5. Phase space description of quantum states (of light)

Let us focus on single mode states

A general state (in the Fock basis)

$$\varrho = \sum_{n,m} |n\rangle\langle n|\varrho|m\rangle\langle m| = \sum_{n,m} \underbrace{\langle n|\varrho|m\rangle}_{C_{n,m}} |n\rangle\langle m|$$

We can also write ϱ in coherent state „basis":

$$\varrho = \underbrace{\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|}_{1} \varrho \int \frac{d^2\beta}{\pi} |\beta\rangle\langle\beta| = \int \frac{d^2\alpha\, d^2\beta}{\pi^2} \langle\alpha|\varrho|\beta\rangle\, |\alpha\rangle\langle\beta|$$

$$R(\alpha^*, \beta) := \langle\alpha|\varrho|\beta\rangle e^{\frac{|\alpha|^2}{2} + \frac{|\beta|^2}{2}}$$ – R-representation

↑ not very illuminating, too many variables; we want a distribution in phase space $P(\alpha)$ just one variable ($\int dz\, dw$)

Single coordinate probability distribution

$\hat{q}$ – position operator $\quad \hat{q} = \int dq\, q\, |q\rangle\langle q|$, $\quad \langle q|q'\rangle = \delta(q-q')$

$$P(q) = \langle q|\varrho|q\rangle = \mathrm{Tr}(\varrho\, |q\rangle\langle q|)$$

Note that: $\quad |q\rangle\langle q| = \delta(q - \hat{q}) := \frac{1}{2\pi} \int dz\, e^{i(q-\hat{q})z}$

Check:

$$\langle q'|\,|q\rangle\langle q|\,|q''\rangle \overset{?}{=} \langle q'|\delta(q-\hat{q})|q''\rangle$$

$$\langle q'|\,|q\rangle\langle q|\,|q''\rangle = \delta(q'-q)\,\delta(q-q'')$$

$$\langle q'|\delta(q-\hat{q})|q''\rangle = \langle q'|\delta(q-q'')|q''\rangle = \delta(q'-q'')\,\delta(q-q'') = \delta(q'-q)\,\delta(q-q'')$$

$$P(q) = \mathrm{Tr}[\varrho\, \delta(q-\hat{q})]$$

We want to define prob. distribution in phase space $P(q,p) = ?$

Let us try :

$$\mathcal{P}'(q,p) \overset{?}{:=} \mathrm{Tr}\left[\varrho\, \delta(q-\hat{q})\, \delta(p-\hat{p})\right]$$

if $\hat{q}$ and