7 - Review of QM

25 października 2010
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States, (pure, mixed), Evolution (Unitary, CP maps), Measurement (von neumann, POVM),
Entangled states, tensor product, reduced density matrices,
separable states, LOCC .. Bloch sphere, single qubit CP maps
(picture of deformation of the Bloch sphere)

7.1 Isolated quantum system

7.1.1 Pure states. Full knowledge about the system

$|\psi\rangle \in \mathcal{H}$ $\quad \dim \mathcal{H} = d$ $\quad |\psi\rangle = \sum_i \alpha_i |i\rangle$

$2(d-1)$ real parameters (Rays in Hilbert space)

7.1.2 Evolution. Schrödinger equation

$$\frac{i}{\hbar}\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle \quad \Rightarrow \quad |\psi(t)\rangle = \underbrace{e^{-\frac{i}{\hbar}Ht}}_{\mathcal{U}}|\psi(0)\rangle$$

Unitary evolution

7.1.3 Measurement (von Neumann)

Defined by a set of orthogonal projectors $\{\Pi_i\}$

$\Pi_i \Pi_j = \delta_{ij}\Pi_i$ $\qquad p_i = \langle\psi|\Pi_i|\psi\rangle$ - probability of measurement $|\psi\rangle$

$$|\psi_i\rangle = \frac{1}{\sqrt{p_i}}\Pi_i|\psi\rangle$$ - state „collapsed" after the measurement

notice we can write $p_i = \mathrm{Tr}(|\psi\rangle\langle\psi|\Pi_i)$

7.1.4 Mixed states. Incomplete knowledge

state $|\psi_k\rangle$ is prepared with probability p_k

For measurement $\{\Pi_i\}$, probability of result i is

$$p_i = \sum_k p_k \langle\psi_k|\Pi_i|\psi_k\rangle = \mathrm{Tr}\Big(\underbrace{\sum_k p_k|\psi_k\rangle\langle\psi_k|}_{\varrho}\,\Pi_i\Big)$$

Evolution: $\varrho(t) = \mathcal{U}\varrho(0)\mathcal{U}^\dagger$

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Set of density matrices: $\rho \in \mathcal{L}(\mathcal{H})$, $\rho \geq 0$ $\mathrm{Tr}\,\rho = 1$

- convex set

7.2 Composite systems

(A) (B)

We describe the state using $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi_{AB}\rangle = \sum_{ij} c_{ij} \underset{\text{basis in } \mathcal{H}_A}{|e_i\rangle} \otimes \underset{\text{basis in } \mathcal{H}_B}{|f_j\rangle}$$

Analogously for more subsystems

Mixed states $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$

7.2.1 Entangled states

<u>Def.</u> We call a pure state $|\psi_{AB}\rangle$ separable iff it can be written as a product

$$|\psi^{AB}\rangle = |\varphi\rangle_A \otimes |\eta\rangle_B$$

otherwise we call it entangled

<u>Def</u> We call a mixed state ρ_{AB} separable iff it can be written as:

$$\rho^{AB} = \sum_k p_k |\varphi_k\rangle\langle\varphi_k| \otimes |\eta_k\rangle\langle\eta_k|$$

otherwise we call it entangled.

In separable state all correlation are classical - can be thought as coming from correlated preparation of product states

7.2.2 Description of a subsystem

Imagine that for a composite system in state ρ_{AB} we only have access to subsystem A.

What mathematical object represent all information we can extract? (Predict our measurement results)

Our measurements are restricted to $\{\Pi_k \otimes \mathbb{1}_B\}$

$$p_k = \mathrm{Tr}(\Pi_k \otimes \mathbb{1} \cdot \rho^{AB}) =$$

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$$= \sum_{ij} \langle i|_A \otimes \langle j|_B \, \Pi_k \otimes \mathbb{1} \cdot \varrho^{AB} \, |i\rangle_A \otimes |j\rangle_B =$$

$$= \sum_{ij} (\Pi_k \otimes \mathbb{1} \cdot \varrho^{AB})^{ij}_{ij} = \sum_{\substack{ij \\ i'j'}} \Pi_k{}^{i}_{i'} \, \delta^{j}_{j'} \cdot \varrho^{AB}{}^{i'j'}_{ij} =$$

$$= \sum_{ii'} \Pi_k{}^{i}_{i'} \sum_j \varrho^{AB}{}^{i'j}_{ij} = \sum_{ii'} \Pi_k{}^{i}_{i'} \, \varrho^{A}{}^{i'}_{i} = Tr(\Pi_k \varrho^A)$$

$$\varrho^{A}{}^{i'}_{i} = \sum_j \varrho^{AB}{}^{i'j}_{ij} \qquad \varrho^A = Tr_B \, \varrho^{AB}$$

reduced density matrix

If we have access only to subsystem A we see just the reduced state $\varrho^A = Tr_B \varrho^{AB}$

- Notice that if $\varrho^{AB} = |\psi^{AB}\rangle\langle\psi^{AB}|$ is pure the ϱ_A in general will be mixed (unless $|\psi_{AB}\rangle$ is product)
  This is a second way in which mixed states enter QM.

7.3 Interaction with the environment

We want to describe measurement and evolution of the system coupled to environment (additional degrees of freedom)

7.3.1 Generalized measurement

Instead of performing von Neuman measurement on the system we let it evolve unitarily and interact with measurement device (environment), after which we perform von Neuman measurement on the measurement device

$$\varrho^{SE} = \varrho^S \otimes \varrho^E \quad \text{- initial state decoupled}$$

$$p_k = Tr(\underbrace{U \varrho^S \otimes \varrho^E U^\dagger} \cdot \mathbb{1} \otimes \Pi^E_k)$$

no limit on number of operators Π_k

$$= Tr(\varrho^S \cdot M_k)$$

$$(*) \; M_k = Tr_E(\mathbb{1} \otimes \varrho^E \cdot U^\dagger \, \mathbb{1} \otimes \Pi^E_k \, U)$$

We can now regard M_k as measurement operators.

We can now regard M_k as measurement operators.

$M_k \geq 0$, $\sum_k M_k = \mathbb{1}$, but they do not have to be projectors

Conversely for every set of operators $\{M_k\}$ there exist ρ^E, U, Π_k^E such that (*) holds

(Steinspring dilation theorem) $\{ U|\psi\rangle|e_0\rangle = \sum_k \sqrt{M_k}|\psi\rangle|e_k\rangle$

This means that when looking for optimal measurement we should only impose:

$M_k \geq 0$, $\sum_k M_k = \mathbb{1}$ (POVM)

7.3.2 Evolution of a system interacting with environment

$\rho_{in}^{SE} = \rho_{in}^S \otimes \rho_{in}^E$

If we have access only to S how does the evolution look like?

$\rho_{in}^{SE} \longrightarrow \rho_{out}^{SE} = U \rho^S \otimes \rho^E U^\dagger$

Without loosing generality we can put $\rho^E = |0\rangle\langle 0|$

$\rho_{out}^S = \mathrm{Tr}_E\, U\rho^S \otimes |0\rangle\langle 0| U^\dagger =$

$\sum_i \langle e_i| U \rho^S \otimes |0\rangle\langle 0| U^\dagger |e_i\rangle =$

$= \sum_i \langle e_i| U |0\rangle \rho^S \cdot \langle 0| U^\dagger |e_i\rangle = \sum_i K_i \rho_{in}^S K_i^\dagger$

$$\boxed{\rho_{out} = \mathcal{E}(\rho_{in}) = \sum_i K_i \rho_{in} K_i^\dagger, \quad \sum K_i^\dagger K_i = \mathbb{1}}$$

Kraus operator

Trace preserving condition

Completely positive trace preserving map.

Conversely for every set of Kraus operators K_i, $\sum_i K_i^\dagger K_i = \mathbb{1}$ there exist U on $\mathcal{H}_S \otimes \mathcal{H}_E$, such that

$K_i = \langle e_i | U | 0\rangle$ (Steinspring dilation theorem)

$\{ U|\psi\rangle|e_0\rangle = \sum_k K_k |\psi\rangle |e_k\rangle$

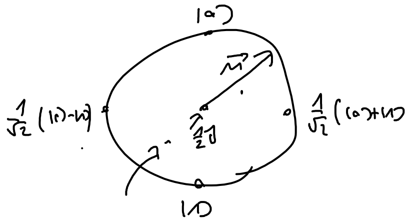
- If one has access to which Kraus operator happened the CP map can be viewed also as generalized measurement where $M_i = K_i^\dagger K_i$

7.4 The most general single qubit evolution

pure state of a qubit $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$
(Bloch sphere)

$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + \vec{m}\cdot\vec{\sigma})$

$\vec{m} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$



Mixed states: $|\vec{m}| \leq 1$

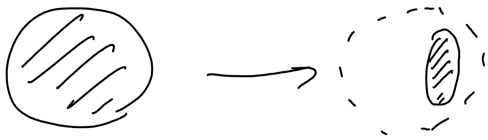
$\vec{m} = r\cdot(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, (Bloch ball)

$r = 0 \Rightarrow \rho = \frac{1}{2}\mathbb{1}$ max mixed state

What can the most general CP map do with the Bloch ball?

- $\rho_{out} = U\rho_{in}U^\dagger$: rotation

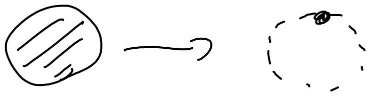
- $\rho_{out} = \sum_i K_i \rho K_i^\dagger$: rotation + shrinking (ball → elipsoid) + translation



Example:

- $K_1 = |0\rangle\langle 1|$, $K_2 = |0\rangle\langle 0|$ $\quad K_1^\dagger K_1 + K_2^\dagger K_2 = \mathbb{1}$ ok

$\mathcal{E}(\rho) = |0\rangle\langle 1|\rho|1\rangle\langle 0| + |0\rangle\langle 0|\rho|0\rangle\langle 0| = |0\rangle\langle 0|$



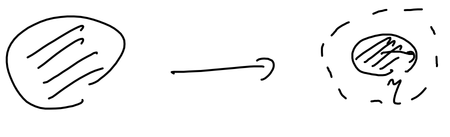
} dissipation

- $K_1 = |0\rangle\langle 0|$, $K_2 = |1\rangle\langle 1|$

} decoherence
destroys coherence
between $|0\rangle$ and $|1\rangle$

- $K_0 = \mathbb{1}\cdot\eta$, $K_1 = \frac{(1-\eta)}{2}|0\rangle\langle 0|$ $\quad K_2 = \frac{(1-\eta)}{2}|0\rangle\langle 1|$

$K_3 = \frac{1-\eta}{2}|1\rangle\langle 0|$ $\quad K_4 = \frac{1-\eta}{2}|1\rangle\langle 1|$



7.5 Linearity of QM

(i) Evolution of pure states is unitary => linear

$$U(a|\psi_1\rangle + b|\psi_2\rangle) = aU|\psi_1\rangle + bU|\psi_2\rangle$$

(ii) General evolution of mixed states is also linear

$$\mathcal{E}(p_1\varrho_1 + p_2\varrho_2) = \sum_i K_i(p_1\varrho_1 + p_2\varrho_2)K_i^\dagger = p_1\mathcal{E}(\varrho_1) + p_2\mathcal{E}(\varrho_2)$$

While it is possible to imagine nonlinear evolution of vectors $U(a|\psi_1\rangle + b|\psi_2\rangle) \neq aU|\psi_1\rangle + bU|\psi_2\rangle$ (nonlinear QM)

it is not possible to have nonlinear evolution of ϱ if ϱ is to used in probability assigning rule $p_i = \mathrm{Tr}(\Pi_i \varrho)$.

Linearity of density matrix evolution is just a necessity. In non-linear QM there is no density matrix!

7.6, LOCC CP maps (Local operations and classical communication)

$$\varrho^{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$\mathcal{E}(\varrho^{AB}) = \sum_i K_i \varrho^{AB} K_i^\dagger \qquad K_i \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

If subsystems A & B are far away it is very hard to perform quantum interactions between them. What we can do in practice is perform local operations and communicate by phone.

– thanks to communication we may correlate our actions (classically). Very important class of CP maps.

- Just operation on A:

$$\mathcal{E}_1(\varrho^{AB}) = \sum_{i_1} K_{i_1} \otimes \mathbb{1}\, \varrho^{AB}\, K_{i_1}^\dagger \otimes \mathbb{1} \qquad \sum_i K_i^\dagger K_i = \mathbb{1}_A$$

- If index i_1 is communicated to B he can perform operation conditioned on i_1

$$\mathcal{E}_2(\rho^{AB}) = \sum_{i_1} \sum_{i_2} \mathbb{1} \otimes K^{(i_1)}_{i_2} \left(K_{i_1} \otimes \mathbb{1}\, \rho^{AB} K_{i_1}^{\dagger} \otimes \mathbb{1}\right) \mathbb{1} \otimes K^{(i_1)}_{i_2}$$

$$\forall_{i_1} \sum_{i_2} K^{(i_1)\dagger}_{i_2} K^{(i_1)}_{i_2} = \mathbb{1}_B$$

- B can communicate back index i_2

$$\mathcal{E}_3(\rho^{AB}) = \sum_{i_1} \sum_{i_2} \sum_{i_3} \left(K^{(i_1, i_2)}_{i_3} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes K^{(i_1)}_{i_2}\right)\left(K_{i_1} \otimes \mathbb{1}\right) \rho^{AB} \text{--}\,||\,\text{--}^{\dagger}$$

$$\vdots$$

Very complicated structure, hard to characterize

- LOCC operations cannot produce entangled states from separable ones (nor increase entanglement)!