Sending qubits via a channel with imperfectly correlated noise - natural SU(2) diffusion model

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Abstract

We present a model of an N-qubit channel where consecutive qubits experience random rotations. Our model is an extension to the standard decoherence-free subsystem approach (DFS) which assumes that all the qubits experience the same disturbance. The variation of rotations acting on consecutive qubits is modeled as diffusion on the SU(2) group. The model may be applied to spins traveling in a varying magnetic field, or to photons passing through a fiber whose birefringence fluctuates over the time separation between photons. We derive an explicit formula describing the action of the channel on an arbitrary N-qubit state. For N=3 we investigate the effects of diffusion on both classical and quantum capacity of the channel. In particular we observe that nonorthogonal states are necessary to achieve the optimal classical capacity. Furthermore we find the threshold for the diffusion parameter above which coherent information of the channel vanishes.

1. Depolarizing channel with perfectly correlated noise

- N qubit depolarizing channel, where each qubit experience the same and completely random disturbance:

\[ \mathcal{E}(\rho) = \int dU^{\otimes N} \rho U^{\otimes N} U^{\dagger \otimes N} \]

N qubit state: SU(2) Haar measure

Applications:
1. photons transmitted through a long fiber (each photon experience the same random rotation of polarization).
2. spins ½ being sent through a slowly varying magnetic field
3. communication in the absence of reference frames

2. Structure of the output state with perfectly correlated noise

Irreducible subspaces under the action of \( U^{\otimes N} \)

\[ \mathcal{T}(\rho) := \mathcal{E}(\rho) = \int dU^{\otimes N} \rho U^{\otimes N} U^{\dagger \otimes N} \]

Twirling operation

- Bidimes (\( d_b \)) times

\[ \rho_j = \sum_{j=1}^{d_b} \rho_j \]

- Faithfully transmitted states - allow for noiseless classical and quantum communication (see e.g. [2])

4. What happens if noise is not perfectly correlated?

- Consecutive qubits experience slightly different rotations

\[ \mathcal{E}(\rho) = \int dU_1 \ldots dU_N p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \rho U_1^{\dagger} \cdots U_N^{\dagger} \]

- The actions described via a stationary Markov process

\[ p(U_1, \ldots, U_N) = p(U_N|U_{N-1} = \cdots = p(U_1|U_0) \]

- Possible applications:
1. photons transmitted through a long fiber (each photon experience comparable to birefringence fluctuation time)
2. spins travelling through relatively quickly varying magnetic field

5. Imperfectly correlated noise model

- Consecutive qubits experience slightly different rotations

\[ \mathcal{E}(\rho) = \int dU_1 \ldots dU_N p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \rho U_1^{\dagger} \cdots U_N^{\dagger} \]

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6. What is the natural choice for conditional probability \( p(U_i|U_{i-1}) = ? \)

7. Natural choice for conditional probability

Diffusion on the SU(2) group

- Isotropic diffusion on SU(2)

\[ \partial p(U; t) = \frac{1}{2} \mathcal{L} p(U; t) \]

- Solution with the initial conditions \( p(U; 0) = \delta(U - I) \)

\[ p(U; t) = \sum_{j=0}^\infty \left( \frac{1}{2} j + 1 \right)^{\frac{1}{2}} \sum_{m=1}^{j+1} \mathcal{D}^{(m)}(U)m \]

- Conditional probability

\[ p(U_i|U_{i-1}) = p(U_i|U_{i+1}; t) \]

\[ t \to 0 \quad \text{perfect noise correlation} \quad t \to \infty \quad \text{no correlation} \]

8. Example: Capacity of the three qubit channel with imperfectly correlated noise

- Three qubit twirled state. Effectively: one qubit + one classical state

\[ \rho = \frac{1}{4} (\rho_{12} + \rho_{13} + \rho_{23}) \]

- Classical capacity. Holevo-Schumacher-Westmoreland formula:

\[ C = \sup (\rho, \psi) S(\mathcal{E}(\rho)) \]

- Ensemble maximizing classical channel capacity \( C \) contain non-orthogonal states:

\[ \frac{1}{2} \rho_{12} \otimes \rho_{13} \otimes \rho_{23} \]

- Appropriate orthogonal states can, however, achieve almost optimal capacity. Best orthogonal states have the form:

\[ \| \psi_1 \|_1 = \sqrt{2} |\psi_2 \|_1 \]

- Additionally for diffusion strength \( t > 0.275 \) the coherent information is zero. This suggests that quantum communication is impossible in this regime (noise correlation is too weak) [1]

Conclusions

- A natural model of an N qubit channel is presented, making use of SU(2) diffusion in order to describe imperfect correlations between random rotations experienced by consecutive qubits.
- In the simplest case of N=3, optimal classical capacity of the channel is calculated. States necessary for optimal communication are found to be non-orthogonal, although appropriately chosen orthogonal states perform almost optimal. The strength of diffusion above which coherent information of the channel is zero is found.

References