Cwiczenia 5

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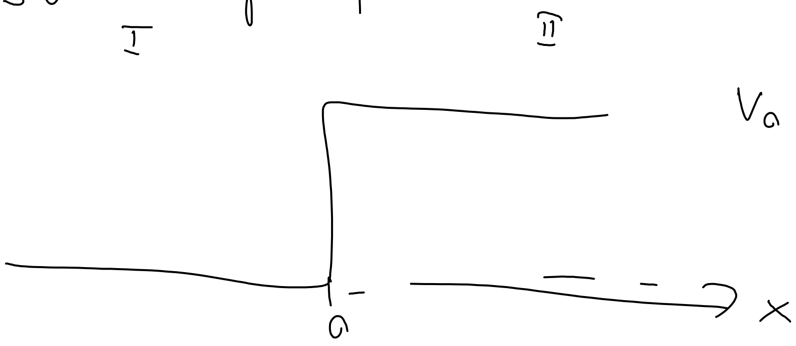
14:59

Stopien potencjalu, trans. Galileusza row. Schroedingera, rown. Falowego i Lorentza row. Falowego.

1. Schodek potencjału



- $E > V_0$

$$\Psi_I = A e^{i\alpha x} + B e^{-i\alpha x} \qquad \alpha = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\Psi_{II} = C e^{i\beta x} \qquad \beta = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\begin{cases} A + B = C \\ i\alpha A - i\alpha B = i\beta C \end{cases}$$

$$A - B = \frac{\beta}{\alpha} C$$

$$2A = \left(1 + \frac{\beta}{\alpha}\right) C \qquad C = \frac{2A}{1+\frac{\beta}{\alpha}} = A\,\frac{2\alpha}{\alpha+\beta}$$

$$B = \frac{2A}{1+\frac{\beta}{\alpha}} - A = \frac{A - \frac{\beta}{\alpha}A}{1+\frac{\beta}{\alpha}} = A\left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$

Współczynniki transmisji i odbicia:

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = \left(\frac{1-\sqrt{1-\frac{V_0}{E}}}{1+\sqrt{1-\frac{V_0}{E}}}\right)^2$$

$$T \cancel{=} \left|\frac{2\alpha}{\alpha+\beta}\right|^2 \quad \text{źle} \qquad T + R \neq 1$$

[illegible]

przypominijmy sobie prądy prawdopodobieństwa

Rownanie ciągłości

$$\begin{cases} \frac{\partial}{\partial t}|\psi|^2 + \vec{\nabla}\vec{j} = 0 \\ \vec{j} = \frac{\hbar}{2mi}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right) \end{cases}$$

$$|j_A| = |j_B| + |j_C| \qquad 1 = \underbrace{\left|\frac{j_B}{j_A}\right|}_{R} + \underbrace{\left|\frac{j_C}{j_A}\right|}_{T}$$

$$j_A = \frac{\hbar}{2mi}(i\alpha + i\alpha)|A|^2 = \frac{\hbar\alpha}{m}|A|^2$$

$$j_\beta = \frac{\hbar}{2mi}(i\beta + i\beta)|B|^2 = \frac{\hbar\beta}{m}|A|^2$$

$$T = \frac{\beta}{\alpha}\left|\frac{B}{A}\right|^2 = \frac{\beta}{\alpha}\left|\frac{2\alpha}{\alpha+\beta}\right|^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2} = \frac{4\sqrt{1-\frac{V_0}{E}}}{\left(1+\sqrt{1-\frac{V_0}{E}}\right)^2} \quad \text{ok}$$

Teraz $T+R = \frac{4\alpha\beta + (\alpha-\beta)^2}{(\alpha+\beta)^2} = 1$ ok

$$\begin{cases} \text{ogólny wtedy } \psi = Ae^{i\alpha x} + Be^{-i\alpha x} \text{ i to wstawmy} \\ \frac{\hbar}{m}\,\mathrm{Im}\left(\left(A^* e^{-i\alpha x} + B^* e^{+i\alpha x}\right)\cdot\left(i\alpha A e^{i\alpha x} - i\alpha B e^{-i\alpha x}\right)\right) = \\ = \frac{\hbar}{m}\,\mathrm{Im}\left(|A|^2 i\alpha - i\alpha|B|^2 \underbrace{- i\alpha A^* B e^{-2i\alpha x} + i\alpha B^* A e^{2i\alpha x}}_{\in \mathbb{R}}\right). \\ = \frac{\hbar}{m}\left(\alpha|A|^2 - \alpha|B|^2\right) \quad \text{ok} \end{cases}$$

• $E < V_0$

$$\psi_I = Ae^{i\alpha x} + Be^{-i\alpha x}$$

$$\psi_{II} = Ce^{-\beta x} \qquad \beta \to i\beta \qquad \beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\psi_{\text{II}} = C e^{-\beta x} \qquad \beta \to i\beta \qquad \beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$C = A \frac{2\alpha}{\alpha + i\beta}$$

$$B = A \frac{\alpha - i\beta}{\alpha + i\beta} \qquad R = \left|\frac{B}{A}\right|^2 = 1$$

$$T = 0 \quad \text{bo} \quad j_C = \frac{\hbar}{m} \operatorname{Im}\left(-\beta e^{-2\beta x}\right) = 0$$

2. Trans Galileusza rów. Schrödingera

$$x = x' + vt'$$
$$t = t'$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t)$$

Teraz w układzie primowanym

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi'(x',t')}{\partial x'^2} + V(x',t')\psi(x',t') = i\hbar \frac{\partial}{\partial t'} \psi(x',t')$$

ale
$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$$
$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \frac{\partial}{\partial x}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi'(x',t')}{\partial x^2} + V(x',t)\psi(x',t') = i\hbar\left(\frac{\partial}{\partial t}\psi(x',t') + v\frac{\partial}{\partial x}\psi(x',t')\right)$$

Widać że nie możemy więc $\psi'(x',t') = \psi(x,t)$. Weźmy

$$\psi'(x',t') = e^{i l(x,t)} \psi(x,t)$$ wtedy ok:

$$\varphi(x,t) = \frac{mvx - \frac{mv^2}{2}\cdot t}{\hbar}$$

Przykład: $\left(V(x,t)=0\right)$

$$\psi(x,t) = e^{\frac{ipx}{\hbar} - \frac{iEt}{\hbar}}$$

$$\psi'(x',t') = e^{\frac{i}{\hbar}\left(px - Et + mvx - \frac{mv^2}{2}t\right)} =$$

$$= e^{\frac{i}{\hbar}\left(p(x'+vt') - Et' + mv(x'+vt') - \frac{mv^2}{2}t'\right)}$$

$$= e^{\frac{i}{\hbar}\left(x'\underbrace{(p+mv)}_{p'} - t'\underbrace{\left(E + \frac{mv^2}{2} + pv\right)}_{E' = \frac{p'^2}{2m}}\right)} \quad \text{ok}$$

Jak to jest relatywistycznie?

Mamy relatywistyczny związek $E^2 = m^2c^4 + p^2c^2$

Rów. Kleina-Gordona: $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\quad E = i\hbar\frac{\partial}{\partial t}$

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2\frac{\partial^2\psi}{\partial x^2}c^2 + m^2c^4\psi$$

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\psi = 0$$

$$\psi(x,t) = e^{\frac{i}{\hbar}(px - Et)}$$

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Wiemy, że $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2}$

Więc iż $\frac{1}{c^2}\frac{\partial}{\partial t^2} - \frac{\partial}{\partial x^2} = \frac{1}{c^2}\frac{\partial}{\partial t'^2} - \frac{\partial}{\partial x'^2}$

niezmiennik.

Czyli $\psi'(x',t') = \psi(x,t)$ i to się

zgadza bo $E\cdot t - px$ też niezmiennik

Lorentzowski.

Dygresja: Granica nierelatywistyczna rów. K-G: wstawmy:

$$\psi = \varphi\, e^{-i\frac{mc^2 t}{\hbar}}$$

wyciągamy oscylacje związane z energią spoczynkową

Wstawiamy:

$$\frac{1}{c^2}\left(\frac{\partial^2\varphi}{\partial t^2} - \cancel{\left(\frac{mc^2}{\hbar}\right)^2}\varphi - 2i\frac{\partial\varphi}{\partial t}\frac{mc^2}{\hbar}\right) - \frac{\partial^2\varphi}{\partial x^2} + \cancel{\frac{m^2c^2}{\hbar^2}}\varphi = 0$$

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2} + \underbrace{\frac{\hbar^2}{2mc^2}}_{\text{małe}}\frac{\partial^2\varphi}{\partial t^2}$$