II. Interactions of q. light with matter

G. Quantum electromagnetic field + nonrelativistic quantum point charge

Lagrangian: $L(\vec{q}, \dot{\vec{q}}, A_\mu, \partial_\nu A_\mu)$

$$L = \int d^3r \, \mathcal{L} + \frac{m\dot{q}^2}{2}$$

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu$$

(A_μ ← Lagrangian variables)

$$j^\mu = [\rho c, \vec{j}] \qquad \rho(\vec{r}) = e\,\delta^{(3)}(\vec{r} - \vec{q})$$

($\vec{q}$ ← Lagrangian variable)

$$\vec{j} = e\dot{\vec{q}}\,\delta^{(3)}(\vec{r} - \vec{q})$$

We use the Coulomb gauge $\vec{\nabla}\vec{A} = 0$

Without charges we could additionally put $\varphi = 0$

In general $\vec{A} = \vec{A}_\perp + \vec{A}_\parallel$ $\quad \{ \nabla \vec{A}_\perp = 0, \quad \vec{\nabla} \times \vec{A}_\parallel = 0$

Gauge transformation: $\vec{A}_c = \vec{A} - \vec{\nabla}\chi$ — In Coulomb Gauge we choose χ

$\varphi_c = \varphi + \frac{\partial \chi}{\partial t}$ — such that $\vec{\nabla}\vec{A}_c = 0$

we remove $\vec{A}_\parallel$ from $\vec{A}$

Electric field $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi = \underbrace{-\frac{\partial \vec{A}_c}{\partial t}}_{\vec{E}_\perp} \underbrace{- \vec{\nabla}\varphi_c}_{\vec{E}_\parallel}$

Without charges: $\vec{E} = \vec{E}_\perp$

With charges we will have additionally $\vec{E}_\parallel = -\vec{\nabla}\varphi_c$,

φ_c - will be determined by charges

Quantization procedure:

$\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$, $\quad \hat{A}_i, \hat{\Pi}_i, \quad \hat{q}_i, \hat{p}_i$

$[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$

$[\hat{A}_i(\vec{r}), \hat{\Pi}_j(\vec{r}')] = i\hbar\,\delta^{tr}_{ij}(\vec{r} - \vec{r}')$

other commutators are zero

The result of quantization procedure:

$$H = \frac{1}{2m}\left(\hat{\vec{p}} - e\,\hat{\Lambda}\right.$$