# Ćwiczenia 6

5 marca 2014
11:30

1. Paczki Gausowskie (funkcjae Greena), funkcje Greena w studni, porownac wynik z tym co się dostale dla sin^3 itp..
2. Potencjal: mgx: -przez transformate Fouriera, i scisle ogolne rozwiazania f. Airyego

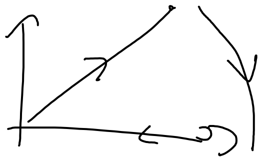
Ogólna całka Gaussowska

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2+\beta x} = \int e^{-\alpha\left(x-\frac{\beta}{2\alpha}\right)^2+\frac{\beta^2}{4\alpha}} = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{\beta^2}{4\alpha}}$$

- $\beta$ może być zespolone
- $\alpha$ też pod warunkiem że $\mathrm{Re}\,\alpha \geq 0$ np.

$$\int_{-\infty}^{+\infty} e^{-ix^2} = 2\int_0^{\infty} e^{-ix^2}, \quad z = e^{i\frac{\pi}{4}}x$$

$$= 2\int_0^{e^{i\frac{\pi}{4}}\infty} e^{-z^2}\, dz$$



$$= 2\left[\int_0^{\infty} e^{-x^2} + \lim_{R\to\infty}\underbrace{\int_0^{\frac{\pi}{4}} e^{-(Re^{i\varphi})^2}\, iR\, d\varphi}_{=0}\right] \text{ ok.}$$

1. Funkcja Greena dla rów. Schrödingera

$$\psi(x,t) = \int dx'\, G(x,t;x',t_0)\, \psi(x',t_0)$$

$$G(x,t_0;x',t_0) = \delta(x-x')$$

$$|\psi(t)\rangle = e^{-\frac{iH(t-t_0)}{\hbar}}\, |\psi(t_0)\rangle$$

Mając stany własne $H = \sum_n E_n |\psi_n\rangle\langle\psi_n|$

możemy zawsze zapisać

$$|\psi(t)\rangle = \sum_n e^{-\frac{iE_n(t-t_0)}{\hbar}}\, |\psi_n\rangle\langle\psi_n|\, |\psi(t_0)\rangle$$

$$i E_n (t-t_0)$$

$$\underbrace{\langle x|\psi(t)\rangle}_{\psi(x,t)} = \int dx' \underbrace{\sum_n e^{-\frac{iE_n(t-t_0)}{\hbar}} \langle x|\psi_n\rangle\langle\psi_n|x'\rangle}_{G(x,t,x',t_0)} \underbrace{\langle x'|\psi(t_0)\rangle}_{\psi(x',t_0)}$$

$$\underbrace{G(x,t,x',t_0) = \sum_n e^{-\frac{iE_n(t-t_0)}{\hbar}} \psi_n(x)\psi_n^*(x')}$$

$$G(x,t_0,x',t_0) = \sum_n \psi_n(x)\psi_n^*(x') = \delta(x-x')$$

zupełności bazy

dla swobodnej ewolucji $\quad H = \int \underbrace{\frac{p^2}{2m}}_{E_p} |p\rangle\langle p|\, dp$

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{\frac{ipx}{\hbar}} dx\, |x\rangle$$

$$G(x,t,x',t_0) = \frac{1}{2\pi\hbar}\int dp\; e^{-\frac{ip^2(t-t_0)}{2m\hbar}}\; e^{i\frac{p}{\hbar}(x-x')} =$$

$$= \frac{1}{(2\pi\hbar)} \sqrt{\frac{i\pi\, 2m\hbar}{(t-t_0)}}\; e^{\frac{i2m}{\hbar(t-t_0)}(x-x')^2} =$$

$$= \sqrt{\frac{im}{2\pi\hbar(t-t_0)}}\; e^{\frac{mi\,(x-x')^2}{2\hbar(t-t_0)}}$$

Dyfuzja z czasem urojonym !?

W przypadku 3D:

$$G(\vec{r},t,\vec{r}\,',t_0) = \left(\frac{im}{2\pi\hbar t}\right)^{\frac{3}{2}} e^{\frac{im|\vec{r}-\vec{r}\,'|^2}{2\hbar t}}$$

2. Ewolucja paczki Gaussowskiej

$$\ldots\; e^{-\frac{x^2}{4\Delta x^2} + \frac{ix\langle p\rangle}{\hbar}} \qquad \ldots \left(\frac{1}{\ldots}\right)^{\frac{1}{4}}$$

$$\psi(x,0)=A\,e^{-\frac{x^2}{4\Delta x^2}+\frac{ix\langle p\rangle}{\hbar}} \qquad A=\left(\frac{1}{2\pi\Delta x^2}\right)^{\frac14}$$

Znajdźmy $\psi(x,t)=?$

$$\psi(x,t)=\underbrace{A\sqrt{\frac{im}{2\pi\hbar t}}}_{A'}\int dx'\,e^{\frac{im(x-x')^2}{2\hbar t}}\,e^{-\frac{x'^2}{4\Delta x^2}+\frac{ix'\langle p\rangle}{\hbar}}$$

$x''=x-x'$

$$=A'\int dx''\,e^{\frac{imx''^2}{2\hbar t}}\,e^{-\frac{(x-x'')^2}{4\Delta x^2}+i\frac{(x-x'')\langle p\rangle}{\hbar}}=$$

$$=A'e^{\frac{ix\langle p\rangle}{\hbar}}\int dx'\,e^{-x''^2\left(\frac{1}{4\Delta x^2}-\frac{im}{2\hbar t}\right)+x''\left(\frac{x}{2\Delta x^2}-\frac{i\langle p\rangle}{\hbar}\right)-\frac{x^2}{4\Delta x^2}}$$

$$=A'\sqrt{\frac{\pi}{\frac{1}{4\Delta x^2}-\frac{im}{2\hbar t}}}\;e^{-\frac{x^2}{4\Delta x^2}+\frac{ix\langle p\rangle}{\hbar}}\;e^{\frac{\left(\frac{x}{2\Delta x^2}-\frac{i\langle p\rangle}{\hbar}\right)^2}{\frac{1}{\Delta x^2}-\frac{2im}{\hbar t}}}$$

$$=A''e^{\frac{ix\langle p\rangle}{\hbar}}\;e^{\frac{-\frac{x^2}{4\Delta x^2}\left(\frac{\cancel{1}}{\Delta x^2}-\frac{2im}{\hbar t}\right)+\frac{x^2}{\cancel{4}\Delta x^4}-\frac{ix\langle p\rangle}{\hbar\Delta x^2}-\frac{\langle p\rangle^2}{\hbar^2}}{\frac{1}{\Delta x^2}-\frac{2im}{\hbar t}}}=$$

$$=A''e^{\frac{ix\langle p\rangle}{\hbar}}\;e^{\frac{x^2\left(\frac{im}{2\Delta x^2\hbar t}\right)-\frac{ix\langle p\rangle}{\hbar\Delta x^2}-\frac{\langle p\rangle^2}{\hbar^2}}{\frac{1}{\Delta x^2}-\frac{2im}{\hbar t}}}=$$

$$=A''e^{\frac{ix\langle p\rangle}{\hbar}}\;e^{\frac{x^2-\frac{2t\langle p\rangle}{m}-\frac{\langle p\rangle^2 2\Delta x^2 t}{i\hbar m}}{-\frac{2\hbar t i}{m}-4\Delta x^2}}=$$

$$=\tilde{A}''\,e^{\frac{ix\langle p\rangle}{\hbar}}\;e^{-\frac{\left(x-\frac{\langle p\rangle}{m}t\right)^2}{4\left(\Delta x^2+\frac{i\hbar t}{2m}\right)}}$$

można bez zamiany zmiennych jednak prościej:

$$A'\int dx'\,e^{-x'^2\left(\frac{1}{4\Delta x^2}-\frac{im}{2\hbar t}\right)+ix'\left(\frac{\langle p\rangle}{\hbar}-\frac{mx}{\hbar t}\right)+\frac{imx^2}{2\hbar t}}$$

$$=A'\sqrt{\frac{\pi}{\frac{1}{4\Delta x^2}-\frac{im}{2\hbar t}}}\;e^{\frac{imx^2}{2\hbar t}}\;e^{-\frac{\frac{m^2}{\hbar^2t^2}\left(x-\frac{\langle p\rangle t}{m}\right)^2}{\frac{1}{\Delta x^2}-\frac{2im}{\hbar t}}}$$

$$=A''e^{\frac{\frac{imx^2}{2\hbar t\Delta x^2}+\cancel{\frac{m^2x^2}{\hbar^2t^2}}-\cancel{\frac{m^2x^2}{\hbar^2t^2}}+\frac{2m\langle p\rangle x}{\hbar^2 t}-\frac{\langle p\rangle^2}{t^2}}{\frac{1}{\Delta x^2}-\frac{2im}{\hbar t}}}=$$

$$=A''e^{\frac{i\langle p\rangle x}{\hbar}}\;e^{\frac{-\frac{i\langle p\rangle x}{\hbar\Delta x^2}-\cancel{\frac{2m\langle p\rangle x}{\hbar^2 t}}+\frac{imx^2}{2\hbar t\Delta x^2}-\frac{\langle p\rangle^2}{t^2}+\cancel{\frac{2m\langle p\rangle x}{\hbar^2 t}}}{\frac{1}{\cdot}-\frac{2im}{\cdot}}}=$$

$$= A'' e^{\frac{i\langle p\rangle x}{\hbar}} e^{\frac{\ldots}{\frac{1}{\Delta x^2} - \frac{2im}{\hbar t}}} =$$

$$= A'' e^{\frac{i\langle p\rangle x}{\hbar}} e^{\frac{x^2 - \frac{2t\langle p\rangle}{m} + \frac{2i\hbar \Delta x^2}{mt}\langle p\rangle^2}{\frac{2\hbar t}{im} - 4\Delta x^2}} =$$

$$= \tilde{A}'' e^{\frac{i\langle p\rangle x}{\hbar}} e^{-\frac{\left(x - \frac{\langle p\rangle t}{m}\right)^2}{4\left(\Delta x^2 + \frac{i\hbar t}{2m}\right)}}$$

$$|\psi(x,t)|^2 = |\tilde{A}''|^2 e^{-\frac{\left(x-\frac{\langle p\rangle t}{m}\right)^2}{4}\cdot\left(\frac{1}{\Delta x^2 + \frac{i\hbar t}{2m}} + \frac{1}{\Delta x^2 - \frac{i\hbar t}{2m}}\right)}$$

$$= |\tilde{A}''|^2 e^{-\frac{\left(x-\frac{\langle p\rangle t}{m}\right)^2}{4}\cdot\left(\frac{2\Delta x^2}{(\Delta x^2)^2 + \frac{\hbar^2 t^2}{4m^2}}\right)} =$$

$$= |\tilde{A}''|^2 e^{-\frac{\left(x-\frac{\langle p\rangle t}{m}\right)^2}{2\,\Delta x(t)^2}}$$

$$\boxed{\Delta^2 x(t) = \Delta^2 x + \frac{\hbar^2 t^2}{4m^2\Delta x^2}}$$

rozmywanie się paczki falowej

$$\Delta^2 x(t) = \Delta^2 x\left(1 + \left(\frac{t}{\tau}\right)^2\right)$$

$$\tau = \frac{2m\Delta^2 x}{\hbar}$$ ~ charakterystyczny czas rozpływania się

a) atom wodoru: $\Delta x \simeq 10^{-10}$ m, $m = 1{,}6\cdot 10^{-27}$

$\tau = 10^{-13}$ s

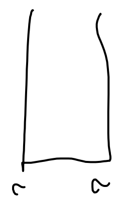
b) $m = 1$ g $\quad \Delta x = 10^{-3}$ m $\quad \tau = 10^{25}$ s

($10^{18}$ s – wiek wszechświata)

3. Ewolucja f. falowej za pomocą f. Greena

w studni.

$$E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \qquad \psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} n x\right)$$



$$G(x,t,x',0) = \sum_{n=1}^{\infty} e^{-\frac{\hbar \pi^2 n^2 t}{2ma^2}} \frac{2}{a} \sin\left(\frac{\pi}{a} n x\right) \sin\left(\frac{\pi}{a} n x'\right) =$$

$= \dots,$ nie sumuje się

więc muszę robić to co zwyczajnie rozkładać
na stany własne i ewoluować ....