Quantum Optics 2019/2020, Problem set 1, 22.10.2020

Problem 1 Consider a standard 1D quantum harmonic oscillator with Hamiltonian $H = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. A coherent state $|\alpha\rangle$ is an eigenstate of the anihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where α is a complex number. Write the coherent state in the basis of energy states of the harmonic oscillator $|n\rangle$.

Problem 2 Compute the probability distribution p_n of projecting a coherent state on a given energy state n. Compute expectation value and variance of this distribution.

Problem 3 Show that if initially the state of a harmonic oscillator is a coherent state $|\psi(0)\rangle = |\alpha\rangle$, then as a result of an evolution for time t it will remain a coherent state with an amplitude $|\psi(t)\rangle = |\alpha e^{-i\omega t}\rangle$.

Problem 4 Prove that coherent state are non-orthogonal $|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2}$, and that they satisfy completeness relation: $\frac{1}{\pi}\int \mathrm{d}\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}$, where $\mathrm{d}\alpha$ denote the integral over the whole complex plane with a natural measure: $\mathrm{dRe}\alpha\,\mathrm{dIm}\alpha$.

Problem 5 Completness condition for coherent states implies that one can decompose any state in terms of coherent states (though this decomposition will not be unique due to overcompleteness of the set) Find an exemplary decomposition of the energy eigenstates $|n\rangle$ in terms of coherent states.

Problem 6 Prove the following identity

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$$

Hint. Consider a function $F(t) = e^{tA} B e^{-tA}$ and differentiate with respect to t.

Problem 7 Prove the special case of the so called Baker-Cambpell-Hausdorff theorem in case $[A, B]$ commutes with both A and B:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}.$$

Hint. Consider a function $F(t) = e^{tA} e^{tB}$ and differentiate with respect to t.

Problem 8 Prove that a coherent state $|\alpha\rangle$ can be written as

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad \text{where } \hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$$

where $D(\alpha)$ is the so called displacement operator. *Hint.* Use Baker-Cambell-Haussdorf theorem

Problem 9 Consider a 1D conducting cavity of length L in the z direction (physically this means we assume that perpendicular dimensions are much larger than L). Consider the standing wave mode corresponding to $k_z = \frac{\pi}{L}$ with polarization in the x direction and denote the corresponding anihilation operator for this mode as a_1. Consider a single photon excitation in this mode $|\psi\rangle = \hat{a}_1^\dagger|0\rangle$. Compute expectation values of $\hat{\vec{E}}(z)$, $\hat{\vec{B}}(z)$ on this state. Compute linear energy density $\varrho(z)$ corresponding to this state. Integrate linear energy density over the length of the cavity and check whether it corresponds to the total energy as it should.

Problem 10 Repeat Problem 9, but replacing the single photon state $|\psi\rangle$ with a coherent state $|\alpha\rangle$.

Problem 11 Consider a cubic cavity (conducting box) $L \times L \times L$. What are the allowed modes for the E-M field in this setup? Setting the vacuum state energy to zero, what is the minimal excitation energy in this system—i.e. in which mode we should excite a single photon to have minimal possible energy.