State estimation on correlated copies
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quant-ph/0412155 (to appear in PRA)

Abstract
State estimation is usually analyzed in the situation when copies are in a product state, either mixed or pure. We investigate here the concept of state estimation on correlated copies. We analyze state estimation on correlated N qubit states, which are permutationally invariant. Using a correlated state we try to estimate as good as possible the direction of the Bloch vector of a single particle reduced density matrix. We derive the optimal fidelity for all permutation invariant states. We find the optimal state, which yields the highest estimation fidelity among the states with the same reduced density matrix. Interestingly this state is not a product state. We also point out that states produced by optimal universal cloning machines are the worst from the point of view of estimating the reduced density matrix.

1. What is state estimation?

- Given N copies of an unknown quantum state, one performs measurements and judging on measurement results tries to guess what was the state.

\[ |\psi^{(N)}\rangle = |\psi\rangle \otimes |\Psi\rangle \otimes \ldots \otimes |\Psi\rangle \]

- Quality of estimation fidelity of a single guess:

\[ F = \sum_{i} \rho_{i} |p_{i} - \rho_{i}| \]

If state |\psi\rangle is totally unknown we require fidelity F not to depend on |\psi\rangle

2. Generalized measurement

- Standard measurement

\[ \mathbb{A} = \sum_{i} \lambda_{i} P_{i} \]

observable measurement outcomes

\[ P_{i} \delta_{P_{i}} \]

orthogonality

\[ \sum_{i} P_{i} = 1 \]

completeness

\[ p_{i} = Tr(P_{i}) \]

probability of the result i

- Generalized measurement

\[ A = \sum_{i} \lambda_{i} P_{i} \]

coupling the system with an external ancillary system (e.g. measuring device), allowing for an evolution and performing a standard measurement on systems-ancilla. This results in an effective generalized measurement on a system, described by operators \( A_{POM} \) (positive operator valued measure).

\[ \rho_{i} = p_{i} \]

probability of the result \( i \)

3. Optimal estimation on product states

- Optimal fidelity of estimating N copies of unknown qubit state [1, 2]

\[ F = \frac{N + 1}{N + d} \]

Optimal estimation strategy requires the use of generalized measurement, performed on all copies simultaneously.

4. Cloning and Estimation – the motivation

- \( N \rightarrow M \) cloning:

\[ |\psi^{(N)}\rangle \otimes |\Psi\rangle \otimes \ldots \otimes |\Psi\rangle \rightarrow \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} |\Psi\rangle \otimes |\psi^{(M)}\rangle \in \mathbb{H}^{N} \otimes \mathbb{H}_{A} \]

input states blank states ancilla

output state of M clones:

\[ \tilde{\rho} = Tr_{A} |\Psi\rangle \langle \Psi| \]

single particle density matrix of each clone is identical and given by:

\[ \rho = \frac{Tr_{A} |\Psi\rangle \langle \Psi|}{M(N-1) + M(d-1)} \]

Clones are correlated:

\[ \tilde{\rho} = \rho^{(M)} \]

- Estimation on clones

\[ N \rightarrow M \text{ cloning} \]

\[ F = \frac{M(N+1) + M(d-1)}{M(N+d)} \]

Optimal fidelity [3]

4. Cloning and Estimation – the motivation

Optimal state is not supported on at most two subspaces with different j

8. Correlated state optimal for state estimation

- Fixing mixedness of single particle density matrix \( \eta \), we look for the highest fidelity \( F \) maximized by \( \eta \) under constraints:

\[ \frac{2}{N+1} \sum_{i} \rho_{i} \leq \eta \]

Subspaces with high j are bad for state estimation, yet necessary to yield high values of \( \eta \)

- Structure of the optimal state in the case of 4 qubits:

Optimal state supported on at most two subspaces with different j

Optimal state is supported on symmetric subspace \( \eta = 1 \)

State supported on symmetric subspace \( \eta = N/2 \)

Such states emerge from optimal cloning machines

Conclusions

- Optimal estimation fidelity found in the case of N qubit permutationally invariant states

- Correlations between copies influence estimation fidelities of single particle density matrix

- Product state is not the optimal one for state estimation

- States supported on symmetric subspace (e.g. states coming out from optimal cloning machines) are the worst from the point of view of state estimation

- States optimal for state estimation are correlated and supported at most on two subspaces with different j

References