Cwiczenia 8

12 marca 2014

1. Własności operatorów $[\hat{x}, \hat{p}] = i\hbar$

a) $[A, BC] = B[A, C] + [A, B]C$

$BAC - BCA + ABC - BAC$

$[AB, C] = -[C, AB] = -A[C, B] - [C, A]B =$

$= A[B, C] + [A, C]B$

b) $[x^n, p] = x[x^{n-1}, p] + [x, p]x^{n-1} =$

$= x^2[x^{n-2}, p] + x[x, p]x^{n-2} + i\hbar x^{n-1} =$

$= i\hbar n x^{n-1}$

c) $[x, p^n] = i\hbar n p^{n-1}$

d) $[f(x), p] = i\hbar \frac{df}{dx}$

pokazać, że $(AB)^\dagger = B^\dagger A^\dagger$

e) $x^\dagger = x$

$\langle \varphi | x^\dagger | \psi \rangle = \langle x\varphi | \psi \rangle = \varphi^* x \psi = \langle \varphi | x | \psi \rangle$

$\frac{d}{dx}^\dagger = -\frac{d}{dx}$

f) $\langle \varphi | \frac{d}{dx}^\dagger | \psi \rangle = \langle \frac{d}{dx}\varphi | \psi \rangle = \int \frac{d\varphi^*}{dx} \psi \, dx$

$= -\int \varphi^* \frac{d\psi}{dx} dx = \langle \varphi | -\frac{d}{dx}\psi \rangle$

g) $p = \frac{\hbar}{i}\frac{d}{dx}$ jest hermitowski

h) $L_x^\dagger = L_x$

$L_x = y \cdot p_z - z \cdot p_y$

$$L_x^+ = p_z \cdot y - p_y \cdot z = L_x$$

2. Rozwiązanie się stanów w obrazie Heisenberga

$$\dot{\hat{A}}(t) = \frac{i}{\hbar}\left[H, \hat{A}(t)\right]$$

$$A(t) = e^{\frac{iHt}{\hbar}} A(0)\, e^{-\frac{iHt}{\hbar}}$$

Uwaga $H(t) = H$

a) cz. swobodna: $H = \frac{\hat{p}^2}{2m}$

$$\hat{p}(t) = \hat{p}(0)$$

$$\dot{\hat{x}}(t) = \frac{i}{\hbar}\left[\frac{\hat{p}^2}{2m}, \hat{x}\right] = \frac{i}{2\hbar m}\left(-2\hat{p}\, i\hbar\right) = \frac{\hat{p}}{m}$$

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)}{m} \cdot t$$

niech $\langle x(0)\rangle = 0$

$\langle p(0)\rangle = 0$

$\langle x(t)\rangle = 0$

$$\left\langle \left(\hat{x}(t) - \langle x(t)\rangle\right)^2 \right\rangle = \left\langle \left(\hat{x}(0) + \frac{\hat{p}(0)}{m}t\right)^2 \right\rangle$$

$$= \left\langle x(0)^2 \right\rangle + \frac{\langle p(0)^2\rangle t^2}{m^2} + \underbrace{\frac{t}{m}\langle xp + px\rangle}$$

Dla Gaussa $\psi(x) = A e^{-\frac{x^2}{4\Delta x^2}}$ $A^2 = \frac{1}{\sqrt{2\pi\Delta x^2}}$

$\langle x^2(0)\rangle = \Delta x^2$ $\langle p(0)^2\rangle = \frac{\hbar^2}{4\Delta x^2}$

$$\langle xp\rangle = A^2 \int dx\, e^{-\frac{x^2}{4\Delta x^2}}\, x\, \frac{\hbar}{i}\frac{d}{dx} e^{-\frac{x^2}{4\Delta x^2}} =$$

$$= -\frac{A^2\hbar}{i}\int dx\, e^{-\frac{x^2}{2\Delta x^2}}\left(\frac{x^2}{2\Delta x^2}\right) = -\frac{A^2\hbar}{2i\Delta x^2}\sqrt{\pi}\cdot\left(\sqrt{2\Delta x^2}\right)^3 = \frac{-A^2\hbar\sqrt{2\pi\Delta x^2}}{i} = -\frac{\hbar}{i}$$

$\langle xp\rangle + \langle px\rangle = 0$

$\langle px\rangle = A^2 \frac{\hbar}{i}\int e^{-\frac{x^2}{4\Delta x^2}} \frac{d}{dx}\left(x e^{-\frac{x^2}{4\Delta x^2}}\right) = A^2\frac{\hbar}{i}\int e^{-\frac{x^2}{2\Delta x^2}} -$

Gdyby były korelacje to mogłoby być kurczenie się paczki falowej

$\simeq \Delta^2 x(0) + \frac{\hbar^2 t^2}{4m^2(\Delta x)}$ to co w obrazie Schrödingera

Można też przez BCH

$$\hat{x}(t) = e^{\frac{i\hat{p}^2 t}{2m}}\,\hat{x}\,e^{-\frac{i\hat{p}^2 t}{2m}}$$

$e^{A} B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots$

przy okazji $e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}$

b) Potencjał liniowy $H = \frac{p^2}{2m} - mgx$

$\dot{\hat{x}}(t) = \frac{i}{\hbar}\left[\frac{p^2}{2m} - mgx, \hat{x}\right] = \frac{\hat{p}(t)}{m}$

$\dot{\hat{p}}(t) = \frac{i}{\hbar}\left[-mg\hat{x}, \hat{p}\right] = mg$

$\hat{p}(t) = \hat{p}(0) + mgt$

$\dot{\hat{x}}(t) = \frac{\hat{p}(0)}{m} + gt \qquad \hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)\cdot t}{m} + \frac{gt^2}{2}$

Weźmy paczkę Gaussowską $\psi(x) = Ae^{-\frac{x^2}{4\Delta x^2}}$

$\langle\hat{x}(t)\rangle = \frac{gt^2}{2}$

$\langle \hat{x}(t) \rangle = \frac{g t^2}{2}$

$\langle (\hat{x}(t) - \langle x \rangle)^2 \rangle = \langle \hat{x}^2(0) + \frac{\hat{p}^2(0) t^2}{m^2} + \frac{\hat{x}\hat{p}\, t}{m} + \frac{\hat{p}\hat{x}\, t}{m} \rangle$

$= \Delta^2 x(0) + \frac{\hbar^2 t^2}{4m^2 \Delta x^2}$ tak samo fala swobodna

Jak to by trzeba robić w obrazie Schrödingera.....

c) Oscylator Harmoniczny.

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$\begin{cases} \dot{\hat{x}} = \frac{\hat{p}}{m} \\ \dot{\hat{p}} = -m\omega^2 \hat{x} \end{cases}$$

$\ddot{\hat{x}} = -\omega^2 \hat{x}$

$\ddot{\hat{p}} = -\omega^2 \hat{p}$

$$\begin{cases} \hat{x}(t) = \cos\omega t\,(\hat{x}(0)) + \sin\omega t \left(\frac{\hat{p}(0)}{m\omega}\right) \\ \hat{p}(t) = \cos\omega t\,(\hat{p}(0)) - \sin\omega t\,(\hat{x}(0)\, m\omega) \end{cases}$$

Weźmy Gaussa $\psi(x) = A e^{-\frac{x^2}{\Delta x^2}}$

$\Delta x^2(t) = \cos^2\omega t\, \Delta^2 x(0) + \sin^2\omega t\, \frac{1}{m^2\omega^2} \Delta^2 p(0) =$

$= \Delta x^2(0) \cdot \left(\cos^2\omega t + \sin^2\omega t\, \frac{\hbar^2}{4m^2\omega^2 \Delta x^4(0)} \right)$

Jeśli $\Delta^2 x(0) = \frac{\hbar}{2m\omega}$ to $\Delta x^2(t) = \Delta^2 x(0)$

$\Delta^2 p(t) = \frac{\hbar^2\, 2m\omega}{4\hbar} = \frac{\hbar m\omega}{2} = \Delta^2 p(0)$

Taki Gauss nie zmienia się w czasie

Tylko Gauss nie zmienia się w czasie

Znaleźliśmy stan własny oscylatora (stan podstawowy)

$$E = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$