Cwiczenia 2

20 lutego 2014
15:36

Przydatne całki

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-tx^2} = -\frac{d}{dt}\int^{+\infty} e^{-tx^2}\Big|_{t=1} =$$

$$= -\frac{d}{dt}\sqrt{\frac{\pi}{t}} = \frac{1}{2}\sqrt{\pi}$$

$$\int x^{2n} e^{-x^2} = \frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2^n}(1\cdot 3\cdots 2n-1)$$

$$\int x^{2n+1} e^{-x^2} = 0$$

Transformata Fouriera

$\psi(x)$ – funkcja na $\mathbb{R}$

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(x)\, e^{ikx}\, dx \quad \text{– trans Fouriera}$$

My w mechanice kwantowej będziemy używać:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\psi(x)\, e^{-\frac{ipx}{\hbar}}\, dx$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\tilde{\psi}(p)\, e^{\frac{ipx}{\hbar}}\, dp \quad \text{– transf odwrotna}$$

Sprawdźmy:

$$\psi(x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dx\, dp\; \psi(x)\, e^{\frac{ip(x-x')}{\hbar}} =$$

$$= \frac{1}{2\pi\hbar}\int dx\, \psi(x) \int dp\, e^{\frac{ip(x-x')}{\hbar}}\, dp$$

$\Big(\int^{+\infty} e^{ikx} \ldots = \delta(x) \ldots$

$$\int_{-\infty}^{+\infty} e^{ikx}\, dk = 2\pi\,\delta(x) \qquad \text{dowód:}$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} e^{ikx}\, e^{-\varepsilon k^2}\, dk = \int_{-\infty}^{+\infty} e^{-\varepsilon\left(k-\frac{ix}{2\varepsilon}\right)^2 - \frac{x^2}{4\varepsilon}}\, dk =$$

$$= \sqrt{\frac{\pi}{\varepsilon}}\; e^{-\frac{x^2}{4\varepsilon}} = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \sqrt{\frac{\pi}{\varepsilon}}\; e^{-\frac{x^2}{4\varepsilon}}\, dx = \sqrt{\frac{\pi}{\varepsilon}}\; 2\sqrt{\pi\varepsilon} = 2\pi \qquad \text{OK.}$$

$$= \frac{1}{2\pi\hbar}\int dx'\, \psi(x')\;\delta\left(\frac{x-x'}{\hbar}\right) = \psi(x) \qquad \text{ok}$$

Tw. Parsevala

$$\int |\psi(x)|^2 dx = \int |\tilde{\psi}(p)|^2 dp$$

interpretacja
zachowanie
prawdopodobieństwa

dowód:

$$\frac{1}{(2\pi\hbar)}\int dx \int dp\; \tilde{\psi}(p)^{*}\, e^{-\frac{ipx}{\hbar}}\, dp'\; \tilde{\psi}(p')\, e^{\frac{ip'x}{\hbar}} =$$

$$= \frac{1}{\hbar}\int dp\, dp'\;\delta\left(\frac{p'-p}{\hbar}\right)\tilde{\psi}^{*}(p)\,\tilde{\psi}(p') =$$

$$= \int |\tilde{\psi}(p)|^2 dp \qquad \text{ok}$$

Fakty:

$$\widetilde{\left(\frac{\hbar}{i}\frac{d\psi}{dx}\right)}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int \frac{\hbar}{i}\,\frac{d}{dx}\psi(x)\; e^{-\frac{ipx}{\hbar}} = p\,\tilde{\psi}(p)$$

$$\widetilde{x\psi} = -\frac{\hbar}{i}\frac{d}{dp}\tilde{\psi}(p)$$

$$\widetilde{x\psi} = -\frac{\hbar}{i}\frac{d}{dp}\tilde{\psi}(p)$$

Trans Fouriera f. Gaussa

$$\psi(x) = N e^{-\frac{x^2}{4\delta_x^2}} \qquad \text{chcemy aby } \int |\psi(x)|^2 = 1$$

$$|\psi(x)|^2 = N^2 e^{-\frac{x^2}{2\delta_x^2}} \qquad N = \frac{1}{(2\pi\delta_x^2)^{\frac{1}{4}}}$$

$$\int x|\psi(x)|^2 = 0, \quad \int x^2 |\psi(x)|^2 = \delta_x^2 \quad \text{ok.}$$

$$\tilde{\psi}(p) = \frac{N}{\sqrt{2\pi\hbar}} \int e^{-\frac{x^2}{4\delta_x^2}} e^{-\frac{ipx}{\hbar}} =$$

$$= \frac{N}{\sqrt{2\pi\hbar}} \int e^{-\frac{1}{4\delta_x^2}\left(x + \frac{2ip\delta_x^2}{\hbar}\right)^2 - \frac{p^2\delta_x^2}{\hbar^2}} =$$

$$= \sqrt{\frac{2}{\hbar}} N \sigma_x \, e^{-\frac{p^2\delta_x^2}{\hbar^2}}$$

$$|\tilde{\psi}(p)|^2 = \frac{2\delta_x^2}{\hbar\sqrt{2\pi\delta_x^2}} e^{-\frac{p^2}{2\frac{\hbar^2}{4\delta_x^2}}} = \frac{1}{\sqrt{2\pi \underbrace{\frac{\hbar^2}{4\delta_x^2}}_{\delta_p^2}}} e^{-\frac{p^2}{2\cdot\frac{\hbar^2}{4\delta_x^2}}}$$

$$= \frac{1}{\sqrt{2\pi\delta_p^2}} e^{-\frac{p^2}{2\delta_p^2}}$$

$$\delta_x^2 \delta_p^2 = \frac{\hbar^2}{4}$$

przypominr zasada nieozn. Heisenb. i studnie

Ogólnie

$$\phi(\lambda) = x\psi(x) + i\lambda\,\hbar\frac{d}{dx}\psi(x) \qquad , \lambda \text{- liczba rzeczywista}$$

$$f(x) = x\,\psi(x) \quad \ldots \quad \frac{\hbar}{i}\frac{d}{dx} \ldots$$

$$\int f^*(x)\, f(x) \geq 0$$

$$\int x^2 |\psi(x)|^2 + \lambda^2 \hbar^2 \left|\frac{d}{dx}\psi(x)\right|^2 dx$$

$$+ \lambda x \psi^*(x)\, \hbar \frac{d}{dx}\psi(x) - \lambda x \psi(x)\, \hbar \frac{d}{dx}\psi^*(x) =$$

$$= \langle x^2 \rangle + \lambda^2 \underbrace{\int p^2 \tilde{\psi}(p)}_{\langle p^2 \rangle} + \int \lambda \hbar\, |\psi(x)|^2 dx$$

$$\left\{ \int |p\,\tilde{\psi}(p)|^2 dp = \int \left|\frac{\hbar}{i}\frac{d}{dx}\psi(x)\right|^2 \right.$$

$$= \langle x^2 \rangle + \lambda^2 \langle p^2 \rangle + \lambda \hbar \geq 0$$

dla każdego λ

$$\Delta = \hbar^2 - 4\langle x^2 \rangle \langle p^2 \rangle \leq 0$$

$$\langle x^2 \rangle \langle p^2 \rangle \geq \frac{\hbar^2}{4}$$

Moglibyśmy zacząć od

$$f(x) = (x - \langle x \rangle)\psi(x) + i\lambda \left(\frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle\right)\psi(x)$$

$$\vdots$$

$$\Delta^2 x\, \Delta^2 p \geq \frac{\hbar^2}{4} \qquad \Delta^2_x = \langle (x - \langle x \rangle)^2 \rangle$$

Widać, że stany Gaussowskie wysycają zasadę nieoznaczoności Heisenberga