Cwiczenia 12

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10:36

1. Dwuwymiarowy oscylator harmoniczny

$$V = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_x x^2\right)\psi(x,y) + \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_y y^2\right)\psi(x,y) = E\psi(x,y)$$

Mamy separację zmiennych czyli możemy szukać rozwiązań w postaci:

$$\psi(x,y) = \psi(x)\cdot\varphi(y)$$

i dzielimy stronami:

$$\underbrace{\frac{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{1}{2}k_x x^2\right)\psi(x)\cancel{\varphi(y)}}{\psi(x)\cancel{\varphi(y)}}}_{=E_x} + \underbrace{\frac{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_y y^2\right)\cancel{\psi(x)}\varphi(y)}{\cancel{\psi(x)}\varphi(x)}}_{=E_y} = E$$

Mamy niezależne problemy w x i y

$$\psi(x) = \psi_{n_x}(x) = A_{n_x} H_{n_x}(\alpha x) e^{-\frac{\alpha^2 x^2}{2}} \quad n_x = 0, \dots, \infty$$

$$\varphi(y) = \psi_{n_y}(y) = A_{n_y} H_{n_y}(\alpha y) e^{-\frac{\alpha^2 y^2}{2}} \quad n_y = 0, \dots, \infty$$

$$E_{n_x} = \hbar\sqrt{\frac{k_x}{m}}\left(n_x + \frac{1}{2}\right)$$

$$E_{n_y} = \hbar\sqrt{\frac{k_y}{m}}\left(n_y + \frac{1}{2}\right)$$

Pełna f. falowa: numerujemy (n_x, n_y)

$$\psi_{(n_x,n_y)} = \psi_{n_x}(x)\cdot\psi_{n_y}(y)$$

$$\Psi_{(n_x,n_y)} = \Psi_{n_x}(x)\cdot\Psi_{n_y}(y)$$

$$E_{(n_x,n_y)} = \hbar\sqrt{\frac{k_x}{m}}\left(n_x+\frac{1}{2}\right) + \hbar\sqrt{\frac{k_y}{m}}\left(n_y+\frac{1}{2}\right)$$

Przypadek izotropowy $k_x = k_y = k$

$$E_{(n_x,n_y)} = \hbar\sqrt{\frac{k}{m}}\left(n_x+n_y+1\right)$$

Mamy degenerację, Energia zależy od sumy $n_x+n_y = n$.

E_n ma stopień degeneracji $n+1$

2. Oscylator 3D

$$E_{(n_x,n_y,n_z)} = \hbar\sqrt{\frac{k_x}{m}}\left(n_x+\frac{1}{2}\right)+\hbar\sqrt{\frac{k_y}{m}}\left(n_y+\frac{1}{2}\right)+\hbar\sqrt{\frac{k_z}{m}}\left(n_z+\frac{1}{2}\right)$$

Jeśli izotropowy: $k_x = k_y = k_z = k$

$$E = \hbar\sqrt{\frac{k}{m}}\left(n_x+n_y+n_z+\frac{3}{2}\right)$$

E_n, $n = n_x+n_y+n_z$ degeneracja:

n_x n_y n_z

$$\frac{n(n+1)}{2}+n+1 = \frac{(n+1)(n+2)}{2}$$

3. Studnia w 2D, 3D

4. Obliczamy L_x, L_y, L_z w bazie stanu

więcej $l=1$:

$$L_z|l,m\rangle = \hbar m\,|l,m\rangle$$

$$L_\pm = L_x \pm i L_y$$

$$L_+|l,m\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\;|l,m+1\rangle$$

$$L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\;|l,m-1\rangle$$

$$L_z = \hbar\begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}$$

$$L_x, L_y, L^2 = \ldots$$

. Dla $l=1$, zapis macierzy obrotu wokół osi z, y o kąt α w bazie $|l,m\rangle_z$

Ogólnie $U = e^{-\frac{i\vec{n}\vec{L}\alpha}{\hbar}}$ – obrót wokół osi $\vec{n}$ o kąt α

$$U_\alpha^{(z)} = e^{-\frac{iL_z\alpha}{\hbar}} = \begin{bmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{+i\alpha}\end{bmatrix}$$

$$L_z = \hbar\begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix},\quad L_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix},\quad L_y = \frac{i\hbar}{\sqrt{2}}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$

$$U_\alpha^{(y)} = e^{-\frac{iL_y\alpha}{\hbar}}$$

szukamy rozkładu własnego L_y:

$$\det\begin{bmatrix}-\lambda & -1 & 0\\ 1 & -\lambda & -1\\ 0 & 1 & -\lambda\end{bmatrix} = -\lambda(\lambda^2+1) - \lambda = -\lambda^3 - 2\lambda = 0$$

$$\lambda(\lambda^2+2)=0 \qquad \lambda_0=0 \quad \lambda_-=i\sqrt{2} \quad \lambda_+=-i\sqrt{2}$$

$$\lambda_0=0 \quad \lambda_-=-\hbar \quad \lambda_+=\hbar$$

λ_0:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=0 \qquad a=c \quad b=0$$

$$|1,0\rangle_y=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

λ_+:

$$\begin{bmatrix} -\hbar & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & -\hbar & -\frac{i\hbar}{\sqrt{2}} \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad -\hbar a-\frac{i\hbar}{\sqrt{2}}b=0 \qquad b=-\frac{a\sqrt{2}}{i}=ia\sqrt{2}$$

$$\frac{i\hbar}{\sqrt{2}}a-\hbar b-\frac{i\hbar}{\sqrt{2}}c=\frac{i\hbar}{\sqrt{2}}a-i\hbar\sqrt{2}a-\frac{i\hbar}{\sqrt{2}}c=0$$

$$-i\hbar a=i\hbar c \qquad c=-a$$

$$|1,+1\rangle_y=\frac{1}{2}\begin{bmatrix} 1 \\ i\sqrt{2} \\ -1 \end{bmatrix}$$

λ_-:

$$\begin{bmatrix} \hbar & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & \hbar & -\frac{i\hbar}{\sqrt{2}} \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$a-\frac{i}{\sqrt{2}}b=0 \qquad b=-i\sqrt{2}a$$

$$\frac{i}{\sqrt{2}}a+b-\frac{i}{\sqrt{2}}c=\frac{i}{\sqrt{2}}a-i\sqrt{2}a-\frac{i}{\sqrt{2}}c=0$$

$$-a-c=0 \qquad c=-a$$

$$|1,-1\rangle_y=\frac{1}{2}\begin{bmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{bmatrix}$$

$$e^{-\frac{iL_y\alpha}{\hbar}}=\sum_{m=-1}^{1}e^{-i\alpha m}\,|1,m\rangle_y\,{}_y\langle 1,m| =$$

$$= e^{i\alpha}\frac{1}{4}\begin{bmatrix}1\\-i\sqrt{2}\\-1\end{bmatrix}[1, i\sqrt{2}, -1] + 1\cdot\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}[1,0,1]$$

$$+ e^{-i\alpha}\frac{1}{4}\begin{bmatrix}1\\i\sqrt{2}\\-1\end{bmatrix}[1, -i\sqrt{2}, -1] =$$

$$= \frac{e^{i\alpha}}{4}\cdot\begin{bmatrix}1 & i\sqrt{2} & -1\\ -i\sqrt{2} & 2 & i\sqrt{2}\\ -1 & -i\sqrt{2} & 1\end{bmatrix} + \frac{1}{2}\cdot\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix} + \frac{e^{-i\alpha}}{4}\begin{bmatrix}1 & -i\sqrt{2} & -1\\ i\sqrt{2} & 2 & -i\sqrt{2}\\ -1 & i\sqrt{2} & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1+\cos\alpha & -\sqrt{2}\sin\alpha & 1-\cos\alpha\\ \sqrt{2}\sin\alpha & 2\cos\alpha & -\sqrt{2}\sin\alpha\\ 1-\cos\alpha & \sqrt{2}\sin\alpha & 1+\cos\alpha\end{pmatrix}$$

5.

- Zasada nieoznaczoności dla momentu pędu na stanach $|l,m\rangle$. Wyznaczyć $\langle L_i\rangle$, $\langle L_i^2\rangle$, $\langle L^2\rangle$

$$L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$$

$$L_z|l,m\rangle = \hbar m|l,m\rangle$$

$$L_+ = L_x + iL_y \qquad L_- = L_x - iL_y$$

$$[L_i, L_j] = i\varepsilon_{ijk}\hbar L_k$$

$$L_\pm|l,m\rangle = [l(l+1) - m(m\pm1)]^{\frac{1}{2}}\hbar|l, m\pm1\rangle$$

$$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$$

$$\langle L_x\rangle = \langle L_y\rangle = 0 \qquad \langle L_z\rangle = m$$

$$\langle L_x^2\rangle = \frac{1}{4}\cdot\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right) =$$

$$= \frac{1}{4} \cdot \Big[\big(j(j+1) - m(m-1) \big)^{\frac{1}{2}} \cdot \big[j(j+1) - (m-1)m \big]^{\frac{1}{2}} \hbar^2 +$$

$$+ \big(j(j+1) - m(m+1) \big)^{\frac{1}{2}} \big[j(j+1) - (m+1)m \big] \Big) \hbar^2$$

$$= \frac{1}{4} \cdot \Big[2j(j+1) - m(m-1+m+1) \Big) =$$

$$= \frac{1}{2} \big[j(j+1) - m^2 \big] \hbar^2$$

$$\langle L_y^2 \rangle = \frac{1}{2} \big[l(l+1) - m^2 \big) \hbar^2$$

$$\langle L_z^2 \rangle = m^2 \hbar^2$$

$$\langle L_x^2 + L_y^2 + L_z^2 \rangle = l(l+1)$$

$$\Delta L_x \cdot \Delta L_y \geq \frac{1}{2} \hbar \, |\langle L_z \rangle|$$

$$\cancel{\frac{1}{2}} \big(l(l+1) - m^2 \big) \hbar^2 \geq \frac{1}{2} \cancel{\hbar} \cdot \hbar |m|$$

$$m = l \qquad l^2 + l - l^2 \underset{=}{\geq} l$$

dla pozostałych m nie uzyskamy

6. $\hat{\vec{L}} = \hat{r} \times \vec{p}$

$$L_i = \varepsilon_{ijk} \, x_j \, p_k =$$

$$= \varepsilon_{ijk} \frac{a_j + a_j^+}{\sqrt{2}} \sqrt{\frac{\hbar}{m\omega}} \cdot \frac{a_k - a_k^+}{i\sqrt{2}} \sqrt{\hbar m \omega} =$$

$$= \frac{\hbar}{2i} \varepsilon_{ijk} (a_j + a_j^+)(a_k - a_k^+) =$$

$$= \frac{\hbar}{2i} \varepsilon_{ijk} \big(a_j^+ a_k - a_j a_k^+ \big) =$$

$$= \frac{\hbar}{2i} \varepsilon_{ijk} (a_j^{+} a_k - a_j a_k^{+})$$

$$= \frac{\hbar}{i} \varepsilon_{ijk} a_j^{+} a_k$$

$$\langle n_1, n_2, n_3 | L_i | n_1, n_2, n_3 \rangle = 0$$

Weźmy $L_3 = \frac{\hbar}{i} \left(a_1^{+} a_2 - a_2^{+} a_1 \right)$

Szukamy st. własnych a następnie energii

$$\frac{1}{\sqrt{2}} \left(|1,0,0\rangle + i|0,1,0\rangle \right) = |\psi^{(1)}\rangle$$

$$L_3 |\psi\rangle = \frac{1}{\sqrt{2}} \frac{\hbar}{i} \left(i|1,0,0\rangle - |0,1,0\rangle \right) =$$

$$= \hbar \left(|1,0,0\rangle + i|0,1,0\rangle \right) = \hbar |\psi\rangle$$

$$|\psi^{(-1)}\rangle = \frac{1}{\sqrt{2}} \left(|1,0,0\rangle - i|0,1,0\rangle \right)$$

$$L_3 |\psi^{(-1)}\rangle = \frac{1}{\sqrt{2}} \frac{\hbar}{i} \left(-|0,1,0\rangle - i|1,0,0\rangle \right) = -\hbar |\psi^{(-1)}\rangle$$

$$|\psi^{(0)}_{l=0}\rangle = |0,0,0\rangle \qquad |\psi^{(0)}_{l=1}\rangle = |0,0,1\rangle$$

$$L^2 |\psi^{(i)}\rangle = ?$$

8. Rotator asymetryczny $I_x = I_y \neq I_z$

$$H = \frac{L_x^2 + L_y^2}{2I_x} + \frac{L_z^2}{2I_z}$$

Znaleźć f. własne i energie własne

$$H = \frac{L_x^2 + L_y^2 + L_z^2}{2I_x} + \frac{L_z^2}{2} \left(\frac{1}{I_z} - \frac{1}{I_x} \right) =$$

$$= \frac{L^2}{2I_x} + \frac{L_z^2}{2}\left(\frac{1}{I_z} - \frac{1}{I_x}\right)$$

Możemy wziąć takie same stany własne jak dla symetrycznego: $|l, m\rangle$

$$H|l,m\rangle = \hbar^2\frac{l(l+1)}{2I_x} + \frac{m^2\hbar^2}{2}\left(\frac{1}{I_z} - \frac{1}{I_x}\right) =$$

$$= \underbrace{\frac{\hbar^2}{2I_x}\cdot\left[\, l(l+1) + m^2\left(\frac{I_x}{I_z} - 1\right)\right]}_{E_{l,m}}|l,m\rangle$$

podwójna degeneracja $\pm m$

Stan $|l=1, m=0\rangle$

Jakie jest prawd., że mierząc L_x wyjdzie nam $\hbar$, po czasie t: . .

Musimy obliczyć $a = \langle l=1, m_x=1 | l=1, m_z=0\rangle =$

$$\left\{\; L_\pm |l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)}\,\hbar\,|l, m\pm 1\rangle\right.$$

$$L_x = \frac{L_+ + L_-}{2} = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

Znajdujemy wekt. własne $\lambda = \hbar$:

$$\begin{bmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} c_+ \\ c_0 \\ c_- \end{bmatrix} = 0$$

$$-2c_+ + \sqrt{2}\,c_0 = 0 \qquad c_+ = \frac{1}{\sqrt{2}}c_0$$

$$\sqrt{2}\, c_+ - 2c_0 + \sqrt{2}\, c_- = 0$$

$$\sqrt{2}\, c_- = 2c_0 - c_0 = c_0 \qquad c_- = \frac{1}{\sqrt{2}} c_0$$

$$|1, m_x = 1\rangle = c_0 \left(\frac{1}{\sqrt{2}} |1,-1\rangle + |1,0\rangle + \frac{1}{\sqrt{2}} |1,1\rangle \right) =$$

$$= \frac{1}{2} |1,-1\rangle + \frac{1}{\sqrt{2}} |1,0\rangle + \frac{1}{2} |1,1\rangle$$

czyli $\langle 1, m_x = 1 | 1, m_z = 0 \rangle = \frac{1}{\sqrt{2}} \qquad p = \frac{1}{2}$