# III. Generalised measurements and distinguishability of quantum states

- Generalised measurements (introduced in Lecture 3)
- General quantum discrimination problem

$\{p_i, \rho_i\}$ given states $\rho_i$ with probabilities $p_i$ our goal is to distinguish between them.

$C(j|i)$ - cost function - penalty for guessing „i" if the true state is „j"

$\{M_j\}$ - measurement, where index „j" corresponds to the state we guess if we obtain this measurement result

Average cost:

$$\overline{C} = \sum_i p_i \sum_j \mathrm{Tr}(\rho_i M_j)\, C(j,i)$$

Goal:

$$\min_{\{M_j\}} \overline{C} = ?$$

In general a very difficult problem

- Simplest case - discrimination of two pure states

$p_0 = \frac{1}{2}$ : $|\psi_0\rangle$

$p_1 = \frac{1}{2}$ : $|\psi_1\rangle$

$$C(j,i) = 1 - \delta_{ij}$$

$$M_0, M_1 \geq 0 \qquad M_0 + M_1 = \mathbb{1}$$

Cost:

$$\overline{C} = \frac{1}{2}\left( \langle\psi_1|M_0|\psi_1\rangle + \langle\psi_0|M_1|\psi_1\rangle\right)$$

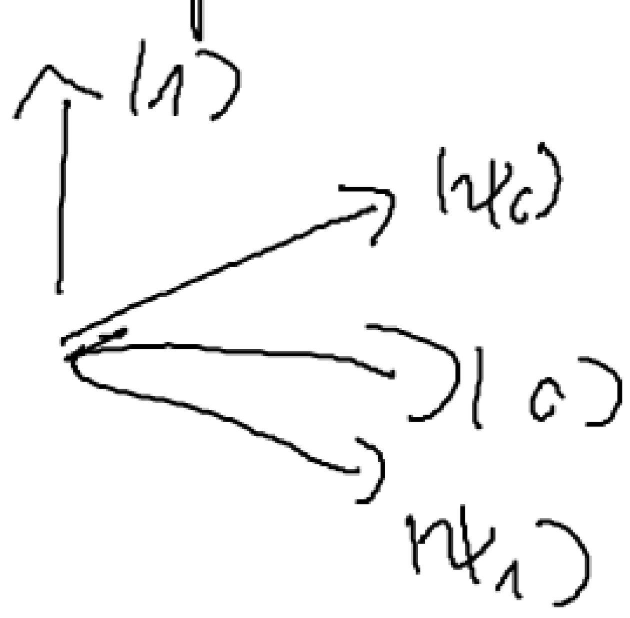
But: $M_1 = \mathbb{1} - M_0$

$$\overline{C} = \frac{1}{2} + \frac{1}{2}\mathrm{Tr}\, M_0\left( |\psi_1\rangle\langle\psi_1| - |\psi_0\rangle\langle\psi_0|\right)$$

$\max\limits_{\substack{M_0 \geq 0 \\ M_0 \leq \mathbb{1}}} \overline{C} = ?$

parametrize $|\psi_0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$

$|\psi_1\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$




$\langle\psi_0|\psi_1\rangle = \cos\theta$

$$\overline{C} = \frac{1}{2}\left[1 + \mathrm{Tr}\left( M_0 \cdot \begin{bmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{bmatrix}\right)\right]$$

$\min \mathrm{Tr}\, M_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ what $0 \leq M_0 \leq 1$ will minimize?

$M_0 = |-\rangle\langle -|$ $\quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \Rightarrow \mathrm{Tr}\left(M_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$

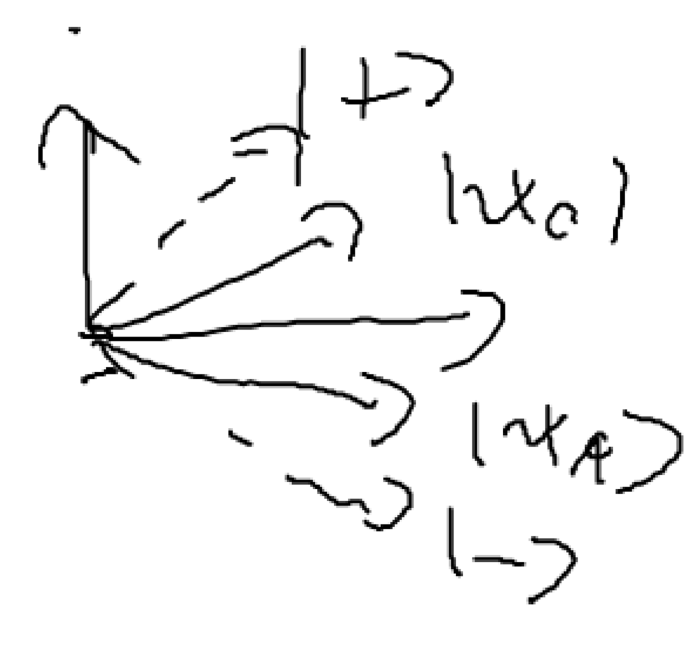
$$\boxed{\overline{C} = \frac{1}{2}(1 - \sin\theta) = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}\right)}$$

If states are non orthogonal $|\langle\psi_1|\psi_2\rangle| > 0$

$\overline{C} > 0$ they cannot be discriminated perfectly

If $|\psi_1\rangle = |\psi_2\rangle \Rightarrow \overline{C} = \frac{1}{2}$ pure guessing.

Optimal measurement basis:



Many copies $|\psi_0\rangle^{\otimes N}$, $|\psi_1\rangle^{\otimes N}$

$$\overline{C_N} = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi_0|\psi_1\rangle|^{2N}}\right) \xrightarrow[|\langle\psi_1|\psi_0\rangle| \neq 1]{N \to \infty} 0$$

this is the „classical" limit. If we have many copies everything becomes distinguishable.

- Unambiguous discrimination.

We know it is impossible to discriminate two non-orthogonal states perfectly, but maybe we can do it with some probability and know when we have not done any mistake

$M_0$ - we guess the state was $|\psi_0\rangle$

$M_1$ - — " — $|\psi_1\rangle$

$M_?$ - we restrain ourselves from making any claim

In order to be sure we make no errors

$$M_0 = \lambda |\psi_1^\perp\rangle\langle\psi_1^\perp| , \qquad M_1 = \lambda |\psi_0^\perp\rangle\langle\psi_0^\perp|$$

$$M_? = 1 - M_0 - M_1$$

If we do like this the probability of success is

$$P_{success} = \frac{1}{2}\left[\lambda |\langle\psi_1^\perp|\psi_0\rangle|^2 + \lambda |\langle\psi_0^\perp|\psi_1\rangle|^2\right]$$

$$\begin{cases} |\psi_0^\perp\rangle = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}|0\rangle \\ |\psi_1^\perp\rangle = \sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|0\rangle \end{cases}$$

$$P_{success} = \frac{1}{2}\lambda\left[2\sin^2\theta\right] = \lambda \sin^2\theta$$

We want to have $\lambda$ as large as possible

But we know that $M_2 \geq 0$ so:

$$\mathbb{1} - \lambda \begin{bmatrix} 2\sin^2\frac{\theta}{2} & 0 \\ 0 & 2\cos^2\frac{\theta}{2} \end{bmatrix} \geq 0 \qquad \lambda \leq \frac{1}{2\cos^2\frac{\theta}{2}}$$

$$P_{success} = \frac{1}{2\cos^2\frac{\theta}{2}} \sin^2\theta = 2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

One can check that this strategy is worse "on average"

Average error $\quad \frac{1}{2}\cos\theta \geq \frac{1}{2}\left(1 - \sqrt{1-\cos^2\theta}\right) \qquad \theta \in [0, \frac{\pi}{2}]$

$$\sqrt{1-\cos^2\theta} \geq (1-\cos\theta)$$

$$1-\cos^2\theta \geq (1-\cos\theta)^2 \qquad \text{ok}$$

$$\cos\theta \geq \cos^2\theta$$