9. Active linear optics

instead of $\vec{a}' = \underset{n\times n \text{ matrix}}{B}\,\vec{a}$ $\left(\begin{array}{c}\text{passive linear optics}\\ B-\text{unitary}\end{array}\right)$

we consider more general transformations

$$\begin{pmatrix}\vec{a}'\\ \vec{a}'^{+}\end{pmatrix} = \underset{2n\times 2n \text{ matrix}}{S}\begin{pmatrix}\vec{a}\\ \vec{a}^{+}\end{pmatrix}$$

n - number of modes

Preservation of commutation relations:

$$\left[\begin{pmatrix}\vec{a}'\\ \vec{a}'^{+}\end{pmatrix}, (\vec{a}'^{+}, \vec{a}')\right] = \begin{pmatrix}[\vec{a}', \vec{a}'^{+}], & [\vec{a}', \vec{a}']\\ [\vec{a}'^{+}, \vec{a}'^{+}], & [\vec{a}'^{+}, \vec{a}']\end{pmatrix} = \overbrace{\left(\begin{array}{c|c}\mathbb{1} & 0\\ \hline 0 & -\mathbb{1}\end{array}\right)}^{G}$$

$$\|$$

$$\left[S\begin{pmatrix}\vec{a}\\ \vec{a}^{+}\end{pmatrix}, (\vec{a}^{+}, \vec{a})S^{+}\right] = S\underbrace{\left[\begin{pmatrix}\vec{a}\\ \vec{a}^{+}\end{pmatrix}, (\vec{a}^{+}, \vec{a})\right]}_{G}S^{+}$$

$$\boxed{S\cdot G S^{+} = G, \quad G = \left(\begin{array}{c|c}\mathbb{1} & 0\\ \hline 0 & -\mathbb{1}\end{array}\right)}$$

S - quasiunitary.

$\Big\{$ in passive case $S = \begin{pmatrix}B & 0\\ 0 & B^{+}\end{pmatrix}$

As before we are looking for $\hat{U}_S$:

$$\begin{pmatrix}\vec{a}'\\ \vec{a}'^{+}\end{pmatrix} = S\begin{pmatrix}\vec{a}\\ \vec{a}^{+}\end{pmatrix} = \hat{U}_S^{+}\begin{pmatrix}\vec{a}\\ \vec{a}^{+}\end{pmatrix}\hat{U}_S$$

Fact

$$\hat{\mathcal{U}}_S = e^{i\hat{H}_S}, \quad \underbrace{\hat{H}_S = \frac{1}{2}(\vec{a}^{\dagger}, \vec{a})\,\mathcal{H}\begin{pmatrix}\vec{a} \\ \vec{a}^{\dagger}\end{pmatrix}}$$

$$\mathcal{H} = -iG\cdot\ln S \quad \{ \; S = e^{iG\mathcal{H}}$$

hermition

let us show that $\mathcal{H}$ is Hermitian.

$$e^{\ln S} = \sum_{k=0}^{\infty}\frac{(\ln S)^k}{k!} = S$$

$$\mathcal{H}^{\dagger} = i(\ln S)^{\dagger}\underset{G}{\underset{\parallel}{G^{\dagger}}} =$$

$$SGS^{\dagger} = G \Rightarrow S^{-1} = GS^{\dagger}G$$

$$G(\ln S)^{\dagger}G = \ln(GS^{\dagger}G) = \ln(S^{-1}) = -\ln S$$

$$G^2 = \mathbb{1}$$

$$G(\ln S)^{\dagger}G = -\ln S \Rightarrow (\ln S)^{\dagger}G = -G\ln S$$

$$\mathcal{H}^{\dagger} = i(\ln S)^{\dagger}G = -iG\ln S = \mathcal{H}.$$

Examples single-mode squeezing

$$\hat{\mathcal{U}}_S = e^{\frac{1}{2}(\zeta^{*}\hat{a}^2 - \zeta\hat{a}^{\dagger 2})}, \quad \hat{H}_S = \frac{1}{2i}(\zeta^{*}\hat{a}^2 - \zeta\hat{a}^{\dagger 2})$$

$$\zeta = re^{i\theta}$$

$$\mathcal{H} = \frac{1}{i}\begin{pmatrix} 0 & -\zeta \\ \zeta^{*} & 0\end{pmatrix}$$

$$\underbrace{i\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}}_{G}\cdot\frac{1}{i}(0 \; -$$