Chapter 3

Uncertainty relations

Figure 3.1: Schemes illustrating different variants of uncertainty relations

3.1 Heisenberg-Robertson uncertainty relation

Let us start by recalling the standard formulation of the uncertainty relation that one may encounter in all quantum mechanics textbooks. Given some observable A one defines its uncertainty on a given state $|\psi\rangle$ as $\Delta A = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2}$. Then one proofs that for two observables A and B the following inequality holds

$$\Delta A \Delta B \geq \frac{|\langle[A,B]\rangle|}{2}, \tag{3.1}$$

which we will refer to as the Heisenberg-Robertson uncertainty relation. In case of position and momentum operators the uncertainty relation reduces to

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{3.2}$$

In order to understand the physical content of the above inequality, we should keep in mind that both ΔA and ΔB are quantities that one determines via measurement of either A *or* B observables performed on *different* realizations of state $|\psi\rangle$. There is no notion of precision of measuring one observable vs. disturbance of the other observable on the same physical system, which was actually the main message of the so called Heisenberg microscope though experiment. Neither there is any notion of joint measurement of the two quantities on a given quantum state. Here, we simply state that the quantum state $|\psi\rangle$ will lead to the following spread of measurement results if observable A *or* observable B is measured independently of each other. In short, this relation reflects the inherent uncertainty that is present in the quantum state itself with respect to measuring different physical quantities.

3.2 Precision vs. Disturbance uncertainty relation

To understand the difference between the Heisenberg-Robertson formulation of the uncertainty relation and the actual Heisenberg microscope thought experiment, let us recall the Heiseberg micropscope setting. Consider a lens with aperture a at a distance l from the illuminated object. For simplicty we assume that the object is imaged using the lens on a very distant screen, see Fig. **??**. We can quantify the precision as a minimal size of a light spot in the object plane that correspond to light travelling at a given angle to the screen. From standard optics considerations one can show, that the precision is then of the order: $\delta x \approx \frac{\lambda}{a} l$. Note that we use a different notation, namely δx rather than Δx, to clearly discriminate between the precision of measurement (δx) vs. the inherent uncertainty of the physical quantity in the state itself (Δx).

In order to asses the disturbance to the momentum of the particle, note that when we collect light arriving at a given angle on the screen, we cannot say exactly at what angle the photon has been emitted from the particle, as ig could travel along different paths through the lens. The angle spread is roughly $\varphi \approx a/l$ (we assume the angle is small), and hence the uncertainty in the x-component of momentum transfer to the particle is $dp \approx \frac{h}{\lambda}\frac{a}{l}$—note again a different notation dp for the disturbance. So finally we can write $\delta x dp \gtrsim \hbar$.

As the reader might have noticed the above reasoning is very qualitative and in fact it is not an easy task to rigorously define the concept of disturbance of the observable (see e.g. a review paper [?]. Moreover, the approach discussed above, breaks the symmetry between the x and p quantities, which is not always desired. Because of this conceptual difficulties we will not study this approach here in detail, and will rather focus on a symmetric and easier to study from a quantitative perspective concept of joint measurement—see Fig. 3 to see the formal difference between these three approaches.

3.3 Joint measurement

Within the concept of joint measurement, our goal is to obtain information on two quantities in the best possible way even if their corresponding observables do not commute. In particular we will want to obtain a limit on the possibility of joint measurement of position and momentum. Clearly, it will not be possible to obtain simultanous measurement of x and p equivalent to sharp measurements of both quantitites. Still, if we allow for a compromise of performing „smeared" position and momentum measurements we may be able to construct the corresponding joint measurement protocol.

We start with a general definition.

Definition. Π_a and Π_b POVMs are jointly measurable iff there exist a POVM $\Pi_{a,b}$ such that $\Pi_a = \int \mathrm{d}b\, \Pi_{a,b}$, $\Pi_b = \int \mathrm{d}a\, \Pi_{a,b}$.

Remark. If Π_x, Π_p commute, we simply take $\Pi_{x,p} = \Pi_x \Pi_p$. If not we cannot make this construction (product will not be hermitian/positive).

In particular, we cannot find a joint measurement operator representing ideal (sharp) position and momentum measurements. But, if we consider smeared position and momentum measurements:

$$\tilde{\Pi}_x = \int \mathrm{d}x'\, \mu(x - x')|x'\rangle\langle x'| \tag{3.3}$$

$$\tilde{\Pi}_p = \int \mathrm{d}p'\, \nu(p - p')|p'\rangle\langle p'|, \tag{3.4}$$

where ν, μ are smearing functions (e.g. gaussians), then maybe it is possible to find $\tilde{\Pi}_{x,p}$ such that $\tilde{\Pi}_x = \int \mathrm{d}p\, \tilde{\Pi}_{x,p}$, $\tilde{\Pi}_p = \int \mathrm{d}x\, \tilde{\Pi}_{x,p}$.

Indeed, consider:

$$\tilde{\Pi}_{x,p} = \frac{1}{2\pi\hbar} D(x,p)\Pi_0 D(x,p)^\dagger, \tag{3.5}$$

where $D(q,p) = e^{\frac{ix\hat{p} - ip\hat{x}}{\hbar}}$ is the displacement operator and $\Pi_0 \geq 0$, $\mathrm{Tr}\Pi_0 = 1$. Equivalently:

$$\tilde{\Pi}_{x,p} = \frac{1}{2\pi\hbar} e^{-\frac{ip\hat{x}}{\hbar}} e^{\frac{ix\hat{p}}{\hbar}} \Pi_0 e^{-\frac{ix\hat{p}}{\hbar}} e^{\frac{ip\hat{x}}{\hbar}}, \tag{3.6}$$

where we have used $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$.

We first check whether the above constructed $\Pi_{x,p}$ is a legal POVM. Positivity is clear, as $D(x,p)$ is unitary and Π_0 is positive. We are therefore left to check the completness condition. Calculating matrix elements in the position eigenbasis

$$\frac{1}{2\pi\hbar}\langle x'| \int \mathrm{d}x\mathrm{d}p e^{-\frac{ipx'}{\hbar}} e^{\frac{ix\hat{p}}{\hbar}} \Pi_0 e^{-\frac{ix\hat{p}}{\hbar}} e^{\frac{ipx''}{\hbar}} |x''\rangle = \delta(x' - x'') \int \mathrm{d}x \langle x|\Pi_0|x\rangle = \delta(x' - x''), \tag{3.7}$$

we see that indeed $\int \mathrm{d}x\mathrm{d}p\, \tilde{\Pi}_{x,p} = \mathbb{1}$.

Let us now calculate marginal probability distributions of x and p resulting from application of this joint measurement to a general state ρ. Writing $\Pi_0 = \int \mathrm{d}x'\mathrm{d}x''(\Pi_0)^{x'}_{x''}|x'\rangle\langle x''|$

in the position eigebasis, we get

$$p(x) = \mathrm{Tr}\left(\rho \int \mathrm{d}p\tilde{\Pi}_{x,p}\right) =$$
$$= \frac{1}{2\pi\hbar}\mathrm{Tr}\left(\rho \int \mathrm{d}p\mathrm{d}x'\mathrm{d}x''(\Pi_0)^{x'}_{x''}|x'+x\rangle\langle x''+x|e^{\frac{ip(x''-x')}{\hbar}}\right) =$$
$$\mathrm{Tr}(\rho \int \mathrm{d}x'(\Pi_0)^{x'-x}_{x'-x}|x'\rangle\langle x'|) = \mathrm{Tr}(\rho\tilde{\Pi}_x), \tag{3.8}$$

from which we see that indeed:

$$\tilde{\Pi}_x = \int \mathrm{d}x'\mu(x'-x)|x'\rangle\langle x'|), \tag{3.9}$$

where $\mu(x) = \langle x|\Pi_0|x\rangle$ is the smearing function. Analogously we can show that the momentum probability distribution corresponds to

$$\tilde{\Pi}_p = \int \mathrm{d}p'\nu(p'-p)|p'\rangle\langle p'|), \tag{3.10}$$

where $\mu(p) = \langle p|\Pi_0|p\rangle$.

Notice that in fact Π_0 can be regarded as a state ($\Pi_0 \geq 0$, $\mathrm{Tr}(\Pi_0) = 1$.) As such it must satisfy standard Heisenberg-Roberstson uncertainty relation: $\Delta_{\Pi_0}x\Delta_{\Pi_0}p \geq \hbar/2$.

What is the uncertainty relation for the actually measured values of x and p? $\tilde{\Delta}x\tilde{\Delta}p$, where˜should remind us that this are distribution obtained using the joined measurement. Using the properties of the convolution we get:

$$\tilde{\Delta}^2x = \Delta^2_{\Pi_0}x + \Delta^2x, \tag{3.11}$$

where Δ^2x is the standard variance of sharp position measurement on ρ. We can write analogous formula for for the momentum distribution. Clearly the final measurement distribution is broadened compared to sharp position measurements. Finally we can write

$$\tilde{\Delta}^2x\tilde{\Delta}^2p = (\Delta^2_{\Pi_0}x + \Delta^2x)(\Delta^2_{\Pi_0}p + \Delta^2p) \geq$$
$$\frac{\hbar^2}{4}(\Delta^2_{\Pi_0}x + \Delta^2x)\left(\frac{1}{\Delta^2_{\Pi_0}x} + \frac{1}{\Delta^2x}\right) \geq \hbar^2 \tag{3.12}$$

Figure 3.2: Model of joint position and momentum measurement

where we used the standard Heisenberg-Robertson uncertainty relations for both ρ and Π_0 states, and also the fact that $(a+b)(1/a+1/b) \geq 4$, for arbitrary $a, b \geq 0$. So finally, we get

$$\tilde{\Delta}x\tilde{\Delta}p \geq \hbar \tag{3.13}$$

and we see that the joint-measurement uncertainty relation differs by a factor of 2 from the standard Heisenberg-Robertson uncertainty relation. Intuitively, this is due to the fact that in final distribution the inherent uncertainty of state we measure is combined togather with the uncertainty of the state which is the building block of the measurement itself, and on which we effectively project the measure state.

Moreover, we can interpret, the widths of the smearing functions Δ_{Π_0}, $\Delta_{\Pi_0}p$ as precisions of the actual measurements, and therefore write $\delta x = \Delta_{\Pi_0}x$ and hence: $\delta x\delta p \geq \hbar/4$, where now we have a tradeoff between the precision of measuring x and the precision of measuring p expressed via the familiar uncertainty relation.

Example. In order to illustrate the above considerations consider the following model of joint position-momentum measurement. Consider a particle S travelling in one dimension, with which we associate position and momentum operators (dimensionless) $\hat{x}_S$, $\hat{p}_S$, satisfying $[\hat{x}_S, \hat{p}_S] = \mathrm{i}$. Initially the particle is in state $|\psi\rangle_S$. Consider a joint position and momentum measurement where particle S interacts with two "measuring devices" M_1 i M_2 through a unitary evolution:

$$|\Psi\rangle_{SM_1M_2} = U|\psi\rangle_S\otimes|0\rangle_{M_1,M_2}, \quad U = e^{-\mathrm{i}(\hat{x}_S\hat{p}_{M_1} - \hat{p}_S\hat{x}_{M_2})}, \tag{3.14}$$

where $|0\rangle_{M_1,M_2}$ is the initial state of the measuring devices. After the action of U, position

(x_{M_1}) and momentum (p_{M_2}) is measured of respectively systems M_1 and M_2 (these measurements commute!). As a result of measurement we obtain a certain joint probability distribution of measuring position and momentum $p(x,p)$ on state $|\psi\rangle_S$

We start by evolving $\hat{x}_1$, $\hat{p}_2$ in the Heisenberg picture:

$$\hat{x}_1^{\text{out}} = \hat{x}_s + \hat{x}_1 - \tfrac{1}{2}\hat{x}_2 \qquad (3.15)$$

$$\hat{p}_2^{\text{out}} = \hat{p}_s + \hat{p}_2 - \tfrac{1}{2}\hat{p}_1, \qquad (3.16)$$

where we have used $e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots$. We can define $\hat{x}_M = \hat{x}_1 - \frac{1}{2}\hat{x}_2$, $\hat{p}_M = \hat{p}_2 - \frac{1}{2}\hat{p}_1$, since $[\hat{x}_M, \hat{p}_M] = i\hbar$. We assume that ${}_{12}\langle 0|\hat{x}_M|0\rangle_{12} = 0$, ${}_{12}\langle 0|\hat{p}_M|0\rangle_{12} = 0$ as thanks to this $\langle \hat{x}_1^{\text{out}}\rangle = \langle x_S\rangle$, $\langle \hat{p}_2^{\text{out}}\rangle = \langle p_S\rangle$. As a result:

$$\Delta^2 x_1^{\text{out}} = \Delta^2 x_{|\psi\rangle} + \Delta^2_{|0\rangle_{12}} x_M \qquad (3.17)$$

$$\Delta^2 p_2^{\text{out}} = \Delta^2 p_{|\psi\rangle} + \Delta^2_{|0\rangle_{12}} p_M \qquad (3.18)$$

and finally, using the Heisenberg-Robertson uncertainty relations for $|\psi\rangle$ and $|0\rangle_{12}$ we get the uncertainty relation for joint measurement of x and p:

$$\tilde{\Delta}_x \tilde{\Delta} p = \Delta x_1^{\text{out}} \Delta p_2^{\text{out}} \geq \hbar. \qquad (3.19)$$

In order to saturate the above inequality we should choose a state $|0\rangle_{12}$ such that it minimizes the standard Heisenberg-Robertson uncertainty relation for the x_M and p_M quantities.