Cwiczenia 3

24 lutego 2014
10:01

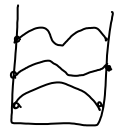
# 1. Nieskończona studnia potencjału


a) Stany własne

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}$$

• $E \geq 0$:

$$\psi(x) = A\sin kx + B\cos kx \qquad k = \frac{\sqrt{2mE}}{\hbar}$$



$$\psi(0) = \psi(a) = 0$$

$$\psi(0) = 0 \Rightarrow B = 0$$

$$\psi(a) = 0 \Rightarrow \frac{\sqrt{2mE}}{\hbar}\cdot a = n\cdot\pi \qquad n = 1\ldots\infty$$

$$\frac{\sqrt{2mE}}{\hbar}a = n\pi \qquad \underbrace{E_n = \overbrace{\frac{\hbar^2\pi^2}{2ma^2}}^{E_1} n^2}$$

$$k_n = \frac{\pi}{a}n$$

$$\psi_n = A_n \sin\left(\frac{\pi}{a}nx\right) ; \qquad \int_0^a dx\,(A_n)^2\sin^2\left(\frac{\pi}{a}nx\right) = \frac{a}{2}|A_n|^2$$

$$\psi_n = \sqrt{\frac{2}{a}}\cdot\sin\left(\frac{\pi}{a}nx\right) \qquad A_n = \sqrt{\frac{2}{a}}$$

Prawdopod. znalezienia cząstki w przedziale $\left[0, \frac{a}{4}\right]$:

$$P = \frac{2}{a}\int_0^{a/4}\sin^2\left(\frac{n\pi x}{a}\right) = \frac{1}{a}\int_0^{a/4}\left(1 - \cos\frac{2n\pi x}{a}\right)dx$$

$$= \frac{1}{4} - \frac{1}{a}\,\frac{a}{2\pi n}\sin\frac{2\pi n a}{4a} = \frac{1}{4} - \frac{1}{2\pi n}\sin\left(\frac{\pi n}{2}\right) \overset{n\to\infty}{\approx} \frac{1}{4}$$

b) Dla stanów własnych znaleźć $\langle x\rangle, \langle p\rangle, \delta^2 x, \delta^2 p$ i sprawdzić zasadę nieoznaczoności

$$\psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}nx\right)$$

$$\langle x\rangle = \frac{2}{a}\int_0^a x\sin^2\left(\frac{\pi}{a}nx\right) = \frac{2}{a}\int_0^a x\,\frac{1}{2}\left(1 - \cos\frac{2\pi nx}{a}\right) =$$

$$= \frac{a}{2} - \frac{1}{a}\int_0^a x\cos\frac{2\pi nx}{a}dx = \frac{a}{2}$$

$a$

$$= \frac{a}{2} - \frac{1}{a}\int_0^a x \cos\frac{2\pi n x}{a}dx = \frac{a}{2}$$

$$\langle x^2\rangle = \frac{1}{a}\int_0^a x^2 \left(1-\cos\frac{2\pi n x}{a}\right) = \frac{a^2}{3} - \frac{1}{a}\int_0^a x^2 \cos\frac{2\pi n x}{a} =$$

$$= \frac{a^2}{3} - \frac{1}{a}\left( x^2 \frac{a}{2\pi n} \sin\frac{2\pi n x}{a}\Big|_0^a - \int_0^a 2x \frac{a}{2\pi n}\sin\frac{2\pi n x}{a}\right) =$$

$$= \frac{a^2}{3} + \frac{1}{\pi n}\int x \sin\frac{2\pi n x}{a} = \frac{a^2}{3} - \frac{1}{\pi n} x \frac{a}{2\pi n}\cos\frac{2\pi n x}{a}\Big|_0^a =$$

$$= \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2} = a^2\left(\frac{1}{3} - \frac{1}{2\pi^2 n^2}\right)$$

$$\delta^2 x = \frac{a^2}{2}\left(\frac{1}{6} - \frac{1}{\pi^2 n^2}\right)$$

$$\langle p\rangle = \frac{2}{a}\int \sin\frac{\pi n x}{a} \frac{\hbar}{i}\frac{d}{dx}\left(\sin\frac{\pi n x}{a}\right) =$$

$$= \frac{2\hbar}{ia}\frac{\pi n}{a}\int \sin\frac{\pi n x}{a}\cos\frac{\pi n x}{a} = 0$$

$$\langle p^2\rangle = -\frac{2}{a}\int \sin\frac{\pi n x}{a} \hbar^2 \frac{d^2}{dx^2}\sin\left(\frac{\pi x n}{a}\right) =$$

$$= + \frac{2\hbar^2}{a}\frac{\pi^2 n^2}{a^2}\int_0^a \sin^2\frac{\pi n x}{a} = \frac{\pi^2\hbar^2 n^2}{a^2} = \delta^2 p$$

$$\delta^2 x\, \delta^2 p = \frac{\pi^2\hbar^2 n^2}{2}\left(\frac{1}{6} - \frac{1}{\pi^2 n^2}\right) = \frac{\hbar^2}{4}\cdot\left(\underbrace{\frac{\pi^2 n^2}{3}}_{>1} - 2\right) > \frac{\hbar^2}{4} \quad \text{ok}$$

gdyby liczyć w repr. pędowej:

$$\tilde{\Psi}_n(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_0^a \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} e^{-\frac{ipx}{\hbar}}dx =$$

$$= \sqrt{\frac{1}{a\pi\hbar}}\frac{1}{2i}\int_0^a e^{i\left(\frac{n\pi x}{a} - \frac{px}{\hbar}\right)} - e^{-i\left(\frac{n\pi x}{a} + \frac{px}{\hbar}\right)} dx$$

$$= \frac{1}{2i\sqrt{a\pi\hbar}}\left(\frac{1}{i\left(\frac{n\pi}{a} - \frac{p}{\hbar}\right)}\left[e^{i\left(\frac{n\pi}{a} - \frac{p}{\hbar}\right)a} - 1\right] + \frac{\left[e^{-i\left(\frac{n\pi}{a} + \frac{p}{\hbar}\right)a} - 1\right]}{i\left(\frac{n\pi}{a} + \frac{p}{\hbar}\right)}\right)$$

$$= -\frac{ia}{\sqrt{a\pi\hbar}}\left(\frac{\sin\left[\frac{1}{2}\left(n\pi - \frac{pa}{\hbar}\right)\right]}{n\pi - \frac{pa}{\hbar}} e^{i\frac{1}{2}\left(n\pi - \frac{pa}{\hbar}\right)} + \dots\right.$$

c) $\psi(x,0) = A \sin^5(\pi x/a)$

Jakie energie i z jakimi prawdopodobieństwami można zmierzyć

Ile jest $\langle E\rangle$.

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$$\sin^5(\pi x/a) = \left(\frac{e^{\frac{i\pi x}{a}} - e^{-\frac{i\pi x}{a}}}{2i}\right)^5 =$$

$$= \frac{1}{(2i)^5} \cdot \left( e^{\frac{5i\pi x}{a}} - e^{-\frac{5i\pi x}{a}} + 5 e^{\frac{i\pi x}{a}\cdot\left(\frac{4i\pi x}{a}\right)} - \binom{5}{2} e^{\frac{2i\pi x}{a} - \frac{3i\pi x}{a}} \right.$$

$$\left. + \binom{5}{3} e^{\frac{3i\pi x}{a} - \frac{2i\pi x}{a}} - 5 \cdot e^{\frac{4i\pi x}{a} - \frac{i\pi x}{a}} \right) =$$

$$= \frac{1}{16} \cdot \left( \sin\frac{5\pi x}{a} - 5 \sin\frac{3\pi x}{a} + 10 \sin\frac{\pi x}{a} \right)$$

unormowanie: $\frac{A^2}{256} \cdot \frac{a}{2}(1 + 25 + 100) = 1$ $\quad 2 \cdot \frac{512}{126} = \frac{256}{63}$ $\quad A = \frac{16}{\sqrt{63 a}}$

Jakie energie z jakimi p:

$E_1$ , $p_1 = \frac{100}{126}$

$E_3 = 9E_1$ $\quad p_3 = \frac{25}{126}$ $\qquad \langle E \rangle = \left(\frac{100}{126} + \frac{225}{126} + \frac{25}{126}\right) E_1 = \frac{350}{126} E_1 =$

$E_5 = 25E_1$ $\quad p_5 = \frac{1}{126}$ $\qquad = \frac{350}{126} \cdot \frac{\hbar^2 \pi^2}{2 m a^2} = \frac{175}{63} \frac{\hbar^2 \pi^2}{2ma^2}$

Obliczmy średnią energii z definicji:

$$\langle E \rangle = -\frac{\hbar^2}{2m} A^2 \int dx \left(\sin\frac{\pi x}{a}\right)^5 \cdot \frac{d}{dx} \cdot \left(5 \sin\left(\frac{\pi x}{a}\right)^4 \cdot \frac{\pi}{a} \cdot \cos\frac{\pi x}{a}\right) =$$

$$= -\frac{\hbar^2 A^2}{2m} \cdot \int \sin^5\frac{\pi x}{a} \cdot \left[ 5 \cdot 4 \cdot \left(\sin\frac{\pi x}{a}\right)^3 \left(\frac{\pi}{a}\right)^2 \cos^2\frac{\pi x}{a} - 5 \left(\sin\frac{\pi x}{a}\right)^5 \left(\frac{\pi}{a}\right)^2 \right] =$$

$$= -\frac{\hbar^2 A^2}{2m} \cdot \left(\frac{\pi}{a}\right)^2 \cdot \int 20 \sin^8\frac{\pi x}{a} \cdot \cos^2\frac{\pi x}{a} - 5 \sin^{10}\frac{\pi x}{a} =$$

$$= -\frac{\hbar^2 A^2}{2m} \left(\frac{\pi}{a}\right)^2 \frac{a}{\pi} \int_0^{\pi} \underbrace{20 \sin^8 t}_{20 \cdot \frac{35}{128}\pi} \underbrace{- 25 \sin^{10} t}_{25 \cdot \frac{63}{256}\pi} =$$

$$= \frac{\hbar^2 \pi^2}{2 m a^2} \frac{256}{63} \cdot \left( \frac{40 \cdot 35 - 25 \cdot 63}{256} \right) = \frac{\hbar^2 \pi^2}{2ma^2} \left( \frac{175}{63} \right) \quad \text{ok.}$$

Ewolucja czasowa: – stan o określonej energii nie ewoluuje fizycznie !

d) $\psi(x,0) = A x (a - x)$

unormowanie $A^2 \int_0^a x^2 (a-x)^2 \, dx = 1$ $\quad A^2 \cdot \int x^2 a^2 - 2ax^3 + x^4 =$

unormowane $A^2\int_0^a x^2(a-x)^2\,dx = 1 \qquad A^2\cdot\int x^2a^2 - 2ax^3 + x^4 =$

$$= A^2\cdot\left(\frac{a^5}{3} - 2a\frac{a^4}{4} + \frac{a^5}{5}\right) = A^2\cdot a^5\cdot\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) = \left(\frac{10-15+6}{30}\right) = \frac{1}{30}A^2a^5$$

$$A=\sqrt{\frac{30}{a^5}}$$

$$\psi(x,0) = \sum_n c_n\psi_n \qquad c_n = \langle\psi_n|\psi\rangle = \int_0^a \psi_n^*(x)\psi(x)\,dx =$$

$$= A\sqrt{\frac{2}{a}}\cdot\int_0^a x(a-x)\sin\left(\frac{\pi n x}{a}\right) = A\sqrt{\frac{2}{a}}\left(a\int x\sin\frac{\pi n x}{a} - \int x^2\sin\frac{\pi n x}{a}\right) =$$

$$= A\sqrt{\frac{2}{a}}\left[a\left(\frac{a}{\pi n}\right)^2\int_0^{\pi n} t\sin t - \left(\frac{a}{\pi n}\right)^3\int_0^{\pi n} t^2\sin t\right] =$$

$$\left\{\int t\sin t = -t\cos t + \int\cos t = -t\cos t + \sin t\right.$$

$$\left\{\int t^2\sin t = -t^2\cos t + \int 2t\cos t = -t^2\cos t + 2t\sin t + 2\cos t\right.$$

$$= A\sqrt{\frac{2}{a}}\cdot\left[a\left(\frac{a}{\pi n}\right)^2\cdot\left(-\pi n\cdot(-1)^n\right) - \left(\frac{a}{\pi n}\right)^3\left(-(\pi n)^2\cdot(-1)^n + 2(-1)^n - 2\right)\right] =$$

$$= A\sqrt{\frac{2}{a}}\cdot\left[-\frac{a^3}{\cancel{(\pi n)}}(-1)^n + \frac{a^3}{\cancel{(\pi n)}}\cdot(-1)^n - \left(\frac{a}{\pi n}\right)^3 2\left[(-1)^n-1\right]\right] =$$

$$= \frac{2\sqrt{60}}{\pi^3 n^3}\left(1-(-1)^n\right) = \frac{4\sqrt{60}}{\pi^3 n^3} \quad n - \text{nieparzyste}$$

$$(c_n)^2 = \frac{960}{\pi^6}\cdot\frac{1}{n^6} = \quad n - \text{nieparzyste.}$$

$$= 0{,}998\cdot\frac{1}{n^6} \quad \text{praktycznie tylko } n=1.$$

$$\langle E\rangle \simeq E_1\cdot\left(\sum_{n=1,3,5}\frac{960}{\pi^6 n^6}\cdot n^2\right) = \frac{10}{\pi^2}E_1$$

Bezpośrednio: $\qquad \left\{\sum_{n=1,3,5}\frac{1}{n^4} = \frac{\pi^4}{96}\right.$

$$\langle E\rangle = -\frac{\hbar^2}{2m}A^2\cdot\int dx\; x(a-x)\frac{d^2}{dx^2}x(a-x) =$$

$$= -\frac{\hbar^2}{2m}A^2\int dx\; x(a-x)\cdot\left\{(-2) + \delta(x) + \delta(x-a)\right\} = \left(\frac{\hbar^2}{m}A^2\int_0^a (xa-x^2) =\right.$$

$$= \frac{\hbar^2}{m}\frac{30}{a^5}\cdot\left(\frac{a^3}{2}-\frac{a^3}{3}\right) = \frac{\hbar^2\, 5}{m a^2} = \frac{\hbar^2\pi^2}{2ma^2}\cdot\underline{\frac{10}{\pi^2}}$$

Odchylenie standardowe $E$:

$$\langle(E-\langle E\rangle)^2\rangle = \langle E^2\rangle - \langle E\rangle^2$$

$$\langle E^2\rangle = A^2 \int dx\, x(a-x)\, \frac{\hbar^4}{4m^2}\frac{d^4}{dx^4}\, x(a-x) =$$

$$= A^2 \int x(a-x)\frac{\hbar^2}{4m^2}\cdot\frac{d^2}{dx^2}\cdot\left(-2+\delta(x)-\delta(x-a)\right) =$$

$$= A^2\cdot\frac{\hbar^4}{4m^2}\int x(a-x)\,\frac{d^2}{dx^2}\left(\delta(x)-\delta(x-a)\right) = ?$$

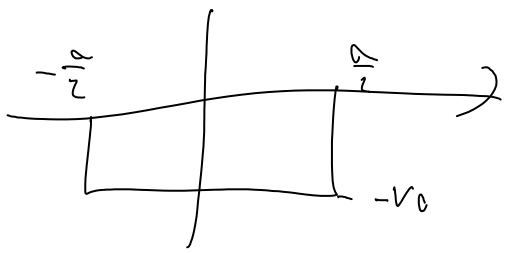
$$\langle E^2\rangle = E_1^2\sum_{n=1,\dots}\frac{960}{\pi^6 n^6}\cdot n^4 = -$$

$$= \frac{960}{\pi^6}\cdot E_1^2\cdot \underbrace{\sum_{n=1,3,\dots}\frac{1}{n^2}}_{\frac{\pi^2}{8}} = \frac{E_1^2}{\pi^4}\,120 = \frac{\hbar^4\, 30}{m^2 a^4}$$

$$\langle E^2\rangle = \frac{30}{a^5}\cdot\frac{\hbar^4}{4m^2}\int\left(-2+\delta(x)-\delta(x-a)\right)\left(-2+\delta(x)+\delta(x-a)\right)$$

$$\psi(x) = A\left(\frac{a}{2}-|x|\right)$$

## 2.5 Skończona studnia potencjału




$-V_0 \le E \le 0$

$$\psi_I = A e^{\alpha x} + \cancel{B e^{-\alpha x}} \qquad \alpha^2 = \frac{2m|E|}{\hbar^2} = -\frac{2mE}{\hbar^2}$$

$B = 0$

$$\psi_{II} = C e^{i\beta x} + D e^{-i\beta x} \qquad \beta^2 = \frac{2m(E+V_0)}{\hbar^2}$$

$$\psi_{III} = \cancel{E e^{\alpha x}} + F e^{-\alpha x}$$

$$\begin{cases} A e^{-\frac{a}{2}\alpha} = C e^{-\frac{ia}{2}\beta} + D e^{\frac{ia}{2}\beta} \\ \alpha A e^{-\frac{a}{2}\alpha} = i\beta C e^{-\frac{ia}{2}\beta} - i\beta D e^{\frac{ia}{2}\beta} \\ C e^{\frac{ia}{2}\beta} + D e^{-\frac{ia}{2}\beta} = F e^{-\frac{a}{2}\alpha} \\ i\beta C e^{i\beta\frac{a}{2}} - i\beta D e^{-i\beta\frac{a}{2}} = -\alpha F e^{-\frac{a}{2}\alpha} \end{cases}$$

$$A = C e^{\frac{a}{2}(\alpha - i\beta)} + D e^{\frac{a}{2}(\alpha + i\beta)}$$

$$\alpha \cdot C e^{-\frac{a}{2} i\beta} + \alpha D e^{\frac{a}{2} i\beta} = i\beta C e^{-\frac{ia}{2}\beta} - i\beta D e^{\frac{ia}{2}\beta}$$

$$D\left(\alpha e^{\frac{a}{2} i\beta} + i\beta e^{\frac{ia}{2}\beta}\right) = C\left(i\beta e^{-\frac{ia}{2}\beta} - \alpha e^{-\frac{a}{2} i\beta}\right) \quad ,$$

$$D = \frac{C e^{-ia\beta}\left(i\beta - \alpha\right)}{\alpha + i\beta} = C\, e^{-ia\beta} \frac{i\beta - \alpha}{i\beta + \alpha}$$

$$A\, e^{-\frac{a}{2}\alpha} = C\left(e^{-\frac{ia}{2}\beta} + e^{-\frac{ia}{2}\beta} \frac{i\beta - \alpha}{i\beta + \alpha}\right)$$

$$C = A\, e^{\frac{a}{2}\cdot(i\beta - \alpha)} \frac{i\beta + \alpha}{2i\beta}$$

$$D = A\, e^{\frac{a}{2}\cdot(-i\beta - \alpha)} \frac{i\beta - \alpha}{2i\beta}$$

$$C\, e^{\frac{a}{2}(i\beta + \cancel{\alpha})} + D\, e^{\frac{a}{2}(\cancel{\alpha} - i\beta)} =$$

$$= -\frac{1}{\alpha} i\beta\left(C e^{\frac{a}{2}(\cancel{\alpha} + i\beta)} - D e^{\frac{a}{2}(\cancel{\alpha} - i\beta)}\right)$$

$$C\, e^{\frac{a}{2} i\beta} \cdot \left(1 + \frac{i\beta}{\alpha}\right) = D e^{-\frac{ia}{2}\beta}\left(\frac{i\beta}{\alpha} - 1\right)$$

$$D = e^{ia\beta} \frac{\alpha + i\beta}{i\beta - \alpha} C$$

$$F = C e^{\frac{a}{2}(\alpha + i\beta)} + D e^{\frac{a}{2}\cdot(\alpha - i\beta)} =$$

$$= A\left(e^{ia\beta} \frac{i\beta + \alpha}{2i\beta} + e^{-ia\beta} \frac{i\beta - \alpha}{2i\beta}\right)$$

Czyli w porównaniu z wcześniejszym wynikiem

$$e^{ia\beta} \frac{\alpha + i\beta}{i\beta - \alpha} = e^{-ia\beta} \frac{i\beta - \alpha}{i\beta + \alpha}$$

$$e^{2ia\beta} = \left(\frac{i\beta - \alpha}{\alpha + i\beta}\right)^2$$

$$e^{ia\beta} = \pm \frac{\alpha - i\beta}{\alpha + i\beta} \qquad \alpha + i\beta = e^{i\varphi}$$

$$e^{ia\beta} = \pm e^{-2i\varphi}$$

• $a\beta = -2\,\mathrm{arctg}\frac{\beta}{\alpha} + 2k\pi$   (+)

$-\mathrm{tg}\frac{a\beta}{2} = \frac{\beta}{\alpha}$   $\boxed{\beta\,\mathrm{ctg}\frac{a\beta}{2} = -\alpha}$

$(C = -D)$

stany asymetryczne $\psi_{II} = C\,2i\sin\beta x$

• $e^{ia\beta} = e^{-2i\varphi+\pi}$   (−)

$ab = -2\,\mathrm{arctg}\frac{\beta}{\alpha} + \pi$

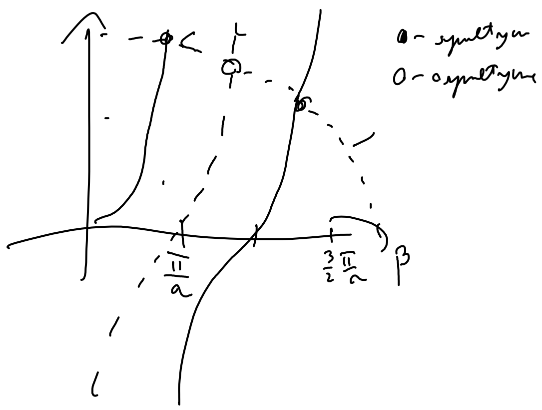
$-\frac{ab}{2} - \frac{\pi}{2} = \frac{\beta}{\alpha}$   $\frac{\beta}{\alpha} = -\mathrm{tg}\frac{ab}{2} + \frac{\pi}{2} = \mathrm{ctg}\frac{a\beta}{2}$

$\boxed{\beta\,\mathrm{tg}\frac{a\beta}{2} = \alpha}$   $(C = D)$

stany symetryczne $\psi_{II} = C\,2\cos\beta x$

$\beta = \sqrt{\frac{2mV_0}{\hbar^2} - \alpha^2}$   $\alpha = \sqrt{\frac{2mV_0}{\hbar^2} - \beta^2}$

$\beta\,\mathrm{tg}\frac{a\beta}{2} = \sqrt{\frac{2mV_0}{\hbar^2} - \beta^2}$, $\frac{2mV_0}{\hbar^2} = R^2$

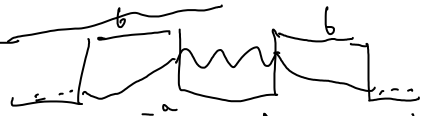



$\beta_n$ - niewiadoma

Może być tylko jeden dla płytkich potencjałów

Jeśli $\sqrt{\frac{2mV_0}{\hbar^2}} < \frac{\pi}{a}$   $V_0 < \frac{\pi^2\hbar^2}{2ma^2}$

## Mój model rozpadu α z Uranu




$e^{-\alpha b} \approx 10^{-22}$

Oszacować: czas życia Uranu $\approx 4\cdot 10^9$ lat czyli je w środku $\simeq 10^{22} \frac{1}{s}$.