Quantum Optics 2019/2020, Problem set 5, 17.11.2020

Problem 1 According to statistical mechanics the thermal state of a quantum system is defined as:

$$\rho_T = \frac{1}{\mathrm{Tr}(e^{-\frac{\hat{H}}{kT}})} e^{-\frac{\hat{H}}{kT}}$$

where $\hat{H}$ is the Hamiltonian of the system. Consider a thermal state of light in a single mode with corresponding frequency ω.

a) Show that the thermal state has a form:

$$\rho_T = (1 - e^{-\xi}) \sum_n e^{-n\xi} |n\rangle\langle n|,$$

where $\xi = \frac{\hbar\omega}{kT}$, and $|n\rangle$ represents the n photon state.

b) Show that the expected photon number in the thermal state reads: $\bar{n} = \frac{1}{e^{\xi}-1}$

c) Prove that the thermal state can be equivalently written as:

$$\rho_T = \frac{1}{\pi\bar{n}} \int \mathrm{d}^2\alpha \, e^{-\frac{|\alpha|^2}{\bar{n}}} |\alpha\rangle\langle\alpha|.$$

Comment on the limit $\bar{n} \to 0$.

d) Inspecting the above formula, would you call the thermal state a *classical* state?

Problem 2 The squeezing operator is defined as:

$$\hat{S}(r,\theta) = e^{\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)},$$

where $\xi = re^{i\theta}$. Prove the following identities that show how the squeezing operator transforms anihilation and creation operators in the Heisenberg picture:

$$\hat{S}^\dagger \hat{a} \hat{S} = \hat{a}\cosh(r) - \hat{a}^\dagger \sinh(r) e^{i\theta}$$

$$\hat{S}^\dagger \hat{a}^\dagger \hat{S} = \hat{a}^\dagger \cosh(r) - \hat{a}\sinh(r) e^{-i\theta}.$$

Show that analogous transformations for quadrature operators $\hat{q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$, $\hat{p} = (\hat{a} - \hat{a}^\dagger)/(i\sqrt{2})$ read:

$$\hat{S}^\dagger \hat{q} \hat{S} = \hat{q}[\cosh(r) - \sinh(r)\cos(\theta)] - \hat{p}\sinh(r)\sin(\theta).$$

$$\hat{S}^\dagger \hat{p}^\dagger \hat{S} = -\hat{q}\sinh(r)\sin(\theta) + \hat{p}[\cosh(r) + \sinh(r)\cos(\theta)].$$

Problem 3 Consider a single standing wave mode with frequency ω in an affective 1D cavity (along the z axis) of volume V, where the mode function is proportional to $\sin kz$, $k = \omega/c$. Consider a squeezed vacuum state in this mode defined as:

$$|r, 0\rangle = \hat{S}(r, 0)|0\rangle.$$

a) Compute the time variation of the expectation value of the electric field at $z_0 = \pi/(2k)$.

b) Compute the time variation of the variance the electric field at z_0.

c) Compute the mean number of photons in the state.

Click to see the answer:

Problem 4 Show that a squeezed vacuum state, when expanded in the Fock basis, reads:

$$|r, \theta\rangle = \hat{S}(r, \theta)|0\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{m=0}^{\infty} (-1)^m \frac{\sqrt{(2m)!}}{2^m m!} e^{im\theta} [\tanh(r)]^m |2m\rangle.$$

Click to see the hint:

Problem 5 (a bit tedious, but worth it) Consider the Mach-Zhender interferometer as in Problem 4 in Probelm Set 4, and consider the situation in which apart from a coherent state $|\alpha\rangle$ sent into the upper input port, one sends a squeezed vacuum state at the other (typically unused) input port of the interferometer, so the input state reads: $|\psi\rangle = |\alpha\rangle \otimes |r, 0\rangle$.

Find the formula for the precision of estimating the phase delay between the arms of the interferometer, which is based on measuring photon number difference at the output ports of the interferometer (for this you need to find the expectation value and the variance of the observable $\hat{A} = \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1$, and use the linear error propagation formula).

Do you see some potential advantage of this scheme compared to the situation when all the light is send just as coherent state at the input?

Click to see the answer: