|^2(1-\cos\varphi)/2$$

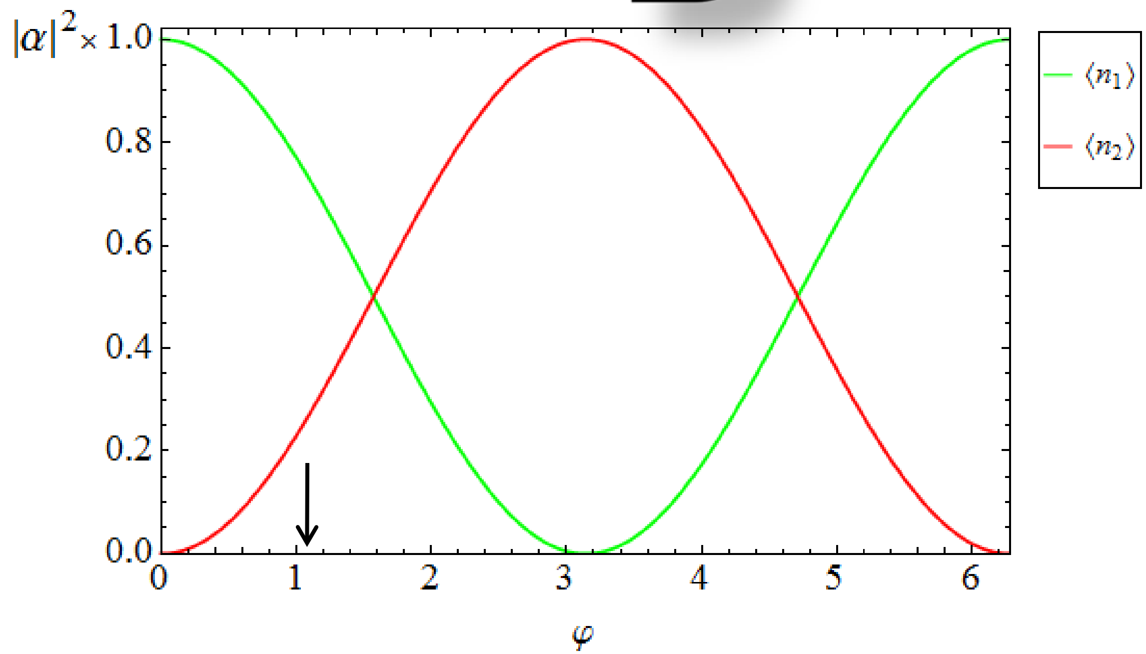
detecting n_1 and n_2

+

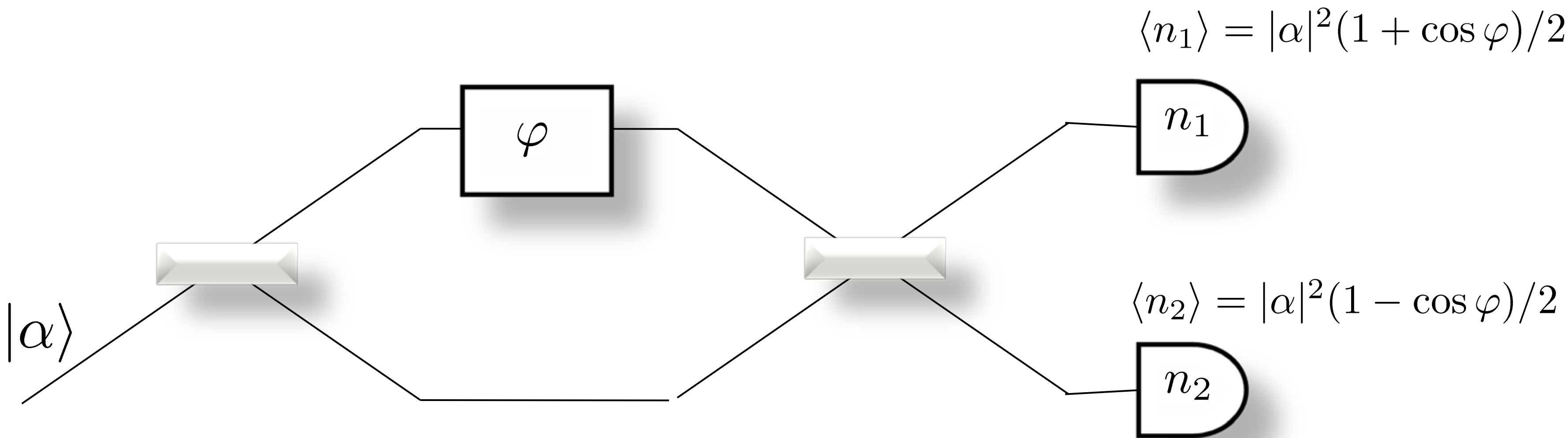
knowing theoretical dependence of n_1, n_2 on φ

↓

we can estimate φ



Classical phase estimation

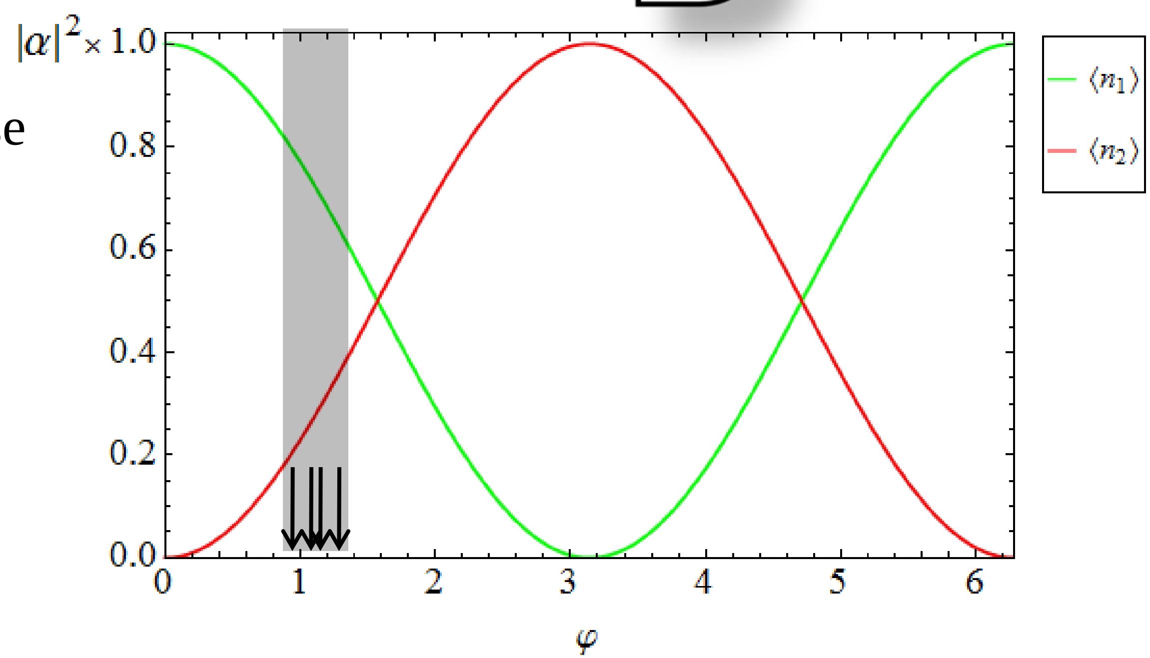


n_1 and n_2 are subject to shot noise

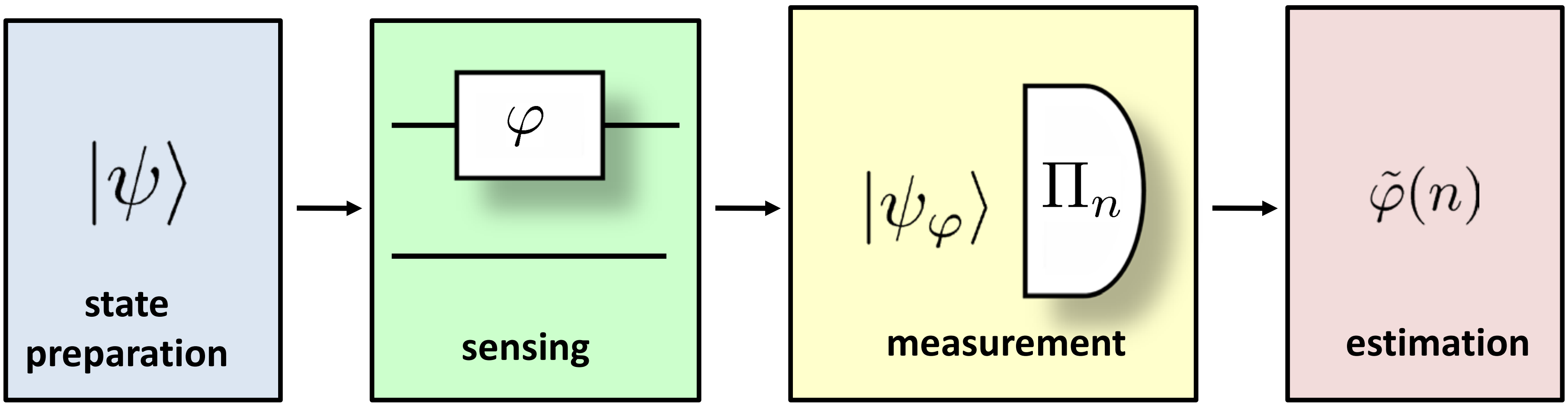
each measurement yields a bit different φ

$$\Delta\varphi \propto \frac{1}{|\alpha|} = \frac{1}{\sqrt{\bar{n}}}$$

Shot noise scaling



Quantum phase estimation



Minimize $\langle(\tilde{\varphi}-\varphi)^2\rangle$ over the choice of $|\psi\rangle$, Π_n and $\tilde{\varphi}$

$$\Delta^2\varphi = \langle(\tilde{\varphi}-\varphi)^2\rangle = \int d\varphi\, p(\varphi) \sum_n p(n|\varphi)[\tilde{\varphi}(n)-\varphi]^2$$

a priori knowledge

$\langle\psi_\varphi|\Pi_n|\psi_\varphi\rangle$

$4\sin^2\left[\frac{\tilde{\varphi}(n)-\varphi}{2}\right]$

In general a very hard problem!

$$\Delta^2\varphi = \int d\varphi\, p(\varphi) \sum_n \langle\psi_\varphi|\Pi_n|\psi_\varphi\rangle[\tilde{\varphi}(n) - \varphi]^2$$

Local approach

we want to sense small fluctuations around a known phase

$$p(\varphi) \approx \delta(\varphi - \varphi_0)$$

Tool: Fisher Information, Cramer-Rao bound

$$\Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F}}$$

$$F = 4[\langle\psi_\varphi|\hat{n}_1^2|\psi_\varphi\rangle - \langle\psi_\varphi|\hat{n}_1|\psi_\varphi\rangle^2]$$

The optimal N photon state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle)$$

$$\boxed{\Delta\tilde{\varphi} \approx \frac{1}{N}}$$

J. J. . Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, *Phys. Rev. A* **54**, R4649 (1996).

Global approach

no a priori knowledge about the phase

$$p(\varphi) \approx \frac{1}{2\pi}$$

Tool: Symmetry implies a simple structure of the optimal measurement

Optimal state: $|\psi\rangle = \sum_{n=0}^{N} \alpha_n |n, N-n\rangle$

$$\alpha_n = \sqrt{\tfrac{2}{N+2}} \sin\left[\tfrac{(n+1)\pi}{N+2}\right]$$

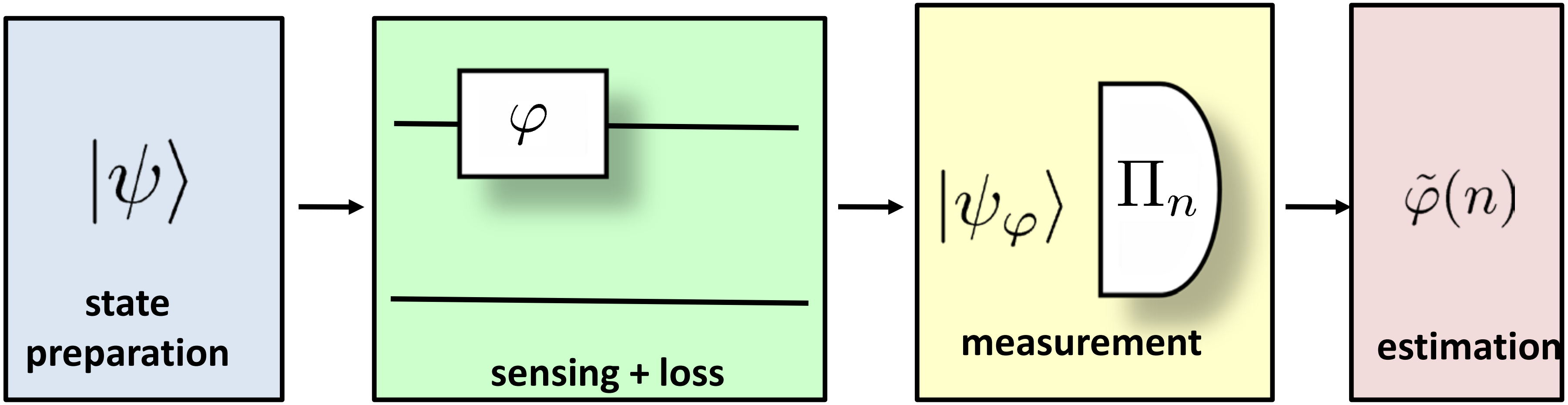
$$\boxed{\Delta\tilde{\varphi} \approx \frac{\pi}{N+2}}$$

D. W. Berry and H. M. Wiseman, *Phys. Rev. Lett*. **85**, 5098 (2000).

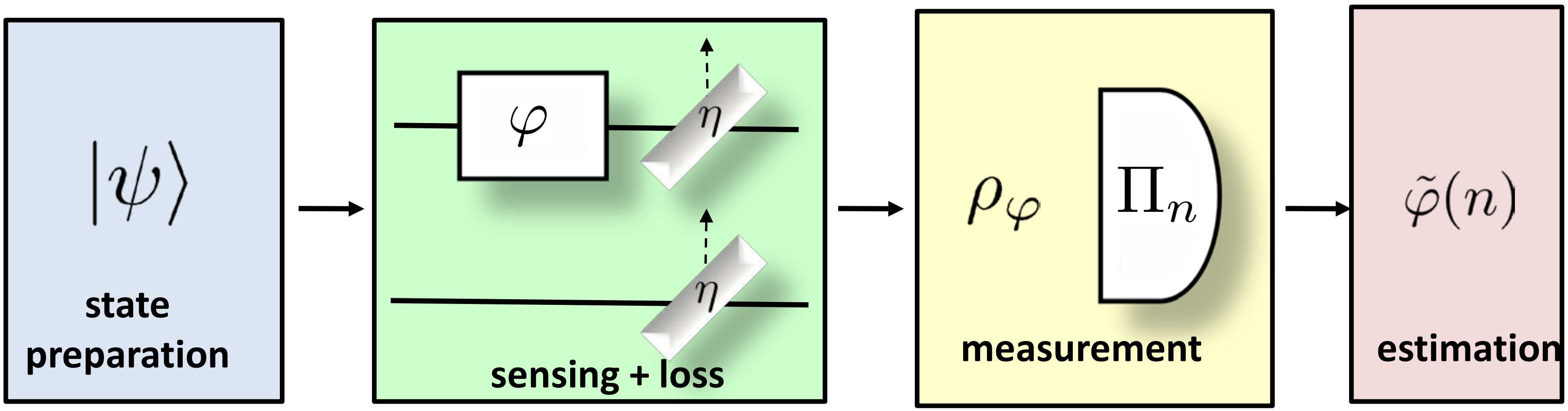
Heisenberg scaling

In reality there is loss…

Phase estimation in the presence of loss



Phase estimation in the presence of loss



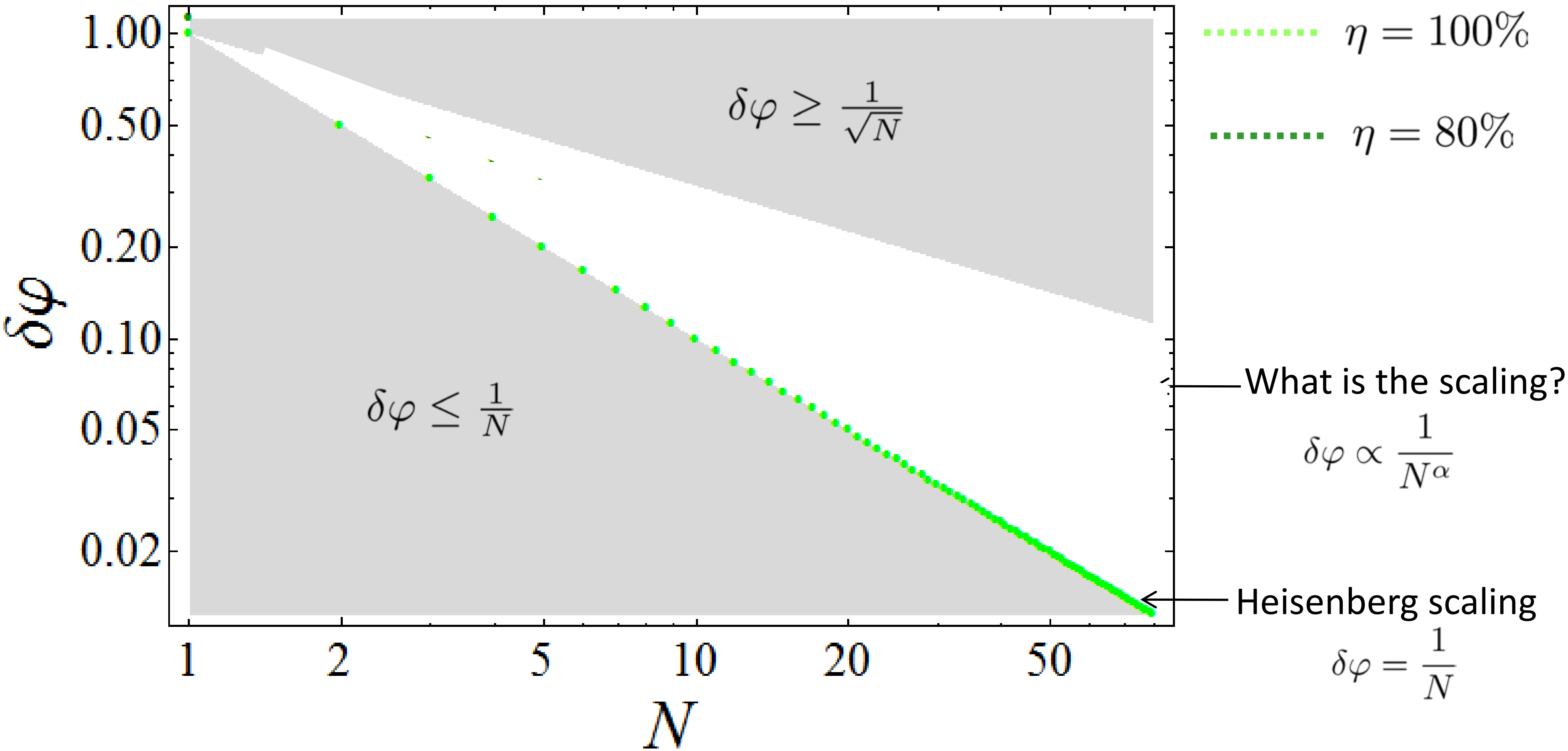
- no analytical solutions for the optimal states and precission
- calculating Fisher information not trivial (symmetric logarithmic derrivative)

- phase sensing and loss commute (no ambiguity in ordering)
- in the global approach the optimal measurements is not altered – the solution is obtained by solving an eigenvalue problem (fast)
- effective numerical optimization procedures yielding global minima

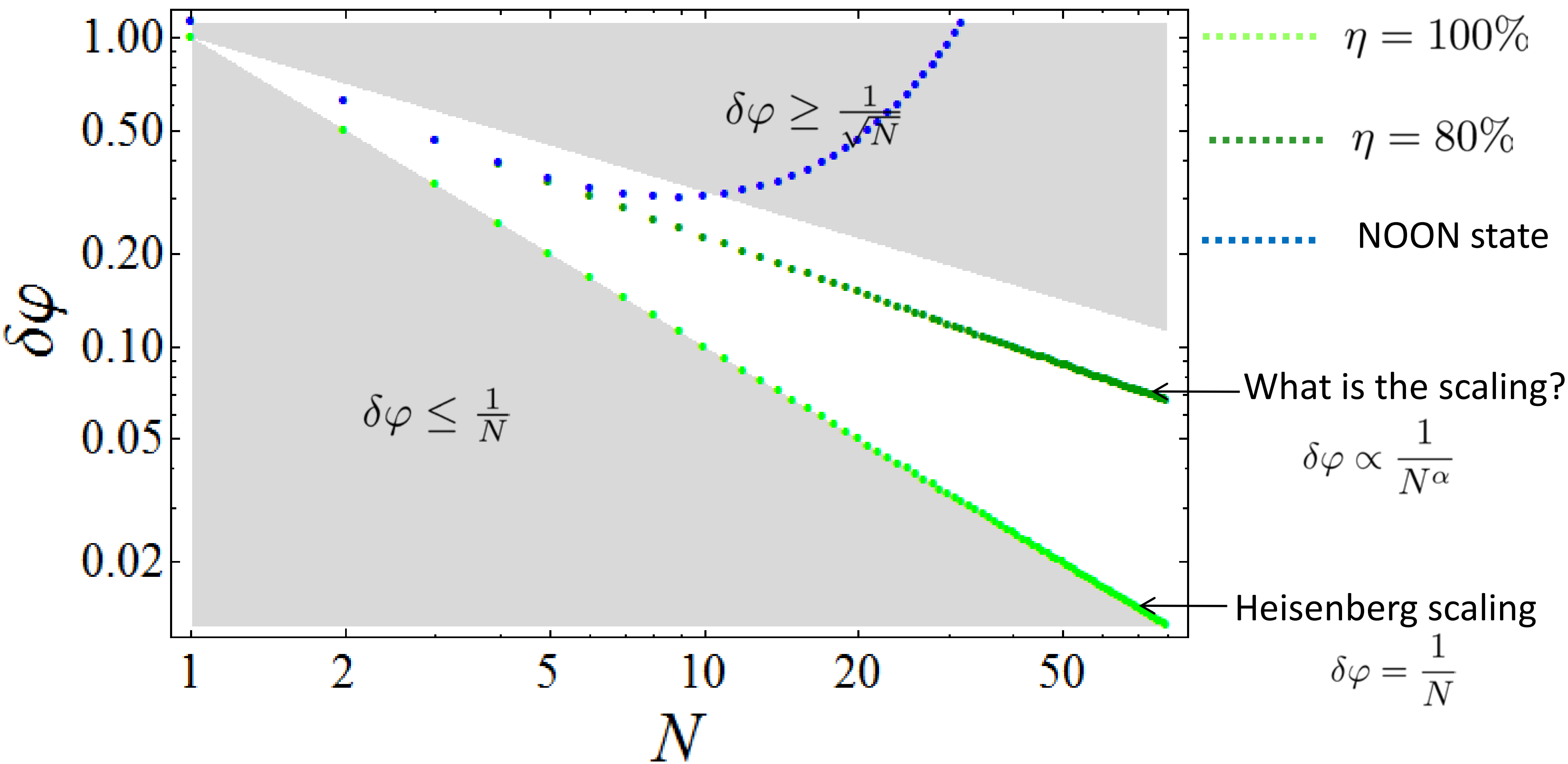
R. Demkowicz-Dobrzanski, et al. ***Phys. Rev.*** **A 80, 013825 (2009)**

U. Dorner, et al.., ***Phys. Rev. Lett.*** **102, 040403 (2009)**

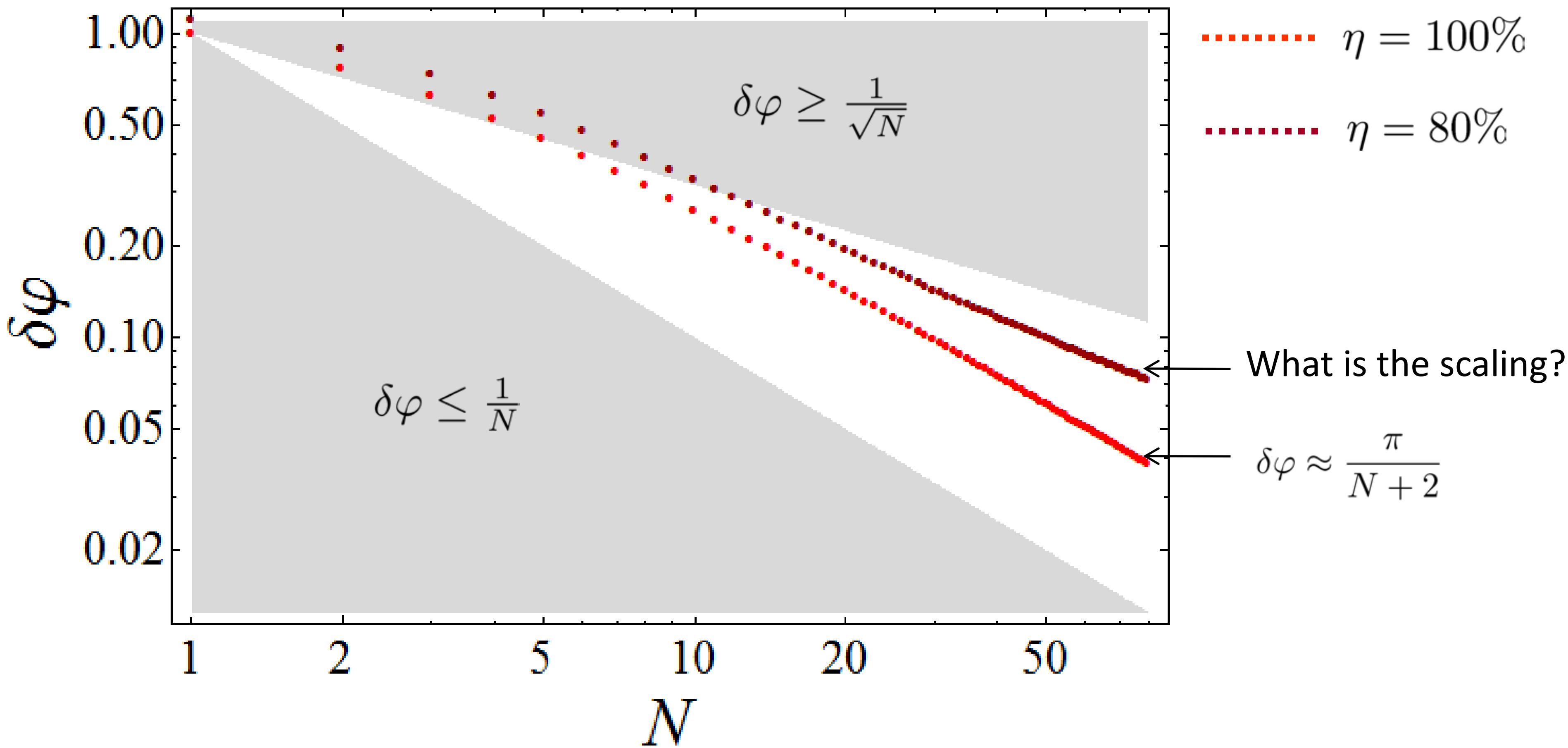
Estimation uncertainty with the number of photons used (local approach)



Estimation uncertainty with the number of photons used (local approach)

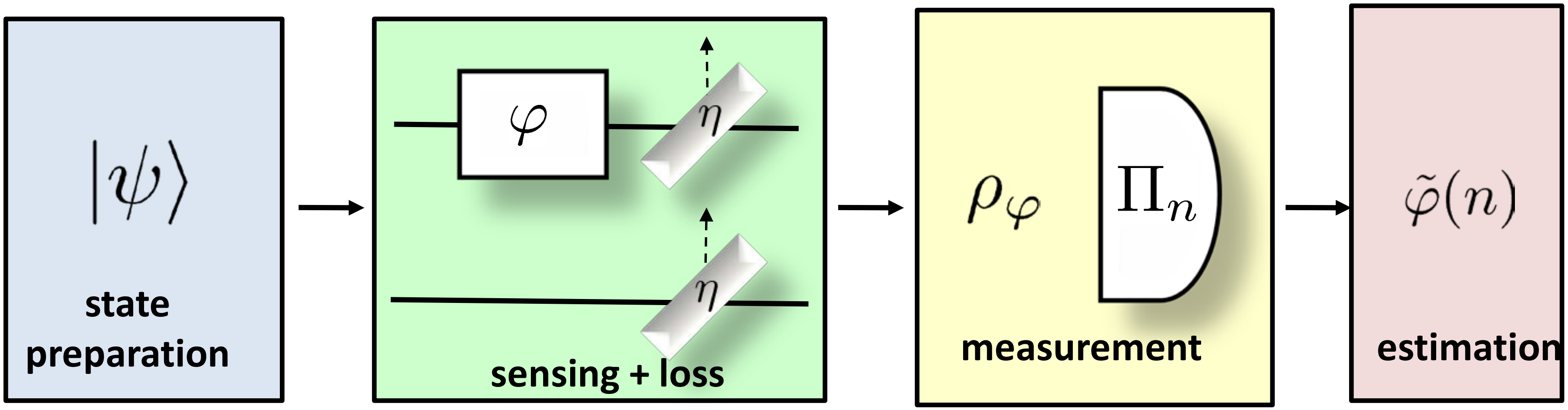


Estimation uncertainty with the number of photons used (global approach)



Do quantum states provide beter scaling exponent in the presence of loss?

Fundamental bound on uncertainty in the presence of loss (global approach)



$$\delta\varphi_{\text{quantum}} \geq \sqrt{\frac{1-\eta}{\eta N}} + O\left(\frac{1}{N}\right)$$

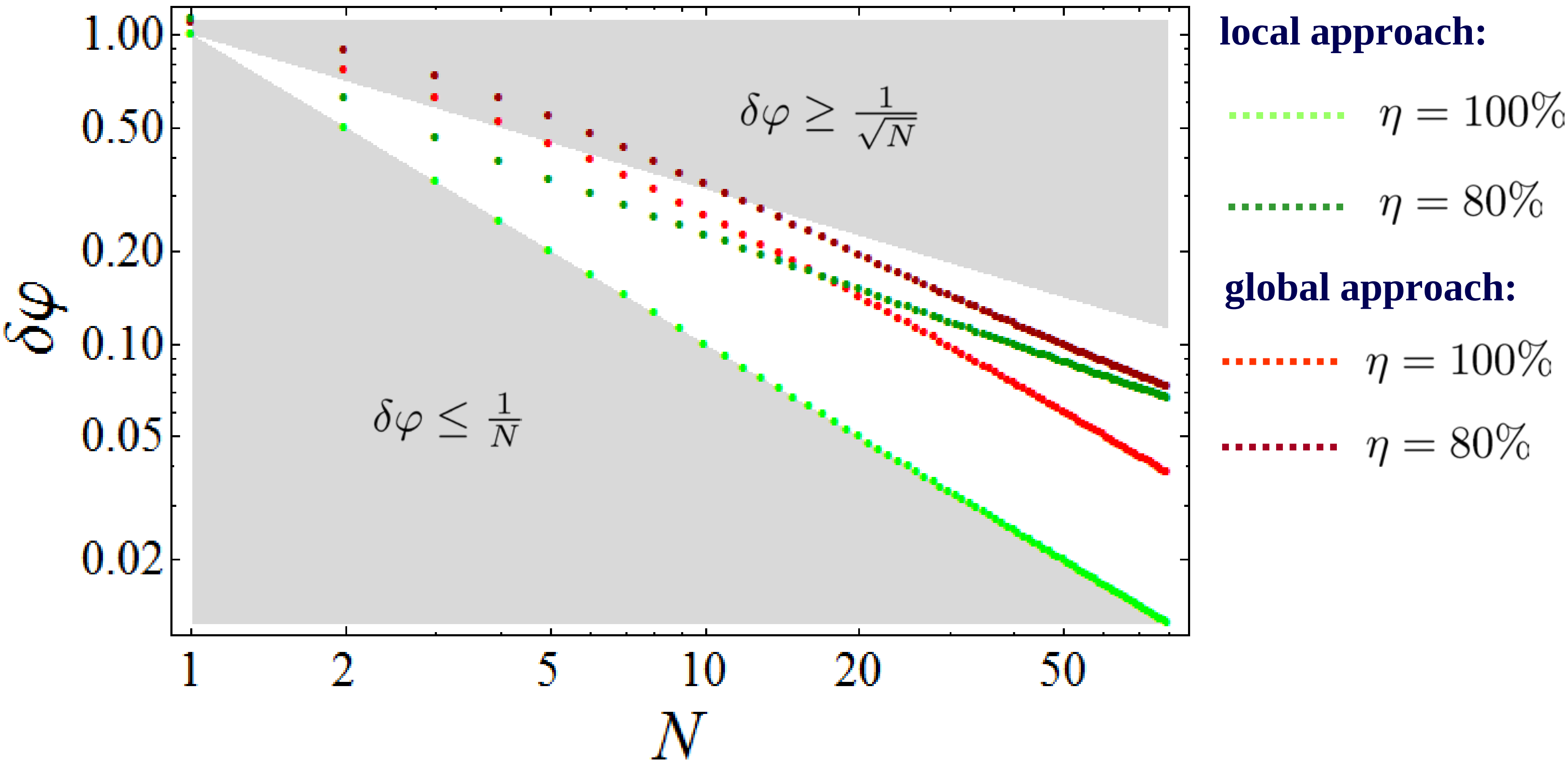
J. Kolodynski and R.Demkowicz-Dobrzanski, *Phys. Rev. A* 82, 053804 (2010)

the same bound can be derived in the local approach:

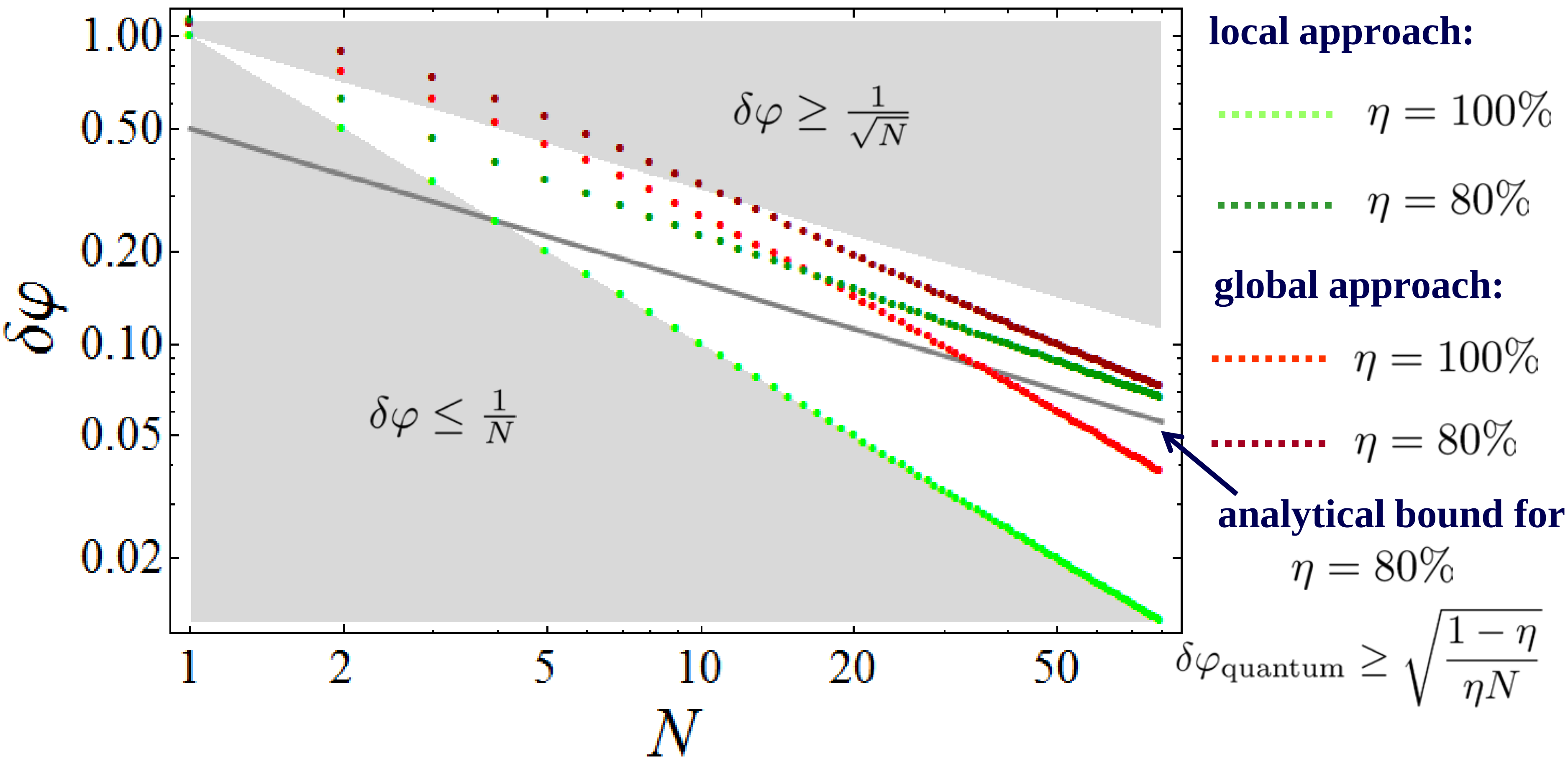
S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, *Phys. Rev. A*, 83, 021804 (2011)

B. M. Escher, R. L. de Matos Filho, and L. Davidovich, *Nature Physics*, 7, 406 (2011).

Fundamental bound on uncertainty in the presence of loss (global approach)



Fundamental bound on uncertainty in the presence of loss (global approach)



J. Kolodynski and R.Demkowicz-Dobrzanski, *Phys. Rev. A* 82, 053804 (2010)

Fundamental bound on asymptotic quantum gain in phase estimation

$$\delta\varphi_{\text{quantum}} \geq \sqrt{\frac{1-\eta}{\eta N}} + O\left(\frac{1}{N}\right) \qquad \delta\varphi_{\text{classical}} = \sqrt{\frac{1}{\eta N}}$$

$$\boxed{\lim_{N\to\infty} \frac{\delta\varphi_{\text{classical}}}{\delta\varphi_{\text{quantum}}} \leq \frac{1}{\sqrt{1-\eta}}}$$

Example: $\eta = 80\%$ $\quad 1/\sqrt{1-\eta} \approx 2.24$

even for moderate loss quantum gain degrades quickly

Summary

• Asymptotically, loss renders quantum phase estimation uncertainty scaling classical and destroys the Heisenberg scaling.

• Quantum states can be practically useful only for very small degree of loss (loss <1% implies gain> 10) or small number of probes

• Neither adaptive measurements, nor photon distinguishability can help

K. Banaszek, R. Demkowicz-Dobrzanski, and I. Walmsley, ***Nature Photonics*** **3, 673 (2009)**
V. Giovannetti, S. Lloyd, and L. Maccone, ***Nature Photonics*****, 5, 222 (2011).**
U. Dorner, et al.   ***Phys. Rev. Lett.*** **102, 040403 (2009)**
R. Demkowicz-Dobrzanski et al ***Phys. Rev.*** **A 80, 013825 (2009)**
M. Kacprowicz, R. Demkowicz-Dobrzanski, W. Wasilewski, and K. Banaszek, **Nature Photonics 4, 357(2010)**
J. Kolodynski and R.Demkowicz-Dobrzanski, ***Phys. Rev. A*** **82, 053804 (2010)**
S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, ***Phys. Rev. A*****, 83, 021804 (2011)**
B. M. Escher, R. L. de Matos Filho, and L. Davidovich, **Nature Physics, 7, 406 (2011).**

Lecture 1
10 sierpnia 2011
10:10

Quantum Enhanced Metrology

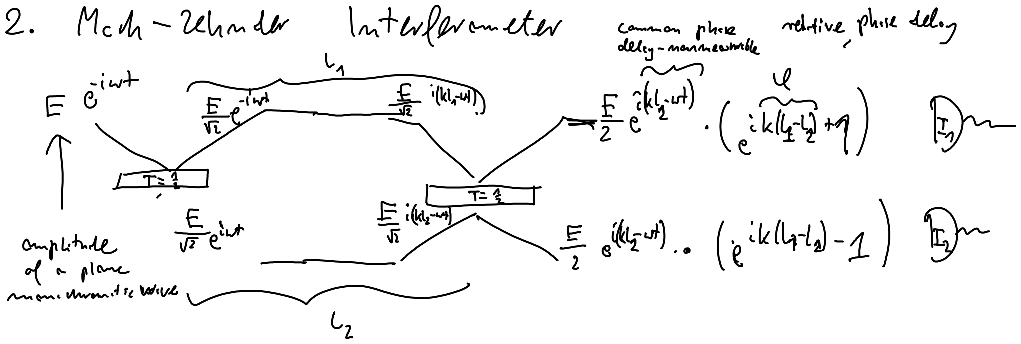
1. Introduction

Metrology - science of measurement
in particular: designing measurement schemes reaching best possible precision.

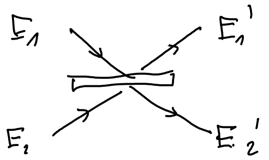
Quantum enhanced metrology - achieve the precision restricted only by the laws of quantum mechanics.

One of the most important tools for high precision measurements of e.g. length is interferometry.

2. Mach-Zehnder Interferometer



Action of an ideal (lossless) beamsplitter



$$\begin{bmatrix} E_1' \\ E_2' \end{bmatrix} = \begin{bmatrix} r & t \\ t & -r \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

$$R = r^2 \quad T = t^2 \quad R+T=1$$

more generally $\begin{bmatrix} r e^{i\theta_1} & t e^{i\theta_2} \\ t e^{-i\theta_2} & -r e^{-i\theta_1} \end{bmatrix}$

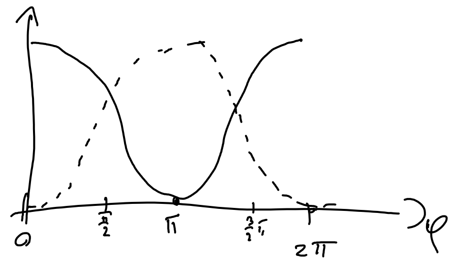
$$\varphi = k \cdot \Delta L = \frac{2\pi}{\lambda}\Delta L$$

$$I_1 = \frac{1}{4}|E|^2 \cdot |e^{i\varphi}+1|^2 = \frac{1}{2}|E|^2(1+\cos\varphi) = |E|^2 \cos^2\frac{\varphi}{2}$$

$$I_2 = \frac{1}{4}|E|^2 \, |e^{i\varphi}-1|^2 = \frac{1}{2}|E|^2(1-\cos\varphi) = |E|^2 \sin^2\frac{\varphi}{2}$$

Measuring I_1, I_2 we can estimate φ, and assuming we know λ we can learn ΔL.
What limits the precision of estimating φ?

3. Estimating φ using classical light



There is an ambiguity φ, $2\pi-\varphi$ give the same I_1, I_2
but this is not a problem it is enough to measure
e.g. two times introducing additional known phase shift.

Apart from that if we knew I_1, I_2 perfectly, we would learn φ perfectly $\cos\varphi = \frac{I_1 - I_2}{|E|^2}$ $\varphi = \arccos\frac{I_1 - I_2}{|E|^2}$

But I_1, I_2 are never known perfectly ...

Light consist of photons, Intensity is proportional to the number of photons absorbed $I \sim n$

But n is discrete so we will not get arbitrary good precision.
And what is more important: classical states of light have Poissonian statistics of photon number distribution.

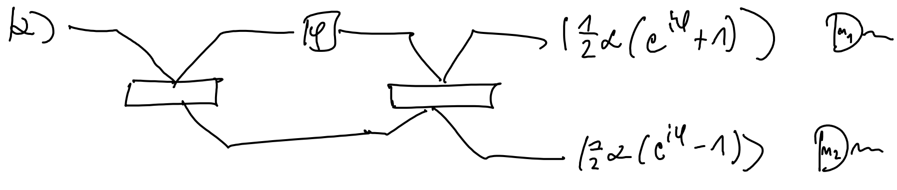
$|\alpha\rangle$ - coherent state representing classical state of light. α - amplitude normalized such that $|\alpha|^2$ - mean number of photons

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad p_n = (\langle n|\alpha\rangle)^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

Poissonian Statistics

$$\langle n\rangle = |\alpha|^2, \quad \Delta^2 n = \langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle = |\alpha|^2$$

shot noise



One registers n_1, n_2 which are governed by Poissonian distributions with $\langle n_1\rangle = \frac{1}{2}|\alpha|^2(1+\cos\varphi)$ $\langle n_2\rangle = \frac{1}{2}|\alpha|^2(1-\cos\varphi)$

$$p_{n_1 n_2} = e^{-\langle n_1\rangle}\frac{\langle n_1\rangle^{n_1}}{n_1!} \cdot e^{-\langle n_2\rangle}\frac{\langle n_2\rangle^{n_2}}{n_2!}$$

If we infer $\cos\tilde{\varphi} := \frac{n_1 - n_2}{|\alpha|^2}$, φ will fluctuate due to n_1, n_2 fluctuations.

What is the estimation uncertainty ?

Calculate the variance of $\cos\varphi$:

$$\Delta^2\cos\tilde{\varphi} = \langle\cos^2\tilde{\varphi}\rangle - \langle\cos\tilde{\varphi}\rangle^2 = \frac{1}{|\alpha|^4}\cdot\left(\langle n_1^2\rangle + \langle n_2^2\rangle - 2\langle n_1 n_2\rangle\right) -$$

$$- \frac{1}{|\alpha|^4}(\langle n_1\rangle - \langle n_2\rangle)^2 = \frac{1}{|\alpha|^4}\left(\Delta^2 n_1 + \Delta^2 n_2\right) = \frac{1}{|\alpha|^4}(\langle n_1\rangle + \langle n_2\rangle)$$

$$- \frac{1}{|\alpha|^4}\cdot \left(\langle n_1\rangle - \langle n_2\rangle\right)^2 = \frac{1}{|\alpha|^4}\cdot\left(\Delta^2 n_1 + \Delta^2 n_2\right) = \frac{1}{|\alpha|^4}\left(\langle n_1\rangle + \langle n_2\rangle\right)$$

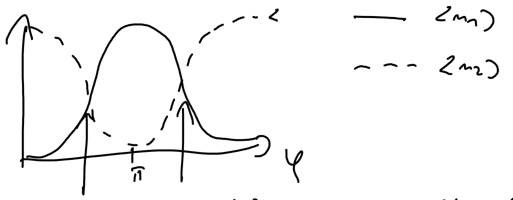
{ since n_1, n_2 independent

$$= \frac{1}{2|\alpha|^2}\left[(1+\cos\varphi) + (1-\cos\varphi)\right] = \frac{1}{|\alpha|^2} = \frac{1}{\langle n\rangle}$$

$\langle n\rangle$ mean number of photons used.

$$\Delta^2\varphi = \frac{\Delta^2 \cos\varphi}{\left(\frac{d\cos\varphi}{d\varphi}\right)^2} = \frac{1}{\langle n\rangle \sin^2\varphi}$$

- $\frac{1}{\langle n\rangle}$ precision scaling (shot noise scaling)
- precision depends on the true value of φ

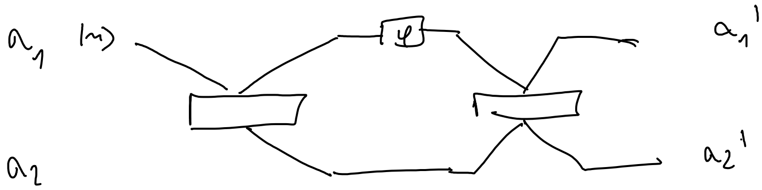


optimal precision curves are the steepest.

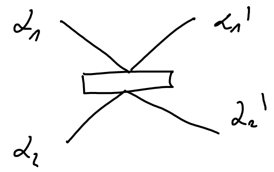
Estimation around π, 2π seems impossible. —
— fluctuations are φ independent while mean photon numbers do not locally change here

Can we improve the precision by using quantum states of light

4. Estimation of φ using Fock states.



How do we describe evolution of quantum states



We know how classical amplitudes evolve

$$\alpha_i' = V_{ij}\alpha_j$$

$|\alpha_i'\rangle = |U_{ij}\alpha_j\rangle$ } dictionary a_i

$\alpha_i = \langle\psi|a_i|\psi\rangle$ $\qquad \alpha_i' = \langle\psi|U^\dagger a_i U|\psi\rangle = V_{ij}\alpha_j$

this implies that $U^\dagger a_i U = V_{ij} a_j = a_i'$ W

obrazie Heisenberga op. anihilacji ewoluja tak jak klasyczne amplitudy

In terms of creation operators: $V_{ij}^* a_j^\dagger = a_i'^\dagger$, $\quad V^{*-1}_{ki} V^*_{ij} a_j^\dagger = (V^{*-1}_{ki})\, a_i'^\dagger$

$$a_k^\dagger = V_{ik}\, a_i'^\dagger$$

To see how a given state of light evolves we just express input operators using the output ones

In Mach-Zehnder:

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}e^{i\varphi} & 0\\ 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} =$$

$$= \frac{1}{2}\cdot\begin{bmatrix}e^{i\varphi} & 1\\ e^{i\varphi} & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\cdot\begin{bmatrix}e^{i\varphi}+1 & e^{i\varphi}-1\\ e^{i\varphi}-1 & e^{i\varphi}+1\end{bmatrix} = e^{\frac{i\varphi}{2}}\cdot\begin{bmatrix}\cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2}\\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{bmatrix}$$

So:

$$a_1^\dagger = V_{11}a_1'^\dagger + V_{21}a_2'^\dagger = \underbrace{\frac{1}{2}\left(e^{i\varphi}+1\right)}_{c_1}a_1'^\dagger + \underbrace{\frac{1}{2}\left(e^{i\varphi}-1\right)}_{c_2}a_2'^\dagger =$$

$$= e^{\frac{i\varphi}{2}}\cdot\left(\cos\frac{\varphi}{2}\,a_1'^\dagger + i\sin\frac{\varphi}{2}\,a_2'^\dagger\right)$$

$$|n\rangle_{in} = \frac{a_1^{\dagger n}}{\sqrt{n!}}|vac\rangle = \frac{\left(c_1a_1'^\dagger + c_2a_2'^\dagger\right)^n}{\sqrt{n!}}|vac\rangle =$$

$$= \sum_{k=0}^{n}\frac{1}{\sqrt{n!}}\binom{n}{k}c_1^k c_2^{n-k}\sqrt{k!}\sqrt{(n-k)!}\,|k,n-k\rangle_{out} =$$

$$= e^{\frac{in\varphi}{2}}\sum_{k=0}^{n}\sqrt{\binom{n}{k}}\,(\cos\varphi)^k(\sin\varphi)^{n-k}(i)^{n-k}|k,n-k\rangle$$

$$p_{n_1,n_2} = \delta_{n_1+n_2,n}\cdot\binom{n}{n_1}\left(\cos^2\frac{\varphi}{2}\right)^{n_1}\left(\sin^2\frac{\varphi}{2}\right)^{n_2}$$

$$\langle n_1\rangle = \sum_{n_1=0}^{\infty}n_1\frac{n!}{n_1!(n-n_1)!}\cos^2\varphi^{n_1}\sin^2\varphi^{n-n_1} = n\cos^2\frac{\varphi}{2}$$

$$\langle n_2\rangle = \quad = n\sin^2\frac{\varphi}{2}$$

We use the same estimator: $\widetilde{\cos\varphi} = \frac{n_1-n_2}{n}$

$$\Delta^2(\widetilde{\cos\varphi}) = \frac{1}{n^2}\cdot\left(\langle(n_1-n_2)^2\rangle - (\langle n_1\rangle-\langle n_2\rangle)^2\right) =$$

$$= \frac{1}{n^2}\cdot\left(\sum_{n_1}(2n_1-n)^2\binom{n}{n_1}\left(\cos^2\frac{\varphi}{2}\right)^{n_1}\left(\sin^2\frac{\varphi}{2}\right)^{n-n_1} - n^2\cos^2\varphi\right) =$$

$$= \frac{1}{n^2}\cdot\left(n^2 - 4n^2\cos^2\frac{\varphi}{2} + 4n\cos^2\frac{\varphi}{2}\cdot\left(1+(n-1)\cos^2\frac{\varphi}{2}\right) - n^2\cos^2\varphi\right) =$$

$$\langle n_1^2\rangle = \sum_{n_1}n_1^2\frac{n!}{n_1!(n-n_1)!}\cos^2\frac{\varphi}{2}^{n_1}\sin^2\frac{\varphi}{2}^{n-n_1} = \cos^2\frac{\varphi}{2}\sum_{n_1}n_1\cdot n\frac{(n-1)!}{(n_1-1)!(n-1-n_1+1)!}\cos^2\frac{\varphi}{2}^{n_1-1}\sin^2\frac{\varphi}{2}^{n-1-n_1+1}$$

$$= \cos^2\frac{\varphi}{2}\,n\cdot\sum_{n_1}\left[(n_1-1)+1\right]\frac{(n-1)!}{(n_1-1)!(n-1-n_1+1)!}\cos^2\frac{\varphi}{2}^{n_1-1}\sin^2\frac{\varphi}{2}^{n-1-n_1+1} =$$

$$= \cos^2\frac{\varphi}{2}\cdot n\cdot\left(1 + (n-1)\cdot\cos^2\frac{\varphi}{2}\right)$$

$$= \frac{1}{n^2}\cdot\left(n^2 - 4n^2\cos^2\frac{\varphi}{2} + 4n\cos^2\frac{\varphi}{2} + 4n^2\cos^4\frac{\varphi}{2} - 4n\cos^4\frac{\varphi}{2} - n^2\left(2\cos^2\frac{\varphi}{2}-1\right)^2\right)$$

$$= \frac{1}{n^2}\cdot\left(n\sin^2\varphi\right) = \frac{\sin^2\varphi}{n}$$ fluktuacje zależą od φ

So:

$$\Delta^2\varphi = \frac{\Delta^2\cos\varphi}{\left(\frac{d\cos\varphi}{d\varphi}\right)^2} = \frac{1}{n}$$ nie zależy od φ

The same scaling as for coherent state but now there is no dependence on φ.

Notice that a coherent state behaves in the same way as incoherent mixture of Fock states with Poissonian statistics. It should not be surprising that mixing introduces some additional difficulties

5. Estimation using squeezed states

Are there states that allow to break the $\frac{1}{n}$ scaling?

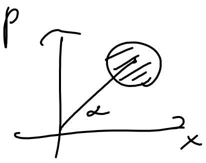
Intuition: when analysing phase estimation with coherent states we have seen that the problem lies in Poissonian fluctuations of photon count. It is known that there are squeezed states that in some settings may reveal sub-Poissonian photon number distribution.

- Squeezed states

$$[a, a^\dagger] = 1$$

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \qquad \hat{p} = \frac{a - a^\dagger}{i\sqrt{2}} \qquad [x, p] = \frac{1}{2}\left(-\frac{1}{i} - \frac{1}{i}\right) = i$$

$$\Delta^2 x \, \Delta^2 p \geq \frac{1}{4}$$



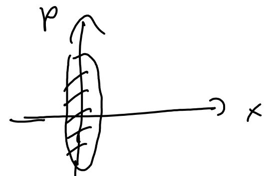
For a coherent state $\Delta_x^2 = \Delta_p^2 = \frac{1}{2}$

$$\langle \alpha | \tfrac{1}{2} (a + a^\dagger)^2 | \alpha \rangle = \tfrac{1}{2} \cdot \left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1 \right)$$

$$\langle \alpha | \tfrac{1}{\sqrt{2}} (a + a^\dagger) | \alpha \rangle = \tfrac{1}{\sqrt{2}} (\alpha + \alpha^*) \qquad \Delta_x^2 = \frac{1}{2}$$

There are states with e.g. $\Delta p^2 < \frac{1}{2}$ $\Delta x^2 > \frac{1}{2}$ such that $\Delta x^2 \Delta p^2 = \frac{1}{4}$

Squeezed vacuum:



$$|r\rangle = S_r |vac\rangle$$

$$S_r = e^{\frac{1}{2} r (a^2 - a^{\dagger 2})}$$

$$\Delta^2 x = \frac{1}{2} e^{-2r} \qquad \Delta^2 p = \frac{1}{2} e^{2r}$$

$$\langle r | a^\dagger a | r \rangle = \langle vac | (a^\dagger \cosh r - a \sinh r)(a \cosh r - a^\dagger \sinh r) | vac \rangle = \sinh^2 r$$

$$\Delta^2 x = ? \quad \langle vac | e^{\frac{1}{2} r (a^{\dagger 2} - a^2)} \tfrac{1}{2} (a + a^\dagger)^2 e^{\frac{1}{2} r (a^2 - a^{\dagger 2})} | vac \rangle$$

$$e^{\frac{1}{2} r (a^{\dagger 2} - a^2)} \, a \, e^{\frac{1}{2} r (a^2 - a^{\dagger 2})} = a + \underbrace{\tfrac{1}{2} r [a^{\dagger 2}, a]}_{\frac{1}{2} r (-a^\dagger - a^\dagger) = -r a^\dagger} + \tfrac{1}{2} \left(\left[\tfrac{1}{2} r a^{\dagger 2}, -r a^\dagger \right] \right) = -\tfrac{1}{2} r^2 a \; \ldots$$

$$= a \cosh r - a^\dagger \sinh r$$

$$S_r^\dagger a S_r = a \cosh r - a^\dagger \sinh r$$

$$S_r^\dagger a^\dagger S_r = a^\dagger \cosh r - a \sinh r$$

$$\langle vac | \tfrac{1}{2} \cdot \left(a \cosh r - a^\dagger \sinh r + a^\dagger \cosh r - a \sinh r \right)^2 | vac \rangle =$$

$$= \tfrac{1}{2} \cdot \left(\cosh r^2 + \sinh^2 r - 2 \cosh r \sinh r \right) = \tfrac{1}{2} \cdot (\cosh r - \sinh r)^2 =$$

$$= \tfrac{1}{2} \cdot \left(e^{-r} \right)^2 = \tfrac{1}{2} e^{-2r}$$

Consider the following setup: [Caves 1981]

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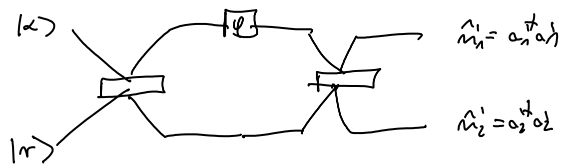


$$a_1' = \cos\frac{\varphi}{2}\, a_1 + i\sin\frac{\varphi}{2}\, a_2\,, \quad a_2' = \cos\frac{\varphi}{2} a_2 + i\sin\frac{\varphi}{2} a_1$$

$$\langle n_1\rangle = \langle\alpha|\langle r|\left(\cos\frac{\varphi}{2} a_1^\dagger - i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 + i\sin\frac{\varphi}{2} a_2\right)|\alpha\rangle|r\rangle =$$

$$= \cos^2\frac{\varphi}{2}|\alpha|^2 + \sin^2\frac{\varphi}{2}\sinh^2 r$$

$$\langle n_2\rangle = \langle\alpha|\langle r|\left(i\sin\frac{\varphi}{2} a_1^\dagger + \cos\frac{\varphi}{2} a_2^\dagger\right)\left(i\sin\frac{\varphi}{2} a_1 + \cos\frac{\varphi}{2} a_2\right)|\alpha\rangle|r\rangle =$$

$$= \sin^2\frac{\varphi}{2}|\alpha|^2 + \cos^2\frac{\varphi}{2}\sinh^2 r$$

$$\langle n_1\rangle - \langle n_2\rangle = |\alpha|^2\cos\varphi + \sinh^2 r\cos\varphi = \cos\varphi\left(|\alpha|^2 - \sinh^2 r\right)$$

Estimator $\quad \cos\tilde{\varphi} = \dfrac{n_1 - n_2}{|\alpha|^2 - \sinh^2 r}$

Now calculate the variance

$$\langle (n_1 - n_2)^2\rangle$$

$$\langle n_1^2\rangle = \cos^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right) + \sin^2\frac{\varphi}{2}\cos^2\frac{\varphi}{2}|\alpha|^2$$

$$\langle n_2^2\rangle = \sin^4\frac{\varphi}{2}\,.$$

$$\langle n_1^2\rangle + \langle n_2\rangle^2 - 2\langle n_1\rangle\langle n_2\rangle =$$

$$= \left(|\alpha|^4 + |\alpha|^2\right)\left(\cos^4\frac{\varphi}{2} + \sin^4\frac{\varphi}{2}\right)$$

$$+ 2\cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}|\alpha|^2 - 2\cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}|\alpha|^4$$

$$\delta^2 n =$$

$$\langle n_1^2\rangle = \langle\alpha, r|\left(\cos\frac{\varphi}{2} a_1^\dagger + i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 - i\sin\frac{\varphi}{2} a_2\right)\left(\cos\frac{\varphi}{2} a_1^\dagger + i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 - i\sin\frac{\varphi}{2} a_2\right)|\alpha r\rangle =$$

$$= \cos^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right) + \sin^4\frac{\varphi}{2}\langle r|\left(a_2^\dagger a_2\right)^2|r\rangle \; +$$

$$\cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\Big(2|\alpha|^2\cdot\sinh^2 r + |\alpha|^2\underbrace{\langle r|a_2 a_2^\dagger|r\rangle}_{(1+\sinh^2 r)} + \sinh^2 r\underbrace{\langle\alpha|a_1 a_1^\dagger|\alpha\rangle}_{(1+|\alpha|^2)} - \left(\alpha^2 + \alpha^{*2}\right)\cosh r\sinh r\Big)$$

$$\langle r|\left(a_2^\dagger a_2\right)^2|r\rangle = \langle vac|\left[\left(\cosh r\, a_2^\dagger - \sinh r\, a_2\right)\left(\cosh r\, a_2 - \sinh r\, a_2^\dagger\right)\right]^2|vac\rangle$$

$$= \sinh^2 r\,\langle 1|\left(\cosh r\, a_2 - \sinh r\, a_2^\dagger\right)\left(\cosh r\, a_2^\dagger - \sinh r\, a_2\right)|1\rangle =$$

$$= \sinh^2 r\left(2\cosh^2 r + \sinh^2 r\right) = \sinh^2 r\left(2 + 3\sinh^2 r\right)$$

$$\langle r|a_2^2|r\rangle = \langle vac|\left(\cosh r\, a_2 - \sinh r\, a_2^\dagger\right)^2|vac\rangle = -\cosh r\sinh r$$

$$\langle n_1^2\rangle = \cos^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right) + \sin^4\frac{\varphi}{2}\sinh^2 r\left(2 + 3\sinh^2 r\right)$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\left(4|\alpha|^2\sinh^2 r + |\alpha|^2 + \sinh^2 r + \left(\alpha^2 + \alpha^{*2}\right)\sinh r\cosh r\right)$$

$$\langle n_2^2\rangle = \cos^4\frac{\varphi}{2}\sinh^2 r\left(2 + 3\sinh^2 r\right) + \sin^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right)$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\Big(4|\alpha|^2\sinh^2 r\; \underbrace{+ |\alpha|^2 + \sinh^2 r}_{\bar{n}} + \left(\alpha^2 + \alpha^{*2}\right)\sinh r\cosh r\Big)$$

$$\langle n_1 n_2\rangle = \langle\alpha, r|\left(\cos\frac{\varphi}{2} a_1^\dagger + i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 - i\sin\frac{\varphi}{2} a_2\right)\left(i\sin\frac{\varphi}{2} a_1^\dagger + \cos\frac{\varphi}{2} a_2^\dagger\right)\left(-i\sin\frac{\varphi}{2} a_1 + \cos\frac{\varphi}{2} a_2\right)|\alpha r\rangle$$

$$= \cos^4\frac{\varphi}{2}|\alpha|^2\sinh^2 r + \sin^4\frac{\varphi}{2}|\alpha|^2\sinh^2 r$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\Big(|\alpha|^4 + |\alpha|^2\;\; \sinh^2 r\left(2 + 3\sinh^2 r\right) - |\alpha|^2\left(1 + \sinh^2 r\right) - \sinh^2 r\left(|\alpha|^2 + 1\right)$$
$$- \left(\alpha^2 + \alpha^{*2}\right)\cosh r\sinh r\Big)$$

$$\langle (n_1 - n_2)^2\rangle = \cos^4\frac{\varphi}{2}\cdot\left(|\alpha|^4 + |\alpha|^2 + \sinh^2 r\left(2 + 3\sinh^2 r\right) - 2|\alpha|^2\sinh^2 r\right)$$

$$+\sin^4\frac{\varphi}{2}\cdot\left(|\alpha|^4+|\alpha|^2+\sinh^2 r(2+3\sinh^2 r)-2|\alpha|^2\sinh^2 r\right)$$

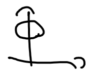
$$+2\sin^2\frac{\varphi}{2}\cos^2\frac{\varphi}{2}\cdot\left(4|\alpha|^2\sinh^2 r+2\bar{n}-|\alpha|^4-|\alpha|^2-\sinh^2 r(2+3\sinh^2 r)+2|\alpha|^2\sinh^2 r-(\alpha^2+\alpha^{*2})\sinh r\cosh r\right)$$

$$=\cos^2\varphi\left(|\alpha|^4+|\alpha|^2+\sinh^2 r(2+3\sinh^2 r)-2|\alpha|^2\sinh^2 r\right)+\sin^2\varphi\left(2|\alpha|^2\sinh^2 r+\bar{n}+(\alpha^2+\alpha^{*2})\cosh r\sinh r\right)$$

$$\Delta^2(n_1-n_2)=\cos^2\varphi\left(\cancel{|\alpha|^4}+|\alpha|^2+\sinh^2 r(1+2\sinh^2 r)-\cancel{2|\alpha|^2\sinh^2 r}-\cancel{|\alpha|^4}-\cancel{2|\alpha|^2\sinh^2 r}-\cancel{\sinh^4 r}\right)$$

$$+\sin^2\varphi\left(2|\alpha|^2\sinh^2 r+\bar{n}+(\alpha^2+\alpha^{*2})\cosh r\sinh r\right)=$$

$\Big\{$ if $\alpha^{*2}=\alpha^2=-|\alpha|^2$



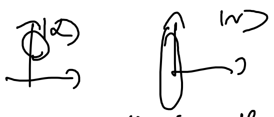
$$=|\alpha|^2\cos^2\varphi+2\cos^2\varphi\cosh^2 r\sinh^2 r+\left(2|\alpha|^2\cdot\frac{1}{2}\cdot(e^{-2r}-1)+|\alpha|^2+\sinh^2 r\right)$$

$$=|\alpha|^2\cos^2\varphi+2\cos^2\varphi\cosh r\sinh r+\sin^2\varphi\left(|\alpha|^2e^{-2r}+\sinh^2 r\right)$$

$$\Delta^2\cos\varphi=\frac{|\alpha|^2\cos^2\varphi+2\cos^2\varphi\cosh r\sinh r+\sin^2\varphi\left(|\alpha|^2e^{-2r}+\sinh^2 r\right)}{\left(|\alpha|^2-\sinh^2 r\right)^2}$$

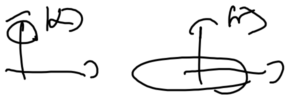
$$\Delta^2\varphi=\frac{\left(|\alpha|^2+2\cosh r\sinh r\right)\operatorname{ctg}^2\varphi+\left(|\alpha|^2e^{-2r}+\sinh^2 r\right)}{\left(|\alpha|^2-\sinh^2 r\right)^2}$$

Intuition:



since we measure quadratures $a_1\cos\frac{\varphi}{2}+ia_2\sin\frac{\varphi}{2}$ we have

$\sim$



smaller fluctuation in the direction α.

Optimal sensitivity is around $\varphi=\frac{\pi}{2}$

$$\Delta^2\varphi=\frac{|\alpha|^2e^{-2r}+\sinh^2 r}{\left(|\alpha|^2-\sinh^2 r\right)^2}$$

We fix $\bar{n}=|\alpha|^2+\sinh^2 r$

$$\Delta^2\varphi=\frac{\left(\bar{n}-\sinh^2 r\right)e^{-2r}+\sinh^2 r}{\left(\bar{n}-2\sinh^2 r\right)^2}$$

$\Big\}$ assume $|\alpha|^2\gg\sinh^2 r\gg 1$

$$\Delta^2\varphi=\frac{\bar{n}e^{-2r}+\frac{1}{4}e^{2r}}{\bar{n}^2}$$

$$\frac{d}{dx}\left(\bar{n}x+\frac{1}{4}\cdot\frac{1}{x}\right)=0$$

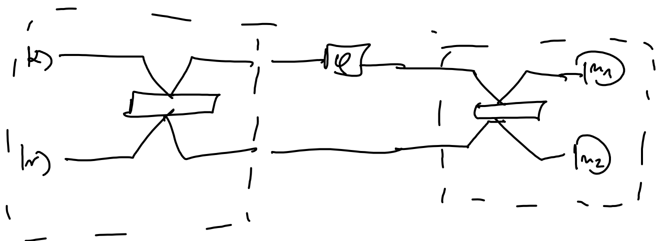
$$\bar{n}-\frac{1}{4x^2}=0\qquad x=\frac{1}{2\sqrt{\bar{n}}}$$

$$\Delta^2\varphi=\frac{\frac{\sqrt{\bar{n}}}{2}+\frac{2\sqrt{\bar{n}}}{2}}{\bar{n}^2}=\frac{1}{\bar{n}^{\frac{3}{2}}}$$

Better scaling. How far we can go?

6. Looking for the optimal estimation schemes

Three element to optimize over:



$$\tilde{\varphi} = \arccos\left(\frac{m_1 - m_2}{|\alpha|^2 - \sinh^2 r}\right)$$

input state preparation — phase sensing — measurement — estimator

$$|\psi_{in}\rangle, \quad \mathcal{U}_\varphi = e^{i\varphi a^\dagger a}, \quad \{\Pi_\xi\} \quad \tilde{\varphi}(m)$$

$$\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$$

general POVM measurement

$$\Pi_\xi \geq 0, \quad \int d\xi\, \Pi_\xi = \mathbb{1}$$

Optimal phase estimation should minimize the average cost.

$C_{\tilde{\varphi},\varphi} = (\tilde{\varphi} - \varphi)^2$, or rather something periodic: $4\sin^2\left(\frac{\varphi - \tilde{\varphi}}{2}\right)$

Average cost:

$$\overline{\delta\varphi^2} = \overline{C} = \underset{\text{a priori distribution}}{\int d\varphi\, p(\varphi)} \int d\xi\, p(\xi|\varphi)\, C_{\tilde{\varphi}(\xi),\varphi}$$

$$p(\xi|\varphi) = \mathrm{Tr}(\Pi_\xi \underbrace{\mathcal{U}_\varphi \rho_{in} \mathcal{U}_\varphi^\dagger}_{\rho_\varphi})$$

problem: $\min\limits_{|\psi_{in}\rangle, \{\Pi_\xi\}, \tilde{\varphi}(\xi)} \overline{C}$ extremely hard

Two extreme ways to approach the problem:

- local approach $p(\varphi) \approx \delta(\varphi - \varphi_0)$ { changes around
- global approach $p(\varphi) = \frac{1}{2\pi}$ sensing small { a known phase; { no apriori knowledge

7. Local approach { Maybe we can skip it if ~~knowledge~~ Mention derives it

If we put $p(\varphi) = \delta(\varphi - \varphi_0)$ $\quad \overline{C} = \int d\xi\, \mathrm{Tr}(\Pi_\xi\, \rho_{\varphi_0})\, C_{\tilde{\varphi}(\xi),\varphi_0}$

trivial way out $\Pi_{\tilde{\varphi} = \varphi_0} = \mathbb{1}$

to avoid trivial solution. We want to force the estimation to be sensitive to small changes of φ around φ_0 — „first order sensitivity".

We impose local-unbiasedness constraint

$\langle\tilde{\varphi}\rangle = \int d\xi\, \tilde{\varphi}(\xi) p(\xi|\varphi_0) = \varphi_0$ unbiasedness condition (to strong)

$$\left.\frac{d\langle\tilde{\varphi}\rangle}{d\varphi}\right|_{\varphi=\varphi_0} = 1 \qquad \int d\xi\, \tilde{\varphi}(\xi) \left.\frac{dp(\xi|\varphi)}{d\varphi}\right|_{\varphi=\varphi_0} = 1 \quad \text{locally unbiased condition}$$

$\delta^2\tilde{\varphi}$ $\quad F$

$$\underbrace{\int d\eta\, p(\eta|\varphi_0)\left(\tilde\varphi(\eta)-\varphi_0\right)^2}\int d\eta\,\frac{1}{p(\eta|\varphi_0)}\left(\frac{dp(\eta|\varphi_0)}{d\varphi_0}\right)^2 \quad \overset{C\text{-}S}{\geq}$$

$$\geq \int d\eta\,\left(\tilde\varphi(\eta)-\varphi_0\right)\sqrt{p(\eta|\varphi_0)}\,\frac{1}{\sqrt{p(\eta|\varphi_0)}}\,\frac{dp(\eta|\varphi_0)}{d\varphi_0} =$$

$$= \underbrace{\int d\eta\,\tilde\varphi(\eta)\,\frac{dp(\eta|\varphi_0)}{d\varphi_0}}_{1} - \underbrace{\varphi_0\int d\eta\,\frac{dp(\eta|\varphi_0)}{d\varphi_0}}_{0} = 1$$

$$\delta^2\tilde\varphi\cdot F \geq 1 \qquad \underbrace{\delta^2\tilde\varphi \geq \frac{1}{F}}_{\text{Cramer-Rao bound}}$$

We have got rid of the estimator problem.
Now we can just look at Fisher

$$F=\int d\eta\,\frac{1}{p(\eta|\varphi)}\left(\frac{dp(\eta|\varphi_0)}{d\varphi_0}\right)^2$$

and maximize F over $|\psi_{in}\rangle$, Π_η.
For k independent realizations $F^{(k)}=k\cdot F$

$$\delta^2\varphi \geq \frac{1}{kF}$$

For $k\to\infty$ Max-likelihood estimator saturates C-R bound.
Notice that F tells just the local variations in $p(\eta|\varphi)$.
We may go further and get rid of the optimization over $\{\Pi_\eta\}$

$$F=\int d\eta\,\frac{1}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)}\left(\frac{d\,(\mathrm{Tr}\,\Pi_\eta\rho_\varphi)}{d\varphi}\right)^2=\int d\eta\,\frac{1}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)}\left[\mathrm{Tr}\left(\Pi_\eta\frac{d\rho_\varphi}{d\varphi}\right)\right]^2$$

$$\left\{\frac{d\rho_\varphi}{d\varphi}=\frac{1}{2}(\Lambda\rho_\varphi+\rho_\varphi\Lambda)\qquad \Lambda_{ij}=\frac{2}{p_i+p_j}\left(\frac{d\rho_\varphi}{d\varphi}\right)_{ij}\quad \text{in } \rho_\varphi \text{ eigenbasis}\right.$$

$$=\int d\eta\,\frac{\left(\mathrm{Tr}\left[\frac{1}{2}\Pi_\eta(\Lambda\rho_\varphi+\rho_\varphi\Lambda)\right]\right)^2}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)} = \qquad \left\{\ |\mathrm{Tr}A^\dagger B|^2\leq \mathrm{Tr}A^\dagger A\cdot\mathrm{Tr}B^\dagger B\right.$$

$$=\int d\eta\,\frac{(\mathrm{Re}\,\mathrm{Tr}(\Pi_\eta\Lambda\rho_\varphi))^2}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)}\leq\int d\eta\,\frac{|\mathrm{Tr}(\Pi_\eta\Lambda\rho_\varphi)|^2}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)}$$

$$\left\{\ A=\sqrt{\Pi_\eta}\sqrt{\rho_\varphi}\qquad B=\sqrt{\Pi_\eta}\Lambda\sqrt{\rho_\varphi}\qquad \mathrm{Tr}A^\dagger B=\mathrm{Tr}\sqrt{\rho_\varphi}\sqrt{\Pi_\eta}\sqrt{\Pi_\eta}\Lambda\sqrt{\rho_\varphi}= \right.$$

$$= \mathrm{Tr}(\rho_\varphi\Pi_\eta\Lambda)$$

$$\leq\int d\eta\,\frac{\mathrm{Tr}\,\rho_\varphi\Pi_\eta}{\mathrm{Tr}\,\rho_\varphi\Pi_\eta}\cdot\mathrm{Tr}\left(\sqrt{\rho_\varphi}\Lambda\sqrt{\Pi_\eta}\sqrt{\Pi_\eta}\Lambda\sqrt{\rho_\varphi}\right)=\mathrm{Tr}(\rho_\varphi\Lambda^2)$$

$$F_Q=\mathrm{Tr}\left(\rho_\varphi\Lambda^2\right)\qquad \underbrace{\delta^2\tilde\varphi\geq\frac{1}{F_Q}}_{\text{Q. C-R bound.}}$$

For pure states it is simple :

$$\frac{d\,|\psi_\varphi\rangle\langle\psi_\varphi|}{d\varphi} = |\psi_\varphi'\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\psi_\varphi'|$$

so it is enought to choose $\Lambda = 2\left(|\psi_\varphi'\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\psi_\varphi'|\right)$

$$\Lambda \cdot |\psi_\varphi\rangle\langle\psi_\varphi| = 2|\psi_\varphi'\rangle\langle\psi_\varphi| + 2|\psi_\varphi\rangle\langle\psi_\varphi'|\psi_\varphi\rangle\langle\psi_\varphi|$$

$$|\psi_\varphi\rangle\langle\psi_\varphi| \cdot \Lambda = 2|\psi_\varphi\rangle\langle\psi_\varphi|\psi_\varphi'\rangle\langle\psi_\varphi| + 2\,|\psi_\varphi\rangle\langle\psi_\varphi'|$$

$$F_Q = 4\Big(\langle\psi_\varphi|\Big(|\psi_\varphi'\rangle\langle\psi_\varphi|\psi_\varphi'\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\psi_\varphi'|\psi_\varphi\rangle\langle\psi_\varphi'| + |\psi_\varphi'\rangle\langle\psi_\varphi'| + \langle\psi_\varphi'|\psi_\varphi\rangle\Big)|\psi_\varphi\rangle$$

$$= 4 \cdot \left(\langle\psi_\varphi|\psi_\varphi'\rangle^2 + \langle\psi_\varphi'|\psi_\varphi\rangle^2 + |\langle\psi_\varphi|\psi_\varphi'\rangle|^2 + \langle\psi_\varphi'|\psi_\varphi'\rangle\right)$$

$$\Big\{ \quad \langle\psi_\varphi|\psi_\varphi'\rangle + \langle\psi_\varphi'|\psi_\varphi\rangle = 0$$

$$= 4 \cdot \left(\langle\psi_\varphi'|\psi_\varphi'\rangle - |\langle\psi_\varphi|\psi_\varphi'\rangle|^2\right)$$

Dependence only on $|\psi_{in}\rangle$!

Now we simply look for the state that maximizes F_Q. For phase estimation:

$$|\psi_\varphi'\rangle = \frac{d}{d\varphi} e^{i a^\dagger a \varphi} |\psi_{in}\rangle$$

$$F_Q = 4 \cdot \left(\langle\psi_{in}|(a^\dagger a)^2|\psi_{in}\rangle - |\langle\psi_{in}|a^\dagger a|\psi_{in}\rangle|^2\right) =$$

$$= 4 \cdot \delta^2 n_a$$

$$\delta\varphi^2 \cdot \delta^2 n_a \geq \frac{1}{4}$$

Heisenberg like relation phase estimation uncertainty vs photon number uncertainty in "phase sensing" arm

The optimal state = the one that maximizes $\delta^2 n_a$

Example Find the optimal N photon state

$$|\psi\rangle = \sum_{m=0}^{N} \alpha_m |m, N-m\rangle$$

number of photons in a, b modes respectively

$$\langle \hat n_a^2\rangle - \langle \hat n_a\rangle^2 \leq \left(\frac{m_{max} - m_{min}}{2}\right)^2 = \quad \text{fact from linear algebra}$$

$$m_{max} = N \qquad m_{min} = 0 \qquad = \frac{N^2}{4}$$

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}} \cdot \left(|N,0\rangle + |0,N\rangle\right) \qquad \delta^2 n = \frac{N^2}{4}$$

$$\underbrace{\delta^2\varphi \geq \frac{1}{N^2}} \quad \text{Heisenberg scaling}$$

Interpretation: $|\psi_\varphi\rangle = \frac{1}{\,}\left(|N,0\rangle + e^{iN\varphi}|0,N\rangle\right)$

Interpretation: $|\psi_\varphi\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + e^{iN\varphi} |0,N\rangle \right)$

- N times faster oscillation than in single photon state, hence N-times increase in precision.

but for practical implementations useless, $\frac{2\pi}{N}$ ambiguity, we need to perform two-stage estimation.
 - rough estimation
 - more precise

If we are given total resource of N photons it requires some thought how to use them optimally ....

8. Global approach.

Since φ is just the label, we may as well label povms with estimated values $\tilde{\varphi}$. $\{\Pi_{\tilde{\varphi}}\}$

$$\int_0^{2\pi} \frac{d\tilde{\varphi}}{2\pi} \Pi_{\tilde{\varphi}} = \mathbb{1}$$

problem: $\min_{|\psi_{in}\rangle, \{\Pi_{\tilde{\varphi}}\}} \overline{C}$ still very difficult

$$\overline{C} = \int d\varphi\, p(\varphi) \int \frac{d\tilde{\varphi}}{2\pi} \mathrm{Tr}\left(\Pi_{\tilde{\varphi}}\ \rho_\varphi \right) C_{\tilde{\varphi},\varphi}$$

We may make use of the U_φ symmetry, no phase is distinguished. $p(\varphi) = \frac{1}{2\pi}$, $C_{\tilde{\varphi}+\varphi_0, \varphi+\varphi_0} = C_{\tilde{\varphi},\varphi}$

Optimal measurement: is covariant:

$$\Pi_{\tilde{\varphi}} = U_{\tilde{\varphi}} \Pi_0 U_{\tilde{\varphi}}^\dagger$$

$$\overline{C} = \int \frac{d\varphi}{2\pi} \frac{d\tilde{\varphi}}{2\pi} \mathrm{Tr}\left(U_{\tilde{\varphi}} \Pi_0 U_{\tilde{\varphi}}^\dagger U_\varphi \rho_{in} U_\varphi^\dagger \right) C_{\tilde{\varphi},\varphi} =$$

$$\begin{cases} \varphi' = \varphi - \tilde{\varphi} \\ \varphi'' = \tilde{\varphi} \end{cases} \underline{\underline{=}} \int \frac{d\varphi'}{2\pi} \mathrm{Tr}\left(\Pi_0 U_{\varphi'} \rho_{in} U_{\varphi'}^\dagger \right) C_{0,\varphi'}$$

Covariant measurements - a detour

General problem.

$g \in G$ - group element $\rightarrow$ encoded via a unitary representation $|\psi_g\rangle = U_g |a\rangle$

In general could be in a mixed state

$$\rho_g = U_g \rho_0 U_g^\dagger$$

$\{\Pi_{\tilde g}\}$ - measurement result denotes the estimated value $\tilde g$

$C_{g,\tilde g}$ - cost function

$$\overline{C} = \int dg \, d\tilde g \; \mathrm{Tr}(\Pi_{\tilde g} \rho_g) \, C_{g,\tilde g}$$

Assumptions. (Estimation problem has symmetry with respect to G)

- dg - Haar measure of group G

  $g' = hg \qquad dg' = dg$

- $C_{hg,h\tilde g}$ - cost function left invariant

Notice all assumptions are trivially satisfied for global approach to phase estimation:

$G = U(1), \qquad U_{\varphi_1} U_{\varphi_2} = U_{\varphi_1+\varphi_2} \qquad d\varphi = d\varphi'$

$\varphi' = \varphi + \varphi_0$

$C_{\varphi,\tilde\varphi} = 4\sin^2\frac{\varphi-\tilde\varphi}{2} = C_{\varphi+\varphi_0,\tilde\varphi+\varphi_0}$ OK

Definition

$\{\Pi_{\tilde g}\}$ is covariant with respect to group $G \iff \forall_{\tilde g, h} \; U_h \Pi_{\tilde g} U_h^\dagger = \Pi_{h\tilde g}$

Corollary

If $\{\Pi_{\tilde g}\}$ is covariant

$$\Pi_{\tilde g} = U_{\tilde g} \Pi_e U_{\tilde g}^\dagger$$

The measurement is fully defined with a single operator Π_e

Theorem

If the estimation problem has symmetry with respect to group G then the optimal measurement can always be

[illegible]

found among measurements covariant with respect to G

<u>Proof</u>

Let $\Pi^{opt}_{\tilde g}$ be the optimal measurement

minimizing $\bar C$:

$$\bar C_{opt} = \int dg\, d\tilde g\, \mathrm{Tr}\left(\Pi^{opt}_{\tilde g}\, \rho_g\right) C_{g,\tilde g}$$

Define

$$\Pi^{cov}_{\tilde g} = \int dg'\, U^{\dagger}_{g'}\, \Pi^{opt}_{g'\tilde g}\, U_{g'}$$

Measurement $\Pi^{cov}_{\tilde g}$ is indeed covariant

$$U_h\, \Pi^{cov}_{\tilde g}\, U_h^{\dagger} = \int dg'\, U_{hg'^{-1}}\, \Pi^{opt}_{g'\tilde g}\, U_{g'h^{-1}}$$

$$\underline{g' \to g'h} \quad \int dg'\, U_{g'}\, \Pi^{opt}_{g'h\tilde g}\, U_{g'} = \Pi^{cov}_{h\tilde g}$$

And gives the same cost as $\Pi^{opt}_{\tilde g}$:

$$\bar C_{cov} = \int dg\, d\tilde g\, \mathrm{Tr}\left(\Pi^{cov}_{\tilde g}\, \rho_g\right) C_{\tilde g, g} =$$

$$= \int dg\, d\tilde g\, \mathrm{Tr}\left(\int dg'\, U^{\dagger}_{g'}\, \Pi^{opt}_{g'\tilde g}\, U_{g'}\, U_g\, \rho_e\, U_g^{\dagger} \right) C_{g,\tilde g}$$

$$= \int dg\, d\tilde g\, dg'\, \mathrm{Tr}\left(U^{\dagger}_{g'g}\, \Pi^{opt}_{g'\tilde g}\, U_{g'g}\, \rho_e \right) C_{g,\tilde g}$$

$$\begin{cases} g \to g'^{-1} g \\ \tilde g \to g'^{-1} \tilde g \end{cases}$$

$$= \int dg\, d\tilde g\, dg'\, \mathrm{Tr}\left(U^{\dagger}_{g}\, \Pi^{opt}_{\tilde g}\, U_g\, \rho_e \right) C_{g'^{-1}g,\, g'^{-1}\tilde g} =$$

$$= \int dg\, d\tilde g\, dg'\, \mathrm{Tr}\left(\Pi^{opt}_{\tilde g}\, \rho_g \right) C_{g,\tilde g} = C_{opt}$$

$\blacksquare$

Problem can be simplified

Problem can be simplified

$$\bar{C} = \int dg\, d\tilde{g}\ \mathrm{Tr}(\Pi_{\tilde{g}}\, \rho_g)\, C_{g,\tilde{g}} =$$

$$= \int dg\, d\tilde{g}\ \mathrm{Tr}\left(U_{\tilde{g}}\, \Pi_e\, U_{\tilde{g}}^{\dagger}\, U_g\, \rho_e\, U_g^{\dagger} \right) C_{g,\tilde{g}}$$

$$= \int dg\, d\tilde{g} \left(\mathrm{Tr}\left(U^{\dagger}_{\tilde{g}^{-1}g}\ \Pi_e\ U_{\tilde{g}^{-1}g}\ \rho_e \right) C_{g,\tilde{g}} \right.$$

$$\left\{ g \to \tilde{g} g \right.$$

$$= \int dg\, d\tilde{g}\ \mathrm{Tr}\left(U_g^{\dagger}\, \Pi_e\, U_g\ \rho_e \right) C_{g,e}$$

$$\bar{C} = \int dg\ \mathrm{Tr}(\Pi_e\, \rho_g)\, C_{g,e}$$

<u>Find form of the problem</u>

$$\min_{|\psi_{in}\rangle,\ \Pi_e \geq 0} \bar{C}$$

$$\int dg\, U_g \Pi_e U_g^{\dagger} = \mathbb{1}$$

we optimize only over one operator!

$$\bar{C} = \int dg\ \mathrm{Tr}(\Pi_e\, \rho_g)\, C_{g,e}$$

<u>Return to phase estimation problem</u>

$$\bar{C} = \int \frac{d\varphi}{2\pi}\ \mathrm{Tr}\left(\Pi_0\ U_\varphi |\psi_{in}\rangle\langle\psi_{in}| U_\varphi^{\dagger} \right) C_{\varphi,0}$$

$$|\psi_{in}\rangle = \sum_m \alpha_m |m, N-m\rangle =: \sum_m \alpha_m |m\rangle$$

$$\bar{C} = \mathrm{Tr}\left(\Pi_0 \int \frac{d\varphi}{2\pi} \sum_{n,m} e^{i(n-m)\varphi}\, \alpha_n \alpha_m^*\, C_{\varphi,0}\, |n\rangle\langle m| \right)$$

$$= \sum_{n,m} \langle m|\Pi_0|n\rangle\ \alpha_n \alpha_m^* \int \frac{d\varphi}{2\pi}\, e^{i(n-m)\varphi}\, C_{\varphi,0}$$

$$\left\{ C_{\varphi,0} = 4 \sin^2\frac{\varphi}{2} = 4 \left(\frac{e^{\frac{i\varphi}{2}} - e^{-\frac{i\varphi}{2}}}{2i} \right)^2 = 2 - e^{i\varphi} - e^{-i\varphi} \right.$$

$$= 2\sum_n \langle n|\Pi_0|n\rangle - \sum_n \langle n+1|\Pi_0|n\rangle \alpha_n \alpha_{n+1}^* - \sum_n \langle n|\Pi_0|n+1\rangle \alpha_n^* \alpha_{n+1}$$

Constraints on Π_c: $\Pi_c \geq 0$

$$\int U_\varphi \Pi_0 U_\varphi^\dagger \, d\varphi = \mathbb{1}$$

$$\Pi_0 = \sum_{n,m} \Pi_{n,m} |n\rangle\langle m| \qquad \int \frac{d\varphi}{2\pi} U_\varphi \Pi_c U_\varphi^\dagger = \sum \Pi_{n,m} |n\rangle\langle m| \int \frac{d\varphi}{2\pi} e^{i(n-m)\varphi}$$

$$= \sum_n \Pi_{n,n} |n\rangle\langle n| = \mathbb{1} \quad \Rightarrow \quad \langle n|\Pi_0|n\rangle = 1$$

$$\bar{C} = 2 - 2\,\mathrm{Re}\sum_n \langle n+1|\Pi_0|n\rangle \alpha_n \alpha_{n+1}^* \geq$$

$$\geq 2 - 2\sum_n |\langle n+1|\Pi_0|n\rangle|\,|\alpha_n||\alpha_{n+1}|$$

since $\Pi_c \geq 0$ and diagonal elements are 1 we know that $|\langle n+1|\Pi_c|n\rangle| \leq 1$ so

$$\bar{C} \geq 2 - 2\sum_n |\alpha_n||\alpha_{n+1}|$$

Moreover we can saturate this bound by choosing: $\alpha_n \geq 0$

$$\Pi_c = |e\rangle\langle e|$$

where $|e\rangle = \sum_{n=0}^{N} |n, N-n\rangle$ we have the optimal measurement Π_c

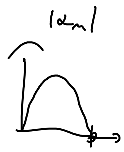
What is the optimal state:

$$\bar{C} = 2 - \langle\psi| \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix} |\psi\rangle$$

we look for eigenvector corresponding to maximal eigenvalue: solution is analytical

$$\bar{C}_{min} = 2\left(1 - \cos\left(\frac{\pi}{N+2}\right)\right)$$

$$|\psi_{in}\rangle = \sqrt{\frac{2}{N+2}} \sum_{n=0}^{N} \sin\frac{(n+1)\pi}{N+2} |n, N-n\rangle$$



[Berry, Wiseman 2000]

$$\delta\varphi^2 = 2\left(1 - \cos\frac{\pi}{N+2}\right) \approx \frac{\pi^2}{(N+2)^2}$$

$$\delta\varphi^2 = 2\left(1-\cos\frac{\pi}{N+2}\right) \approx \frac{\pi^2}{(N+2)^2}$$

Heisenberg scaling

Completely different state than in local approach
here we are really sure we could in
practical implementation reach the Heisenberg —
if we knew how to implement the
optimal measurement.

Lecture 2

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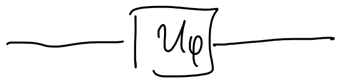
9. A Bigger picture on quantum metrology
[Giovannetti, Lloyd, Maccone 2006]

Consider a probe system (e.g. photon) which experiences $\mathcal{U}_\varphi = e^{i\varphi \hat{G}}$

$\hat{G}$ - generator of a phase shift



C-R bound

$$\delta\varphi \, \delta G \geq \frac{1}{2}$$

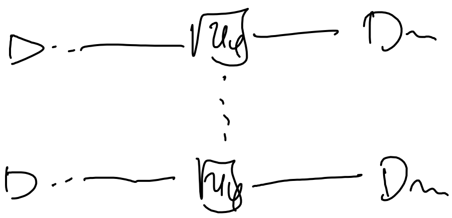
$$\delta G \leq \frac{(\lambda_+ - \lambda_-)}{2}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|e_{max}\rangle + |e_{min}\rangle\right)$$

$$\delta^2 G = \frac{1}{2}(\lambda_+^2 + \lambda_-^2) - \frac{1}{4}(\lambda_+ + \lambda_{-})^2 =$$

$$= \frac{1}{4}(\lambda_+ - \lambda_-)^2$$

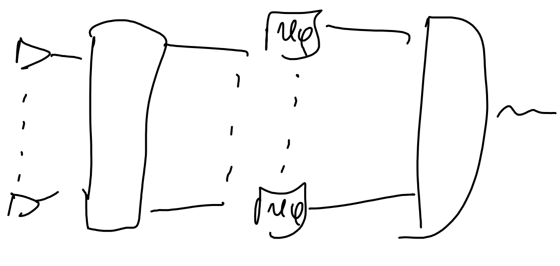
If we use N probes independently



$$\delta\varphi \geq \frac{1}{\sqrt{N}(\lambda_+ - \lambda_-)}$$

This corresponds to e.g. N single photons sent through the interferometer.

If we allow entangled input states and arbitrary measurement



$$\mathcal{U}_\varphi^{\otimes N} |\psi_N\rangle \qquad G^{(N)} = \sum_{i=1}^{N} G_i$$

$$\delta\varphi \, \delta G^{(N)} \geq \frac{1}{2}$$

$$\lambda_+^{(N)} = N \cdot \lambda_+ \qquad \lambda_-^{(N)} = N\lambda_-$$

$$\delta\varphi \geq \frac{1}{N(\lambda_+ - \lambda_-)}$$

Remark: local measurement are sufficient to reach the bounds

Example: Frequency standards

Cs fountain: two level atoms

$\omega_0 \updownarrow$ $|1\rangle$, $|0\rangle$

$$|0\rangle^{\otimes N} \xrightarrow{\text{pulse}} \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)^{\otimes N} \longrightarrow \left(\frac{1}{\sqrt{2}}(|0\rangle + e^{-i\Delta t}|1\rangle)\right)^{\otimes N} \longrightarrow \frac{1}{2}\cdot\left((1+e^{-i\Delta t})|0\rangle + (1-e^{-i\Delta t})|1\rangle\right)^{\otimes N}$$

$\Delta = \omega - \omega_0$ - frequency detuning

t - time of flight

exactly the same mathematical structure as in M-Z interferometer (Ramsey interferometry) now $\varphi \to \Delta \cdot t$

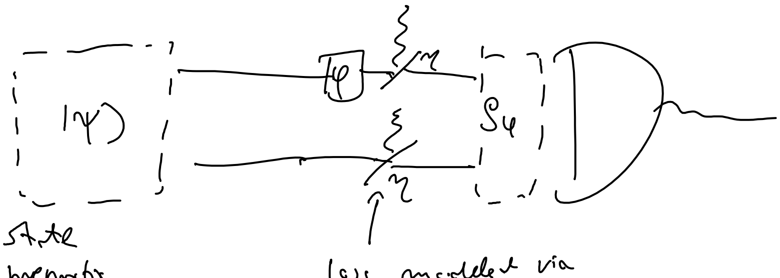
If we use independent atoms $\delta\Delta \geq \frac{1}{t\sqrt{N}}$

Using entangled states (spin squeezed states) could possibly lead to $\delta\Delta \geq \frac{1}{tN}$ ..

10. Role of decoherence

For practical implementations we need to take into account decoherence. Most common models

- photon loss (in Mach-Zehnder interferometer)
- dephasing (in Ramsey interferometry).



loss modelled via fictitious beam-splitters with transmission η

Mathematically problem more difficult since we deal with mixed states at the output

$$|n_a, n_b\rangle \longrightarrow \sum_{l_a=0}^{n_a}\sum_{l_b=0}^{n_b} \underbrace{\sqrt{\binom{n_a}{l_a}\eta_a^{n_a-l_a}(1-\eta_a)^{l_a}}}_{B^{n_a}_{l_a}(\eta_a)} \cdot \underbrace{\sqrt{\binom{n_b}{l_b}\eta_b^{n_b-l_b}(1-\eta_b)^{l_b}}}_{B^{n_b}_{l_b}(\eta_b)} \cdot$$

$$\cdot |n_a - l_a, n_b - l_b, \underbrace{l_a, l_b}_{\text{loss modes}}\rangle$$

$$|\psi\rangle = \sum_n \alpha_n |n, N-n\rangle \xrightarrow{\varphi} \sum_n \alpha_n e^{in\varphi}|n, N-n\rangle$$

$$\xrightarrow{\text{loss}} \sum \alpha_n e^{in\varphi} \sum^{n} \sum^{N-n} \sqrt{B^{n}_{l_a}(\eta_a)} \cdot \sqrt{B^{N-n}_{l_b}(\eta_b)} \cdot$$

$$\xrightarrow{\text{loss}} \sum_n \alpha_n e^{in\varphi} \sum_{l_a=0}^{n} \sum_{l_b=0}^{N-n} \sqrt{B^n_{l_a}(\eta_a)} \cdot \sqrt{B^{N-n}_{l_b}(\eta_b)} \cdot$$

$$\cdot |n-l_a, N-n-l_b, l_a, l_b\rangle =: |\Phi\rangle$$

$$\rho_\varphi = \mathrm{Tr}_{l_a, l_b} |\Phi^\varphi\rangle\langle\Phi^\varphi| =$$

$$= \sum_{l_a=0}^{N} \sum_{l_b=0}^{N-l_a} |\psi^\varphi_{l_a,l_b}\rangle\langle\psi^\varphi_{l_a,l_b}|$$

conditional state (unnormalized) provided l_a and l_b photons were lost

$$|\psi^\varphi_{l_a,l_b}\rangle = \sum_{n=l_a}^{N-l_b} \alpha_n e^{in\varphi} \sqrt{B^n_{l_a}(\eta_a) B^{N-n}_{l_b}(\eta_b)}\, |n-l_a, N-n-l_b\rangle$$

Alternatively we may represent the channel using the Kraus representation. :

$$\rho_\varphi = \sum_{l_a=0}^{N} \sum_{l_b=0}^{N-l} K^\varphi_{l_a,l_b} |\psi\rangle\langle\psi| K^{\dagger\varphi}_{l_a l_b}$$

$$K^\varphi_{l_a,l_b} = (K_{l_a} \otimes K_{l_b}) \cdot U_\varphi$$

$$K_{l_a} = \sum_{n=l_a} \sqrt{B^n_{l_a}(\eta_a)}\, |n-l_a\rangle\langle n| \, , \quad K_{l_b} = \sum_{n=l_b} \sqrt{B^n_{l_b}(\eta_b)}\, |n-l_b\rangle\langle n|$$

$$\sum_{l_a} K^\dagger_{l_a} K_{l_a} = \sum_{l_a} \sum_{n=l_a} |n\rangle\langle n|\, B^n_{l_a}(\eta_a) = \mathbb{1} .$$

To find the fundamental bounds we need to either calculate F_Q (local approach) or design optimal covariant estimation scheme (global approach)

Most interesting question. Do we still have qualitative precision enhancement i.e. $\frac{1}{N}$ (or $\frac{1}{N^\alpha}, \alpha > \frac{1}{2}$) instead of $\frac{1}{\sqrt{N}}$?

11. Bounds in the local approach in presence of

decoherence

In general it is impossible to write analytical formula for F_Q since for mixed states $F_Q = \mathrm{Tr}(\rho_\varphi \Lambda^2)$. It is even more unlikely to be possible to perform optimization over input states analytically.

Therefore we need to find more tractable bounds..

[Escher, Filho, Davidovich 2011)

General setup:

$$\rho_\varphi = \sum_i K_i^\varphi |\psi\rangle\langle\psi| K_i^{\varphi\dagger}$$

We can always look at it as a unitary transformation in an extended space $S+E$

$$\rho_\varphi = \mathrm{Tr}_E |\Phi^\varphi\rangle\langle\Phi^\varphi|$$

$$|\Phi^\varphi\rangle = U^\varphi_{S,E} |\psi\rangle_S \otimes |0\rangle_E = \sum_i K_i^\varphi \otimes 1 \, |\psi\rangle \otimes |i\rangle$$

Kraus representation is not unique. We have freedom to apply local unitary V on the E subsystem. This is equivalent to new Kraus representation $K_{i'}' = V_{i'i} K_i$.

Intuitive fact: Tracing out E can only reduce the information available on φ

$$F_Q(\rho_\varphi) \leq \underbrace{F_Q(|\Phi^\varphi\rangle)}_{\text{easy to calculate}}$$

More formally

$$F_Q(\rho_\varphi) = \max_{\Pi_m^S} \underbrace{F}_{\text{classical Fisher}}(\rho_\varphi, \Pi_m^S) = \max_{\{\Pi^S\}} F(|\Phi^\varphi\rangle, \Pi_m^S \otimes 1^E)$$

$$\leq \max_{\{\Pi^{S,E}\}} F(|\Phi^\varphi\rangle, \Pi^{S,E}) = F_Q(|\Phi^\varphi\rangle)$$

$| \quad \{\Pi^{S,E}\}$

$$F_Q(|\Phi^\varphi\rangle) = 4\left(\langle\Phi^{\varphi\prime}|\Phi^{\varphi\prime}\rangle - |\langle\Phi^\varphi|\Phi^{\varphi\prime}\rangle|^2\right)$$

$$|\Phi^{\varphi\prime}\rangle = \sum_i \frac{dK_i^\varphi}{d\varphi}\otimes 1\, |\psi\rangle \otimes |i\rangle$$

$$F_Q(|\Phi^\varphi\rangle) = 4\cdot\Big(\langle\psi|\overbrace{\sum_i \frac{dK_i^{\varphi\dagger}}{d\varphi}\frac{dK_i^\varphi}{d\varphi}}^{H_2}|\psi\rangle - \Big|\langle\psi|\underbrace{\sum_i \frac{dK_i^{\varphi\dagger}}{d\varphi}K_i^\varphi}_{H_1}|\psi\rangle\Big|^2\Big)$$

Is it useful?

Theorem:

$$F_Q(\rho_\varphi) = \min_{\{K_i^\varphi\}} F_Q(|\Phi^\varphi\rangle)$$

Proof based on the fact that second order expansion of Bures fidelity corresponds to F_Q:

$$F_B(\rho_\varphi, \rho_{\varphi+d\varphi}) = \left[\mathrm{Tr}\left(\sqrt{\rho_\varphi^{\frac{1}{2}}\rho_{\varphi+d\varphi}\rho_\varphi^{\frac{1}{2}}}\right)\right]^2 \approx$$

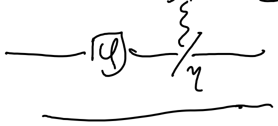
$$\approx 1 - \frac{\delta x^2}{4}\cdot F_Q(\rho_\varphi)$$

$$F_B(\rho_\varphi, \rho_{\varphi+d\varphi}) = \max_{|\Psi_\varphi\rangle, |\Psi_{\varphi+d\varphi}\rangle} |\langle\Psi_\varphi|\Psi_{\varphi+d\varphi}\rangle|^2$$

Intuition: there is always a purification in which access to environment is not helpful in estimating φ.

Example: Interferometry with loss

for simplicity only in one arm



• Let us take Kraus decomposition

$$K_l^{\varphi} = (K_l \otimes \mathbb{1}) \cdot U_\varphi$$

$$\frac{dK_l^{\varphi}}{d\varphi} = i(K_l \otimes \mathbb{1})\, a^\dagger a\, U_\varphi$$

$$\langle\psi|\sum_{l=0,\dots} \frac{dK_l^{\varphi\dagger}}{d\varphi} \frac{dK_l^{\varphi}}{d\varphi}|\psi\rangle = \langle\psi|U_\varphi^\dagger a^\dagger a \underbrace{\sum_l K_l^{\varphi\dagger} K_l^{\varphi}}_{\mathbb{1}} a^\dagger a U_\varphi|\psi\rangle$$

$$= \langle\psi|U_\varphi^\dagger (a^\dagger a)^2 U_\varphi|\psi\rangle = \langle\psi|(a^\dagger a)^2|\psi\rangle$$

similarly for $\langle\psi|\sum_{l=0,\dots} \frac{dK_l^{\dagger}}{d\varphi} K_l|\psi\rangle = \langle\psi|a^\dagger a|\psi\rangle$

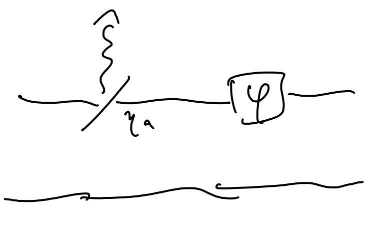
The same as in lossless case, this result is useless !
It just tells us that Fisher ~~in~~ lossy case will be less then Fisher in lossless case

• Let us take another Kraus representation for a lossy interferometer

$$K_l'^{\varphi} = e^{-ila\varphi} K_l^{\varphi}$$

It is exactly the same as if we change the order of U_φ and K_l

$$K_l'^{\varphi} = U_\varphi (K_l \otimes \mathbb{1})$$ } first loss then phase sensing



It is obvious that environment "knows nothing" about the phase φ,
So this is a good try!
We should not be sure, however, that we will get a strict bound. Monitoring E may still be helpful as we may make

E may still be helpful as we may make use of coherences that are lost if we look only at $S_\cdot$

$$\frac{d K_l'^{\varphi}}{d\varphi} = i a^+ a\, U_\varphi \left(K_l \otimes \mathbb{1}\right)$$

$$\langle\psi|\sum_l K_l^+ U_\varphi (a^+a)^2 U_\varphi^+ K_l |\psi\rangle =$$

$$= \langle\psi| \sum_l \sum_{n\geq l} \sqrt{B_l^n(\eta)}\, |n\rangle\langle n-l| (a^+a)^2 \sum_{n'\geq l} |n'-l\rangle\langle n'| \sqrt{B_l^{n}(\eta)}\, |\psi\rangle$$

$n = n'$

$$= \langle\psi|\, \hat{n}_a^2\, |\psi\rangle - \langle\psi| \sum_{n=0} \sum_{l=0}^{n} |n\rangle\langle n|\, 2nl\, B_l^n(\eta) |\psi\rangle$$

$$+ \langle\psi| \sum_{n,l} |n\rangle\langle n|\, l^2\, B_l^n(\eta)\, |\psi\rangle =$$

$$\left\{ \begin{array}{l} \sum_{l=0}^{n} l\, B_l^n(\eta) = \sum_{l=0}^{n} l \binom{n}{l} \eta^{n-l} (1-\eta)^l = \sum_l \frac{n!}{(l-1)!(n-l)!} \eta^{n-l} (1-\eta)^l = \\ = n(1-\eta) \end{array} \right.$$

$$\left\{ \begin{array}{l} \sum_{l=0} l^2\, B_l^n(\eta) = \sum_l (1-\eta)\, l \cdot n \frac{(n-1)!}{(l-1)!(n-l)!} \eta^{n-l} (1-\eta)^{l-1} = \\ = + (1-\eta) n \cdot + (n-1) n (1-\eta)^2 \end{array} \right.$$

$$\langle H_2 \rangle = \langle n_a^2 \rangle - 2(1-\eta) \langle n_a^2 \rangle + (1-\eta) \langle n_a \rangle + (1-\eta)^2 \left(\langle n_a^2 \rangle - \langle n_a \rangle\right)$$

$$\langle H_1 \rangle = \langle n \rangle - (1-\eta) \langle n \rangle = \eta \langle n \rangle$$

$$F_Q = 4\Big(\langle n^2 \rangle \cdot \left(1 - 2 + 2\eta + (1-\eta)^2\right) + \langle n \rangle \cdot \left(1 - \eta - (1-\eta)^2\right)$$
$$- \langle n \rangle^2 \eta^2 \Big)$$

$$= 4\eta^2 \langle \Delta n^2 \rangle + 4\eta(1-\eta) \langle n \rangle$$

A bit better bound but still would suggest that $\frac{1}{N}$ scaling survives

But, we could use more general Kraus representation

$$K_l'^{\varphi} = e^{-i\alpha l \varphi} K_l^{\varphi} \qquad \text{then}$$

?

$$F_Q = 4\cdot\left(1-(1-\eta)\alpha\right)^2\langle\Delta n^2\rangle + 4\eta(1-\eta)\alpha^2\langle n\rangle$$

If we now take $\alpha = \frac{1}{1-\eta}$ then

$$F_Q = \frac{4\eta}{1-\eta}\langle n\rangle$$

Scales only linearly for the number of photons

So precision will scale $\delta\varphi \gtrsim \sqrt{\frac{1-\eta}{4\eta N}}$

We lose the Heisenberg scaling

12 Bounds in the global approach in the presence of loss (again for simplicity $\eta_G = 1$)

We can still use covariant measurement

$$\overline{C} = \int \frac{d\varphi}{2\pi}\, \mathrm{Tr}\left(\Pi_C\, \varrho_\varphi\right) C_{\varphi,0}$$

$$\varrho_\varphi = \mathrm{Tr}_L\, |\Phi^\varphi\rangle\langle\Phi^\varphi| =$$

$$= \sum_{l=0}^{N} |\psi_l^\varphi\rangle\langle\psi_l^\varphi|$$

$$|\psi_l^\varphi\rangle = \sum_{n=l}^{N} \alpha_n e^{in}\sqrt{B_l^n(\eta_a)}\,|n-l, N-n\rangle$$

One can argue that the optimal choice are $\alpha_n \in \mathbb{R}$ and $\Pi_C = |e\rangle\langle e|$

$$e = \sum_n |n, N-n\rangle$$

So linearly:

$$\overline{C} = 2 - \langle\psi|\, A\, |\psi\rangle \qquad A = \begin{bmatrix} a & a & & & \\ a & a & a & & \\ & a & a & a & \\ & & a & a & a \\ & & & a & a \end{bmatrix}$$

$n-1$

$$A_{n-1,n} = A_{n,n-1} = \sum_{l=0} \sqrt{B_l^n(\eta_a) B_l^{n-1}(\eta_a)}$$

Looking for maximal eigenvalue is no longer possible analytically ....

Fact:

$$\lambda_{max} \leq \lambda_{max}'$$

wher λ_{max}' is the maximum eigenvalue of matrix A' where all element $A_{n-1,n}$ are replaced by the maximal one $A_{N-1,N}$

Proof:

For a matrix with all entries ≥ 0 eigenvector corresponding to λ_{max} has $|v\rangle = \sum \alpha_n |n\rangle$ where all $\alpha_n \geq 0$

$$\lambda_{max} = \langle v|A|v\rangle \leq \langle v|A'|v\rangle \leq \lambda'_{max}$$

$$\bar{C} \geq 2 - 2A_{N-1,N} \cdot \cos\left(\frac{\pi}{N+2}\right) =$$

$$= 2\left[1 - \cos\left(\frac{\pi}{N+2}\right) \cdot \sum_{l=0}^{N-1} \sqrt{B_l^N(\eta)\, B_l^{N-1}(\eta)}\right]$$

expanding in $\frac{1}{N}$

$$\bar{C} \gtrsim 2\Big[1 - \Big(1 - \underbrace{\frac{\pi^2}{2(N+2)^2}}_{\text{not relevant}}\Big) \cdot \Big(1 - \frac{1-\eta}{8\eta N} + \ldots\Big)\Big] = \frac{1-\eta}{4\eta N}$$

$$\delta\varphi \gtrsim \sqrt{\frac{1-\eta}{4\eta N}}$$

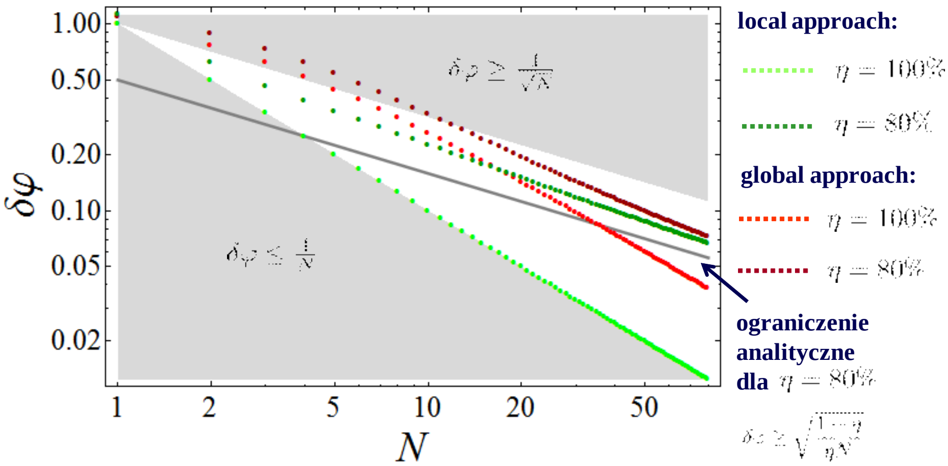
The same bound as in the local approach !

For losses in both arms, one can derive:

$$\delta\varphi \underset{\eta}{\gtrsim} \sqrt{\frac{1-\eta}{N}}$$

Summary in a plot :

Summary in a plot:



12. Outlook

- Is this behaviour typical in all relevant decoherence models
- looking for practical applications