Quantum Optics 2019/2020, Problem set 2, 20.10.2020

Problem 1 - Not obligatory. If you were assigned this problem by the algorithm you can choose a different one. But if you are brave enough, give it a try! Consider two infinite conducting plates placed at a distance a from each other (in direction z). Compute the effective force per unit area which is a result of non-zero vacuum fluctuations of electromagnetic field—the so called Casimir force.

Hints: Compute the difference of energy of electromagnetic field in the volume between the plates in case the plates are present (discrete modes) and in case they are not present (remember about two polarizations, except for the case when $k_z = 0$.). Use Euler-McLaurin formula:

$$\sum_{n=n_1}^{n_2} f(n) - \int_{n_1}^{n_2} \mathrm{d}n\, f(n) = \sum_{k=1}^{\infty} \frac{B_k}{k!}(f^{(k-1)}(n_2) - f^{(k-1)}(n_1))$$

where B_k are Bernoulli numbers defined as $\frac{y}{e^y-1} = \sum_{k=0}^{\infty} B_k \frac{y^k}{k!}$, and $f^{(k)}$ denotes the k-th derivative of f. Since the sums and integrals, one needs to deal with, are divergent, one has to introduce some converging factors e.g. in the form of exponentially damping terms that only in the end are set to 1.

Answer: $\frac{F}{S} = -\frac{\hbar c \pi^2}{240 a^4}$

Problem 2 Electric and magnetic field operators in free space are given as:

$$\hat{\vec{E}} = \int \mathrm{d}^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\epsilon_0}} \sum_{\sigma=1,2} \vec{e}(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c.$$

$$\hat{\vec{B}} = \int \mathrm{d}^3k \sqrt{\frac{\hbar}{(2\pi)^3 2\omega_k \epsilon_0}} \sum_{\sigma=1,2} \vec{k} \times \vec{e}(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c.$$

Show that

$$[\hat{E}_j(\vec{r},t), \hat{B}_k(\vec{r'},t)] = \frac{i\hbar}{\epsilon_0}\varepsilon_{jkl}\partial_l \delta^{(3)}(\vec{r}-\vec{r'}).$$

Problem 3 The Hamiltonian of the electromagnetic field is give by:

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3r \left(\epsilon_0 \hat{\vec{E}}^2 + \frac{\hat{\vec{B}}^2}{\mu_0}\right).$$

Show that for a free field, it can be written as:

$$\hat{H} = \sum_{\sigma} \int \mathrm{d}^3k\, \hat{a}^\dagger(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)\, \hbar\omega_k + E_0$$

where E_0 denotes the vaccum energy contribution, which and can be written as:

$$E_0 = \int \mathrm{d}^3r \int \frac{\mathrm{d}^3k}{(2\pi)^3}\hbar\omega_k.$$

What is the vacuum spectral energy density?

Answer: $\varrho(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3}$

Problem 4 Show that the total momentum of the field operator defined as

$$\vec{P} = \int \mathrm{d}^3 r \frac{1}{c^2 \mu_0} \left(\hat{\vec{E}} \times \hat{\vec{B}} \right)$$

can be written as:

$$\vec{P} = \sum_{\sigma} \int \mathrm{d}^3 k \, \hat{a}^\dagger(\vec{k}, \sigma) \hat{a}(\vec{k}, \sigma) \, \hbar \vec{k}.$$

How would you define a operator corresponding to the Poynting vector of the field (momentum density operator)?

Hint: At a first glance in analogy to classical electromagnetism one would like to define the Poynting vector operator as: $\hat{\vec{S}} = \frac{1}{\mu_0} \left(\hat{\vec{E}} \times \hat{\vec{B}} \right)$, but since E and B field do not commute, this operator is not hermitian...

Problem 5 Starting from the following decomposition of the vector field operator $\hat{\vec{A}}$ and the electric field $\vec{E}$ (which up to a constant is the canonical conjugated momentum):

$$\hat{\vec{A}}(\vec{r}, t) = i \int \mathrm{d}^3 k \sqrt{\frac{\hbar}{(2\pi)^3 2 \epsilon_0 \omega_k}} \sum_{\sigma=1,2} \vec{e}(\vec{k}, \sigma) \hat{a}(\vec{k}, \sigma) e^{i(\vec{k}\vec{r} - \omega_k t)} + h.c.$$

$$\hat{\vec{E}}(\vec{r}, t) = \int \mathrm{d}^3 k \sqrt{\frac{\hbar \omega_k}{(2\pi)^3 2 \epsilon_0}} \sum_{\sigma=1,2} \vec{e}(\vec{k}, \sigma) \hat{a}(\vec{k}, \sigma) e^{i(\vec{k}\vec{r} - \omega_k t)} + h.c.$$

where $[\hat{a}(\vec{k}, \sigma), \hat{a}^\dagger(\vec{k}', \sigma')] = \delta^{(3)}(\vec{r} - \vec{r}') \delta_{\sigma, \sigma'}$ show that

$$[\hat{A}_i(\vec{r}, t), \hat{E}_j(\vec{r}', t)] = -\frac{i\hbar}{\epsilon_0} \delta^{(3)}_{\mathrm{tr}}(\vec{r} - \vec{r}')_{ij},$$

where $\delta^{\mathrm{tr}}_{\mathrm{tr}}(\vec{r} - \vec{r}')$ is the transversal delta defined as:

$$\delta^{(3)}_{\mathrm{tr}}(\vec{r} - \vec{r}')_{ij} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(\delta_i j - \frac{k_i k_j}{\vec{k}^2} \right) e^{i\vec{k}(\vec{r} - \vec{r}')}.$$