Quantum Optics 2019/2020, Problem set 3, 29.10.2020

Problem 1 When instead of linear polzarization, we decompose electric and magnetic fields using circular polarization modes we can write

$$\vec{\hat{E}} = \int \mathrm{d}^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\epsilon_0}} \sum_{\sigma=\pm} \vec{e}(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c.$$

$$\vec{\hat{B}} = \int \mathrm{d}^3k \sqrt{\frac{\hbar}{(2\pi)^3 2\omega_k\epsilon_0}} \sum_{\sigma=\pm} \vec{k}\times\vec{e}(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c.,$$

where circular polarization vectors are complex vectors which are expressed in terms of linear polarization vectors as $\vec{e}(\vec{k},\pm) = \frac{\vec{e}(\vec{k},1)\pm i\vec{e}(\vec{k},2)}{\sqrt{2}}$. For a given circular polarization (helicity) we define the Riemann-Silberstein vector operator as:

$$\vec{\hat{\Psi}}_\sigma(\vec{r},t) = \sqrt{\frac{\epsilon_0}{2}}\left(\vec{\hat{E}}_\sigma(\vec{r},t) + i\sigma c\vec{\hat{B}}_\sigma(\vec{r},t)\right),$$

where $\vec{\hat{E}}_\sigma$, $\vec{\hat{B}}_\sigma$ represent electric and magnetic field operators corresponding to a given helicity σ. Show that the operator takes the form:

$$\vec{\hat{\Psi}}_\sigma(\vec{r},t) = \int \mathrm{d}^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\vec{e}(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)}.$$

Consider a single photon state:

$$|\psi\rangle = \sum_{\sigma=\pm}\int \mathrm{d}^3k\, \tilde{\psi}(\vec{k},\sigma)\hat{a}^\dagger(\vec{k},\sigma)|0\rangle$$

Show that the "photon wave function" corresponding to the above state, defined as:

$$\vec{\Psi}_\sigma(\vec{r},t) = \langle 0|\vec{\hat{\Psi}}_\sigma(\vec{r},t)|\psi\rangle$$

reads:

$$\vec{\Psi}_\sigma(\vec{r},t) = \int \mathrm{d}^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\tilde{\psi}(\vec{k},\sigma)\vec{e}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)}.$$

Problem 2 Following the notation from Problem 1, one may define an alternative photon wave function using the following formula:

$$\vec{\Psi}^{\mathrm{LP}}(\vec{r},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\tilde{\psi}(\vec{k},\sigma)\vec{e}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)},$$

which is called the Landau-Peierls (LP) photon wave function. Note the missing $\sqrt{\omega_k}$ in the enumerator. What will be the proper (i.e. in agreement with momentum representation) form of the scalar product for the Landau-Peierls wave functions. Prove that the relation between Landau-Peierls wave function and Riemann-Silberstein (RS) based wave function is the following:

$$\vec{\Psi}^{\mathrm{LP}}(\vec{r}) = \frac{1}{2\sqrt{\hbar c}(2\pi)^{3/2}} \int \mathrm{d}^3 r' \, \frac{\vec{\Psi}(\vec{r'})}{|\vec{r} - \vec{r'}|^{5/2}}.$$

Since, RS wave function has a direct relation with field energy density, we see that LP based density will not be properly representing energy density as it is a non-local transformation of the RS wave function. Morover, it does not have any reasonable transformation properties and hence is rarely regarded as a satisfactory definition of a photon wave function.

Problem 3 When integrating over momentum space, we did not pay attention to the fact that integration measure $\int \mathrm{d}^3 k$ is actually not Lorenz invariant, and hence it may be hard to tell what transformation properties the objects we deal with have. Prove that the proper Lorenz invariant measure is $\int \frac{\mathrm{d}^3 k}{k}$. Think, how we could change our convention for photon wave function in momentum representation and commutation relation between anihilation and creation to reflect better the Lorenz transformation properties.

Hint. Start by considering the Lorenz invariant measure in 4-dimensional space time $\int d^4 k$, where a four vector $\bar{k} = [k_0, k_1, k_2, k_3]$, and consider integrating a function $f(\bar{k})$, with an imposed constraint that $k_0 = \sqrt{k_1^2 + k_2^2 + k_3^2}$: $\int \mathrm{d}^4 k \, f(\bar{k}) \delta(k_0^2 - \vec{k}^2)$. Note that both $\delta(k_0^2 - \vec{k}^2)$ and $\int \mathrm{d}^4 k$ are proper relativistic invariant objects and hence the resulting prescription for integrating $f(\bar{k})$ on the „photon mass shell" $k_0^2 - \vec{k}^2 = 0$ should be relativistically invariant.

Problem 4 Show that

$$\int \frac{\mathrm{d}^3 k}{k} e^{i\vec{k}(\vec{r} - \vec{r'})} = \frac{4\pi}{|\vec{r} - \vec{r'}|^2}.$$

Use the above property to prove that the proper scalar product (i.e. the one compatible with the Hilbert space scalar product) for the RS wave function should be defined as:

$$\langle \vec{\Psi} | \vec{\Phi} \rangle = \frac{1}{2\pi^2 \hbar c} \sum_\sigma \int \mathrm{d}^3 r \mathrm{d}^3 r' \frac{\vec{\Psi}_\sigma^\dagger(\vec{r}) \vec{\Phi}_\sigma(\vec{r'})}{|\vec{r} - \vec{r'}|^2}.$$

Problem 5 In order to justify that the RS wave function yields a proper energy density probablity distribution show that:

$$\sum_\sigma \vec{\Psi}_\sigma^\dagger(\vec{r}) \vec{\Psi}_\sigma(\vec{r}) = \langle \psi | {:}\mathcal{H}(\vec{r}){:} | \psi \rangle$$

where $|\psi\rangle$ is a single photon state, while $:\mathcal{H}:$ is the normally ordered energy density operator $\mathcal{H}$:

$$\mathcal{H}(\vec{r}) = \frac{1}{2} \left(\epsilon_0 \hat{\vec{E}}(\vec{r})^2 + \frac{1}{\mu_0} \hat{\vec{B}}(\vec{r})^2 \right).$$

Normal ordering means, that all the creation operators are moved to the left and all the anihilation operators to the right—this leads to an effective removal of the vacuum energy: