Canonical Quantization of E-M field

General

Fields $\varphi^i(\vec{r},t) \longrightarrow \mathcal{L}(\varphi^i, \partial_\nu \varphi^i)$

$\uparrow$ lagrangian density

$$\begin{cases} L = \int d^3\vec{r}\, \mathcal{L}, \quad \frac{\partial \mathcal{L}}{\partial \varphi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} = 0 \end{cases}$$

$$\longrightarrow \mathcal{H} = \sum_i \Pi^i \partial_t \varphi_i - \mathcal{L}, \quad H = \int d^3\vec{r}\, \mathcal{H}$$

Hamiltonian density

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi^i)}$$

Quantization: $\varphi^i \to \hat{\varphi}^i$, $\Pi^i \to \hat{\Pi}^i$

$$[\hat{\varphi}^i(\vec{r},t), \hat{\Pi}^j(\vec{r}',t)] = i\hbar\, \delta^{(3)}(\vec{r}-\vec{r}')\, \delta^{ij}$$

$$\dot{\hat{\Pi}}^i = -\frac{i}{\hbar}[\hat{\Pi}^i, \hat{H}] \qquad \dot{\hat{\varphi}}^i = -\frac{i}{\hbar}[\hat{\varphi}^i, \hat{H}]$$

In Electrodynamics

$$A^\mu = \left[\frac{\varphi}{c}, \vec{A}\right]^T$$

$\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}\varphi - \partial_t \vec{A}$

$\Downarrow$

$\vec{\nabla}\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Remaining equations: $\vec{\nabla}\vec{E} = 0 \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t \vec{E}$

$\Downarrow$

$\Box \vec{A} = 0$

We should choose gauge

$$A^\mu \to A^\mu - \partial_\mu \chi$$

We choose Coulomb gauge $\vec{\nabla}\vec{A} = 0$, $\overset{\text{no charges}}{\Longrightarrow}$ $\varphi = 0$

Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & B_x & 0 \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu A^\nu} = 0$$

$$\Downarrow$$

$$\Box A^\nu = 0$$

$$\left\{ \begin{array}{l} A^0 = \frac{\varphi}{c} \\ \partial_{\nu=0} = \frac{1}{c}\partial_t \end{array} \right.$$

$$\Pi^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\nu)} = \frac{1}{c^2\mu_0}\left(\partial^t A^\nu - \partial^\nu \varphi\right)$$

$$\Pi^0 = 0 \quad , \quad \Pi^k = \frac{1}{c^2\mu_0}\left(-\partial^t A^k + \partial^k \varphi\right) = -\frac{E^k}{c^2\mu_0} = -\varepsilon_0 F$$