8 - Nonorthogonal states

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Discrimination of nonorthogonal states

$|\psi\rangle, |\varphi\rangle$ what is the optimal procedure to distinguish between $|\psi\rangle$, $|\varphi\rangle$

Π_0, Π_1 . $0 \to$ we guess $|\psi\rangle$

$1 \to$ we guess $|\varphi\rangle$

probability of error

$$e = \frac{1}{2}\langle\varphi|\Pi_0|\varphi\rangle + \frac{1}{2}\langle\psi|\Pi_1|\psi\rangle \qquad \Pi_0 + \Pi_1 = \mathbb{1} \quad \Pi_0, \Pi_1 \geq 0$$

$$\min_{(\Pi_0, \Pi_1)} e$$

$$e = \frac{1}{2}\left(\langle\varphi|\Pi_0|\varphi\rangle + 1 - \langle\psi|\Pi_0|\psi\rangle\right) = \frac{1}{2}\left(1 + \mathrm{Tr}\left[\Pi_0 \cdot \underbrace{(|\varphi\rangle\langle\varphi| - |\psi\rangle\langle\psi|)}_{A}\right]\right)$$

useful parametrization:

$$|\varphi\rangle \quad \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle \qquad \langle\psi|\varphi\rangle = \cos\theta$$

$$A = \begin{bmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix} - \begin{bmatrix} \cos^2\frac{\theta}{2} & -\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix} = \sin\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e = \frac{1}{2}\left[1 + \mathrm{Tr}\left(\Pi_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right]$$

we need to minimize $\mathrm{Tr}\left(\Pi_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ $\qquad \Pi_0 \geq 0$, $\mathbb{1} - \Pi_0 \geq 0$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle\langle+| - |-\rangle\langle-| \qquad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\langle+|\Pi_0|+\rangle - \langle-|\Pi_0|-\rangle$$

we know that with any $|\psi\rangle$ $\quad \langle\psi|\Pi_0|\psi\rangle \geq 0$

$\langle\psi|\Pi_0|\psi\rangle \leq 1$

the best option is to have $\langle+|\Pi_0|+\rangle = 0 \quad \langle-|\Pi_0|-\rangle = 1$

$$\Pi_0 = |-\rangle\langle-| \quad \Rightarrow \quad \Pi_1 = |+\rangle\langle+|$$

$$\mathrm{tr}\left(\Pi_0 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = -1$$

$$\mathrm{tr}\left(\Pi_0 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = -1$$

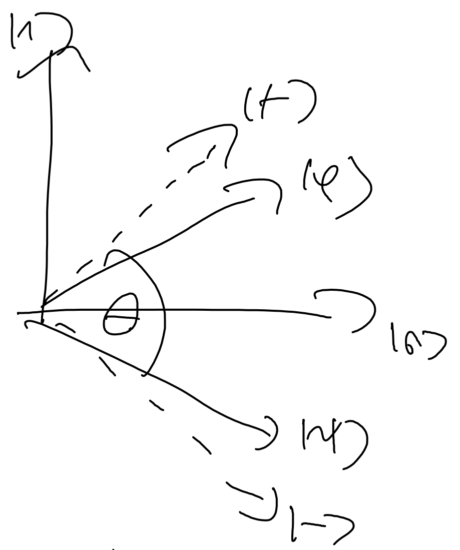
$$e = \frac{1}{2}\left(1 - \sin\theta\right) = \frac{1}{2}\left(1 - \sqrt{1-\cos^2\theta}\right)$$

$$\boxed{e = \frac{1}{2}\left(1 - \sqrt{1-|\langle\psi|\varphi\rangle|^2}\right)}$$

If $|\langle\psi|\varphi\rangle| > 0$ it is not possible to distinguish without errors e

- Examples:
  - $|\leftrightarrow\rangle, |\updownarrow\rangle$ $\quad e = 0$
  - $|\leftrightarrow\rangle, |\swarrow\rangle$ $\quad \langle\leftrightarrow|\swarrow\rangle = \frac{1}{\sqrt{2}}$

$$e = \frac{1}{2}\left(1 - \sqrt{\frac{1}{2}}\right) \approx 14{,}6\,\%$$



- What if we have N copies of a state

$$|\Phi\rangle = \underbrace{|\varphi\rangle \otimes \ldots \otimes |\varphi\rangle}_{N} \qquad |\Psi\rangle = \underbrace{|\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{N}$$

$$e_N = \frac{1}{2}\left(1 - \sqrt{1 - \langle\Phi|\Psi\rangle^2}\right) = \frac{1}{2}\left(1 - \sqrt{1-|\langle\psi|\varphi\rangle|^{2N}}\right)$$

$$|\langle\psi|\varphi\rangle|^{2N} \xrightarrow{N\to\infty} 0$$

the more copies we have the more distinguishable

The more copies we have the more distinguishable states are. $N \to \infty$ = classical limit where all states are distinguishable

Cloning of quantum states

Let $|\psi\rangle$ be an unknown state. Is there a quantum operation producing many copies of the state

$$\forall_{\psi} \quad |\psi\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_M \xrightarrow{\mathcal{U}} |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |A_\psi\rangle$$

cloning machine

— if it existed we could first clone and then estimate violating the bound derived previously.

. No cloning theorem. It is not possible to clone two nonorthogonal states $0 < |\langle\varphi|\psi\rangle| < 1$

$$|\psi\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle \xrightarrow{\mathcal{U}} |\psi\rangle \otimes |\psi\rangle \otimes |A_\psi\rangle$$

$$|\varphi\rangle \otimes |0\rangle \otimes |0\rangle \longrightarrow |\varphi\rangle \otimes |\varphi\rangle \otimes |A_\varphi\rangle$$

by unitarity

$$\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle^2 \cdot \langle A_\psi|A_\varphi\rangle$$

$$\langle\psi|\varphi\rangle\left(1 - \langle\psi|\varphi\rangle\langle A_\psi|A_\varphi\rangle\right) = 0$$

either $\langle\psi|\varphi\rangle = 0$ or $|\langle\psi|\varphi\rangle| = 1$ contradiction

Non-cloning and nondistinguishability of q. states $\to$
$\to$ essential for QKD.