Quantum Optics 2019/2020, Problem set 4, 10.11.2020

Problem 1 Consider a Mach-Zehnder interferometer, with a phase delay between the arms equalt to φ, for which the transformation of the anihiliation operators of the respective modes reads:

$$\begin{bmatrix} \hat{a}'_1 \\ \hat{a}'_2 \end{bmatrix} = \begin{bmatrix} i\sin(\varphi/2) & \cos(\varphi/2) \\ -\cos(\varphi/2) & -i\sin(\varphi/2) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}.$$

a) Consider an N-photon input state $|\psi\rangle = |N, 0\rangle$, where all N photons are sent into the interferometer in the upper arm. Find the probability of detecting n_1, n_2 photons in the upper and lower arm respectively of the interferometer at the output.

b) Imagine that N photon states as above are sent into the interferometer with probabilities $p(N) = \frac{e^{-\bar{N}}}{N!}\bar{N}^N$, where $\bar{N}$ is the average number of photons sent. Find the probabilities of detecting n_1, n_2 photons in the upper and lower arm of the interferometer at the output.

c) Compare the result with the case in which one sends a coherent state $|\alpha\rangle$ into the upper arm where $\alpha = \sqrt{\bar{n}}$. Try to contemplate the result.

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Problem 2 The model of Hong-Ou-Mandel interference with non-perfectly overlapping temporal modes. Consider a two-photon state

$$|\psi\rangle = \hat{b}_1^\dagger \hat{b}_2^\dagger |0\rangle,$$

where

$$\hat{b}_1 = \int \mathrm{d}\omega\, \tilde{f}_1(\omega)\hat{a}_1(\omega), \quad \hat{b}_2 = \int \mathrm{d}\omega\, \tilde{f}_2(\omega)\hat{a}_2(\omega),$$

where $a_1(\omega), a_2(\omega)$ represent anihilation operators corresponding to monochromatic modes of respectively upper and lower inputs of a beam splitter $\mathcal{B}$, so that

$$\begin{bmatrix} \hat{a}'_1(\omega) \\ \hat{a}'_2(\omega) \end{bmatrix} = \mathcal{B} \begin{bmatrix} \hat{a}_1(\omega) \\ \hat{a}_2(\omega) \end{bmatrix}, \quad \mathcal{B} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

a) After the two photons passed through the beam splitter, find the probabilities of coincidence events (i.e. cases where two photons go into different output ports) and double-click events (when both photons go together to one output port)

b) Compute the above quantities in case the temporal modes in which photons travel can be described by Gaussians so that the uncertainty of time-localization of the photon is σ_τ and assume that the temporal wave packets of the two photons are shifted with respect to each other by time τ. Discuss the special cases $\tau = 0$ and $\tau \gg \sigma_\tau$.

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Problem 3 Consider two pairs of photons travelling respectively in mode 1 and mode 2 (so we have four photons in total). The photons impinge on a balanced beamsplitter $\mathcal{B} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Find the probabilities $p(n_1, n_2)$ of different click patterns of the output detectors in case:

a) The two photons travelling in the same mode are indistinguishable and the two modes are overlapping perfectly on the beam splitter.

b) The two photons in a given mode are indistinguishable but the modes 1 and 2 are in fact non-overlapping at all on the beamsplitter (e.g. they have ortohogonal temporal profiles)

c) All four photons are actually in orthogonal temporal modes.

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Problem 4 Consider the Mach-Zehnder interferometr as described in Problem 1. Let us imagine we want to send light into this interferometer in order to find out what is the relative phase delay between the arms φ. At the output we will be measuring difference in photon numbers $n_2 - n_1$ detected in the two output modes. In other words we will be interested in measuring observable $\hat{A} = \hat{n}_2 - \hat{n}_1 = \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1$. Based on the value of the measurement we will be trying to find out what was the value of φ. We will be comparing two cases:

a) When as a input we send a coherent state in one of the input modes of the interferometer $|\psi\rangle = |\alpha\rangle \otimes |0\rangle$, where $|\alpha|^2 = \bar{n}$.

b) When we send exactly n photons into the upper arm of the interferometer: $|\psi\rangle = |n\rangle \otimes |0\rangle$.

In both cases compute the expectation value of the observable at the output $\langle \hat{A} \rangle$ and its variance $\Delta^2 A = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$. Using linear propagation of error formula we may asses the error of estimating φ in both of these cases:

$$\Delta\varphi = \frac{\sqrt{\Delta^2 A}}{\left| \frac{\mathrm{d}\langle A \rangle}{\mathrm{d}\varphi} \right|}.$$

Comment which state of light yields lower estimation uncertainty for the same (average) number of photons used i.e. $n = \bar{n}$.

Important Hint: It is much easier and more illuminating to solve this problem using the Heisenberg picture: evolve observable $\hat{A}$ and compute required quantities on the input states.

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