Cwiczenia 10

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15:25

1. Udowodnić że w oscylatorze harmonicznym $\langle E_{kin} \rangle = \langle E_{pot} \rangle$ w stanach $|n\rangle$

Na poprzednich ćwiczeniach obliczyliśmy:

$$\langle x^2 \rangle = \frac{1}{\alpha^2}\left(n+\frac{1}{2}\right) = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

$$\hat{E}_{pot} = \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\langle E_{pot} \rangle = \frac{1}{2} m\omega^2 \frac{1}{m\omega}\hbar\left(n+\frac{1}{2}\right) = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)$$

$$= \frac{1}{2} E$$

Czyli pozostałe to

$$\langle E_{kin} \rangle = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)$$

$$\langle \hat{p}^2 \rangle = m\hbar\omega\left(n+\frac{1}{2}\right)$$

$$\Delta x^2 \Delta p^2 = \hbar^2\left(n+\frac{1}{2}\right)^2$$

Gdybyśmy liczyli bezpośrednio

$$\langle p^2 \rangle = -A_n^2 \int dx\, \hbar^2 H_n(\alpha x) e^{-\frac{\alpha^2 x^2}{2}} \frac{\partial^2}{\partial x^2}\left(H_n(\alpha x) e^{-\frac{\alpha^2 x^2}{2}}\right) =$$

$$= -A_n^2 \alpha \hbar^2 \int d\xi\, H_n(\xi) e^{-\frac{\xi^2}{2}} \frac{\partial^2}{\partial \xi^2}\left(H_n(\xi) e^{-\frac{\xi^2}{2}}\right)$$

$$S(\xi,s) = \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n = e^{-s^2+2s\xi}$$

$$\int S(\xi,s) e^{-\frac{\xi^2}{2}} \frac{\partial^2}{\partial \xi^2}\left(S(\xi,t) e^{-\frac{\xi^2}{2}}\right) =$$

$$\int e^{-s^2+2s\xi-\frac{\xi^2}{2}} \frac{\partial^2}{\partial \xi^2} e^{-t^2+2t\xi-\frac{\xi^2}{2}} =$$

$$= \int d\eta\, e^{-s^2+2s\eta-\frac{\eta^2}{2}} \frac{\partial^2}{\partial\eta^2} e^{-t^2+2t\eta-\frac{\eta^2}{2}} =$$

$$= \int d\eta\, e^{-s^2+2s\eta-\frac{\eta^2}{2}} \cdot \frac{\partial}{\partial\eta}\cdot\left((2t-\eta)\, e^{-t^2+2t\eta-\frac{\eta^2}{2}}\right) =$$

$$= \int d\eta\, e^{-s^2+2s\eta-\frac{\eta^2}{2}}\left[(2t-\eta)^2 e^{-t^2+2t\eta-\frac{\eta^2}{2}} - e^{-t^2+2t\eta-\frac{\eta^2}{2}}\right] =$$

$$= \int d\eta\, e^{-(s^2+t^2)+2\eta(s+t)-\eta^2}\left[(2t-\eta)^2 - 1\right] =$$

$$= \int d\eta\, e^{-(\eta-(s+t))^2+2st}\left[(2t-\eta)^2-1\right] =$$

$$= e^{2st}\int d\eta\, e^{-\eta^2}\left[(t-s-\eta)^2-1\right] =$$

$$= e^{2st}\sqrt{\pi}\left[(t-s)^2-\frac{1}{2}\right] = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2st)^n}{n!}\left[(t-s)^2-\frac{1}{2}\right]$$

$$\doteq \sqrt{\pi}\cdot\left(-\frac{1}{2}2^n n! - 2^n n\, n!\right) = -\sqrt{\pi}\,2^n n!\left(n+\frac{1}{2}\right)$$

$$\langle p^2\rangle = A_n^2\,\alpha\cdot\hbar^2\sqrt{\pi}\,2^n n!\left(n+\frac{1}{2}\right) =$$

$$= \frac{\alpha^2}{\cancel{\sqrt{\pi}2^n n!}}\hbar^2\cancel{\sqrt{\pi}2^n n!}\left(n+\frac{1}{2}\right) = m\hbar\omega\left(n+\frac{1}{2}\right)$$

2. Stany koherentne

Operatory kreacji i anihilacji

$$\hat{a} = \frac{1}{\sqrt{2}}\left[\hat{x}\sqrt{\frac{m\omega}{\hbar}} + i\frac{\hat{p}}{\sqrt{\hbar m\omega}}\right]$$

$$\sqrt{\frac{m\omega}{\hbar}} = \sqrt{\frac{kg\cdot\frac{1}{s}}{kg\cdot\frac{m^2}{s}}}$$

$$\sqrt{\frac{1}{\dots}} = \sqrt{kg^2\cdot m^2 \dots}$$

$$a^{+} = \frac{1}{\sqrt{2}}\left[\hat{x}\sqrt{\frac{m\omega}{\hbar}} - i\frac{\hat{p}}{\sqrt{m\omega\hbar}}\right]$$

$$\sqrt{\hbar m\omega} = \sqrt{kg^2 \cdot \frac{m^2}{s^2} \cdot - kg\frac{m}{s}}$$

$$[a, a^{+}] = 1$$

$$\hat{x} = \frac{a + a^{+}}{\sqrt{\frac{2m\omega}{\hbar}}} \qquad \hat{p} = \frac{a - a^{+}}{i\sqrt{2}} \cdot \sqrt{m\omega\hbar}$$

$$\hat{H} = \hbar\omega\left(\frac{1}{2} + a^{+}a\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a^{+}a|n\rangle = n|n\rangle$$

Stan koherentny - stan własny op. anihilacji

b) $\hat{a}|z\rangle = z|z\rangle$

$$|z\rangle = \sum_n c_n |n\rangle$$ Szukamy c_n

$$\sum_n c_n \hat{a}|n\rangle = z\sum_n c_n \cdot |n\rangle$$

$$\sum_{n=1}^{\infty} c_n\sqrt{n}\,|n-1\rangle = z\sum_{n=0}^{\infty} c_n|n\rangle$$

$$\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle = \sum_{n=0}^{\infty} z c_n|n\rangle$$

$$c_{n+1}\sqrt{n+1} = z c_n$$

$$\frac{c_{n+1}}{c_n} = \frac{z}{\sqrt{n+1}}$$

$$\frac{c_n}{c_0} = \frac{z^n}{\sqrt{n!}} \qquad |z\rangle = c_0\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle$$

Normalizacja

$$\langle z|z\rangle = |c_0|^2 \sum_n \frac{|z|^{2n}}{n!} = |c_0|^2 e^{|z|^2} = 1$$

$$c_0 = e^{-\frac{|z|^2}{2}}$$

$$\boxed{|z\rangle = e^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle}$$

$$p_n = e^{-|z|^2} \frac{(|z|^2)^n}{n!}$$ statystyka Poissonowska

średnio $\sim$ $|z|^2$ kwantów średnia energia $\hbar\omega\left(|z|^2 + \frac{1}{2}\right)$

a) W repr. położeniowej $\xi = \underbrace{\sqrt{\frac{m\omega}{\hbar}}}_{\alpha} x$

$$\frac{1}{\sqrt{2}} \cdot \left(\hat{x}\sqrt{\frac{m\omega}{\hbar}} + \frac{i\hat{p}}{\sqrt{m\omega\hbar}} \right) \psi(x) = z\,\psi(x)$$

$$\frac{1}{\sqrt{2}} \left(\xi + \frac{d}{d\xi} \right) \psi(\xi) = z\,\psi(\xi)$$

$$\frac{d}{d\xi}\psi = \psi\left(\sqrt{2}z - \xi\right)$$

$$\frac{d}{d\xi}\ln\psi = \sqrt{2}z - \xi$$

$$\ln\psi = \sqrt{2}z\xi - \frac{\xi^2}{2} + C$$

$$\psi(\xi) = A\,e^{\sqrt{2}z\xi - \frac{\xi^2}{2} + C}$$

$$\psi(x) = A\,e^{\sqrt{2}z\alpha x - \frac{\alpha^2 x^2}{2} + C} =$$

$$\left\{ z = \frac{1}{\sqrt{2}} \cdot \left(x_0\alpha + \frac{ip_0}{\hbar\alpha} \right) \right.$$

$$= A\,e^{\alpha^2 x_0 x + \frac{ip_0}{\hbar}x - \frac{\alpha^2 x^2}{2} + C} =$$

$$= A\,e^{-\frac{\alpha^2}{2}\cdot(x-x_0)^2 + \frac{ip_0 x}{\hbar} + \underbrace{\frac{\alpha^2 x_0^2}{2} + C}_{\text{wciągamy do } A}}$$

Normalizacja

$$A^2 \int e^{-\alpha^2 (x-x_0)^2} = \sqrt{\frac{\pi}{\alpha^2}}\, A^2 = 1 \qquad A^2 = \frac{\alpha}{\sqrt{\pi}}$$

Stan koherentny:

$$|\psi(x)\rangle = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\, e^{-\frac{\alpha^2}{2}(x-x_0)^2 + \frac{ip_0 x}{\hbar}}$$

Zinterpretujmy x_0 i p_0:

$$\langle x \rangle = A^2 \int x\, e^{-\alpha^2 (x-x_0)^2} = x_0$$

$$\langle p \rangle = A^2 \int e^{-\frac{\alpha^2}{2}(x-x_0)^2 - \frac{ip_0 x}{\hbar}} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{-\frac{\alpha^2}{2}(x-x_0)^2 + \frac{ip_0 x}{\hbar}} =$$

$$= A^2 \frac{\hbar}{i} \int e^{-\alpha^2 (x-x_0)^2}\, \frac{i p_0}{\hbar} = p_0$$

Pokazać, że

- $|z\rangle = \hat{D}(z)|0\rangle$ gdzie $\hat{D}(z) = \exp(z a^\dagger - z^* a)$

Używamy Tw. BCH

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \qquad \text{jeśli } [A,[A,B]] = [B,[A,B]] = 0$$

$\hat{D}(z)|0\rangle =$

$$= e^{z a^\dagger} e^{-z^* a} e^{-\frac{1}{2}|z|^2} |0\rangle =$$

$$= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n a^{\dagger n}}{n!} |0\rangle =$$

$$= e^{-\frac{1}{2}|z|^2} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle = |z\rangle$$

- Pokazać, że $\hat{D}(z)$ jest unitarny.

$$\hat{D}^{+}(z)D(z) = e^{z^* a - z a^+} e^{z a^+ - z^* a} = \mathbb{1}$$

- $$\langle z | z' \rangle = e^{-\frac{1}{2}(|z|^2 + |z'|^2)} \sum_{n=0}^{\infty} \frac{(z^* z')^n}{n!} =$$

$$= e^{-\frac{1}{2}(z^* z + z'^* z' - 2 z^* z')}$$

$$|\langle z | z' \rangle|^2 = e^{-|z - z'|^2}$$

- Stany koherentne są zupełne

dowolny wektor można nie rozłożyć:

$$|\psi\rangle = \int dz \; c_z |z\rangle$$

wystarczy pokazać, że

$$\int dz \, |z\rangle\langle z| = a \cdot \mathbb{1}$$

$$\int dz \, |z\rangle \underbrace{\langle z|\psi\rangle}_{c_z \cdot a} = a |\psi\rangle$$

$$\int dz \sum_{n,m} \frac{z^n z^{*m}}{\sqrt{n!\,m!}} e^{-|z|^2} |n\rangle\langle m| =$$

$$= \int dr\, d\varphi \; r \sum_{n,m} \frac{r^{n+m} e^{i(n-m)\varphi}}{\sqrt{n!\,m!}} |n\rangle\langle m| \, e^{-r^2} =$$

$$= 2\pi \int dr \; r \sum_n \frac{r^{2n}}{n!} |n\rangle\langle n| \, e^{-r^2} =$$

$$= 2\pi \sum_n |n\rangle\langle n| \cdot \int_0^{\infty} dr \, \frac{r^{2n+1}}{n!} e^{-r^2} =$$

$$\overset{t = r^2}{=} \pi \sum_n |n\rangle\langle n| \underbrace{\int_0^{\infty} dt \, \frac{t^n}{n!} e^{-t}}_{1} = \pi \cdot \mathbb{1}$$

3. Ewolucja czasowa stanu koherentnego

Niech $|\psi(0)\rangle = e^{-\frac{g^2}{2}} \sum_{n=0}^{\infty} \frac{g^n}{\sqrt{n!}} |n\rangle \qquad g \in \mathbb{R}$

$$|n\rangle \longrightarrow e^{-iE_n t} |n\rangle$$

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right)$$

$$|g\rangle = |\psi(t)\rangle = e^{-\frac{g^2}{2}} \sum_{n=0}^{\infty} \frac{g^n e^{-i\hbar\omega n t - \frac{\hbar\omega}{2} t}}{\sqrt{n!}} |n\rangle$$

$$= e^{-\frac{g^2}{2}} \sum_{n=0}^{\infty} \frac{\left(g e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle e^{-\frac{i\omega}{2} t} =$$

$$= |g e^{-i\omega t}\rangle$$

Wciąż stan koherentny. Jakie średnie położenia i pędy:

$$g e^{-i\omega t} = \langle x(t)\rangle \sqrt{\frac{m\omega}{2\hbar}} + i \langle p(t)\rangle \cdot \frac{1}{\sqrt{2m\omega\hbar}}$$

$$\langle x(t)\rangle = \overbrace{g\sqrt{\frac{2\hbar}{m\omega}}}^{x(0)} \cos\omega t = x_0 \cos\omega t$$

$$\langle p(t)\rangle = -g\sqrt{2m\omega\hbar} \sin\omega t = -m\omega x_0 \sin\omega t$$

Oscylują jak klasyczne wielkości

Stany koherentne najbardziej klasyczne ze wszystkich stanów oscylatora