Quantum enhanced phase estimation in the presence of loss

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Interferometry

- **Mach-Zehnder Interferometer**

\[
\langle n_1 \rangle = |\alpha|^2 (1 + \cos \varphi) / 2
\]

\[
\langle n_2 \rangle = |\alpha|^2 (1 - \cos \varphi) / 2
\]

detecting \( n_1 \) and \( n_2 \) + knowing theoretical dependence of \( n_1, n_2 \) on \( \varphi \) ↓ we can estimate \( \varphi \)
Interferometry

- **Mach-Zehnder Interferometer**

\[
\langle n_1 \rangle = |\alpha|^2 (1 + \cos \varphi)/2
\]

\[
\langle n_2 \rangle = |\alpha|^2 (1 - \cos \varphi)/2
\]

\(n_1\) and \(n_2\) are subject to shot noise

each measurement yields a bit different \(\varphi\)

\[
\delta \varphi \propto \frac{1}{|\alpha|} = \frac{1}{\sqrt{n}}
\]

Standard scaling
**Interferometry**

- **Mach-Zehnder Interferometer**

In general the exact formula for precision depends on $\varphi$

The steeper curves, the better precision
Heisenberg scaling - Quantum Cramer-Rao bound

For arbitrary measurement on $|\psi_\varphi\rangle$ and arbitrary estimator $\tilde{\varphi}(n)$, the minimal variance of the estimator is bounded by:

$$\delta\tilde{\varphi} \geq \frac{1}{\sqrt{F}}, \quad F = 4\Delta n_1 = 4[\langle \psi | \hat{n}_1^2 | \psi \rangle - \langle \psi | \hat{n}_1 | \psi \rangle^2]$$

Given $N$ photons

$$\Delta n_1 \leq N^2/4$$

$$\delta\tilde{\varphi} \geq \frac{1}{\sqrt{N^2}} = \frac{1}{N}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle)$$
What are the optimal $N$ photon states if there are losses?
Lossy interferometry

\[ l \text{ photons lost} \]

\[ \eta - \text{transmission coefficient} \]

- **NOON state**  
  \[ |\psi_\varphi\rangle = \frac{1}{\sqrt{2}} (|N, 0\rangle e^{-iN\varphi} + |0, N\rangle) \]

- **Even if a single photon is lost**  
  \[ |\psi\rangle \rightarrow |\psi_{\varphi}^{(1)}\rangle \propto |N - 1, 0\rangle \quad \text{we lose all phase information!} \]
Looking for the optimal state

\[ |\psi\rangle = \sum_{n=0}^{N} \alpha_n |n, N - n\rangle \]

\[ |\psi^{(l)}\rangle = \sum_{n=0}^{N} e^{-in\phi} \alpha_n |n, N - n\rangle \]

\[ |\psi^{(l)}_\varphi\rangle = \sum_{n=l}^{N} e^{-in\phi} \alpha_n \sqrt{\binom{n}{l}} \eta^{n-l} (1 - \eta)^l |n-l, N-n\rangle \]
Cramer-Rao bound for mixed states

\[ |\psi\rangle = \sum_{n=0}^{N} \alpha_n |n, N-n\rangle \rightarrow \rho_\varphi = \sum_{l=0}^{N} p_l |\psi_\varphi^{(l)}\rangle \langle \psi_\varphi^{(l)}| \]

- Subspaces with different \( l \) are orthogonal

\[ \delta \tilde{\varphi} \geq \frac{1}{\sqrt{F}} \quad F = \sum_{l=0}^{N} p_l F_l \]

\[ F_l = 4(\Delta n_1)_l = 4(\langle \psi^{(l)}|\hat{n}_1^2|\psi^{(l)}\rangle - (\langle \psi^{(l)}|\hat{n}_1|\psi^{(l)}\rangle)^2) \]

\[ F = F(x_0, \ldots, x_N) \quad x_n = |\alpha_n|^2 \quad \text{concave function} \]

- Problem: maximization of a concave function over a convex set

\[ x_n \geq 0 \quad \sum_{n=0}^{N} x_n = 1 \]
Optimal $N=6$ photon state

$$|\psi\rangle = \sum_{n=0}^{N} \sqrt{x_n} |n, N - n\rangle$$

Diagram showing the N00N state with various curves representing different loss cases.
Optimal vs N00N states, $N=20$

Quantum enhanced precision

Heisenberg limit = $1/N$

100% losses

no losses
Scaling for large $N$, $\eta = 0.9$

For arbitrary small losses, if $N$ is large enough, the scaling becomes $\delta \varphi \propto \frac{1}{\sqrt{N}}$
Experimental lossy phase estimation using the optimal $N=2$ states
Optimal $N=2$ state

$$|\psi\rangle = \sqrt{x_0}|0, 2\rangle + \sqrt{x_1}|1, 1\rangle + \sqrt{x_2}|2, 0\rangle$$

$$\eta_{th} = \frac{1 + 2\sqrt{2}}{7} \approx 0.5469$$
Linear optical scheme - preparation

\[ |\psi\rangle = \sqrt{x_0}|0, 2\rangle + \sqrt{x_1}|1, 1\rangle + \sqrt{x_2}|2, 0\rangle \]
Linear optical scheme - sensing

\[ |\psi^{(0)}\rangle \propto \sqrt{x_0} |0, 2\rangle + \sqrt{x_1} \sqrt{\eta} e^{-i\phi} |1, 1\rangle + \sqrt{x_2} \eta e^{-2i\phi} |2, 0\rangle \]

no photon lost

\[ |\psi^{(1)}\rangle \propto \sqrt{x_1} \sqrt{1 - \eta} |0, 1\rangle + \sqrt{x_2} \sqrt{2\eta(1 - \eta)} e^{-i\phi} |1, 0\rangle \]

one photon lost

\[ |\psi^{(2)}\rangle \propto \sqrt{x_2} (1 - \eta) |0, 0\rangle \]

two photons lost

No phase information
Linear optical scheme - detection

Optimal measurement saturating the Cramer-Rao bound
Theoretical coincidence probabilities

- $\eta = 0.361$, the optimal state

- Registering given number of coincidences, we perform Max-Likelihood estimation of $\varphi$
Experimental setup

measurement

interferometer

state preparation

interferometer visibility: 98%
Experimental results

- $\eta = 0.361$

the optimal state
Experimental results

- $\eta = 0.361$
Estimation for two phases separated by 0.12 rad

• After estimating phase 100 times using the maximum likelihood estimator:

the optimal state

\[ \delta \varphi = 0.02060 \]

the NOON state: \[ \frac{1}{\sqrt{2}} (|2, 0\rangle + |0, 2\rangle) \]

\[ \delta \varphi = 0.02556 \]

the Cramer-Rao bound: \[ \delta \varphi \geq 0.01906 \]

the Cramer-Rao bound (with 98% interferometer visibility): \[ \delta \varphi \geq 0.02001 \]
Achieving the Cramer-Rao bound

- Precision rescaled by the square root of the number of coincidences for various transmissions $\eta$, for five phases: 0, $\pm 0.02$, $\pm 0.04$ rad
Maybe one day...

Gravitational wave detectors

Laser gyroscopes
Summary


- Optimal $N$-photon states for lossy phase estimation found
- Numerical evidence for lack of asymptotic better-than-standard scaling of precision when losses are present
- ...

**Experiment** [arXiv: 0906.NEXT_WEEK (2009)]

- Design of preparation and measurement scheme able to reach the Cramer-Rao bound
- Experimental two photon phase estimation performed using coherent, NOON and the optimal states