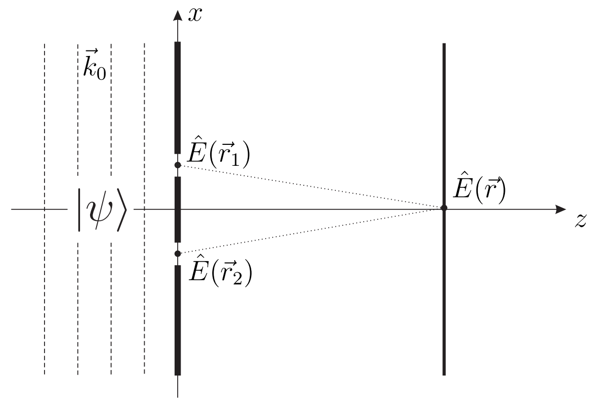
Quantum Optics 2019/2020, Problem set 7, 11.12.2020

Prologue Consider a two-slit interferometric experiment as depicted below:



During the lecture, we have computed the inteference patter on the screen by expressing the electric field operator at the screen $\hat{E}(\vec{r}, t)$ as a linear combination of electric field operators at positions $\vec{r}_1$, $\vec{r}_2$ at appropriately delayed times $t_1 = t - |\vec{r}_1 - \vec{r}|/c$, $t_2 = t - |\vec{r}_2 - \vec{r}|/c$:

$$\hat{E}^{(+)}(\vec{r}, t) = K_1 \hat{E}^{(+)}(\vec{r}_1, t_1) + K_2 \hat{E}^{(+)}(\vec{r}_2, t_2),$$

where K_i are some constant related with the classical propagation of light from the slits to the screen, the exact form of which is not relevant for our considerations, and for simplicity we set $K_1 = K_2$. We then acted with these operators on state $|\psi\rangle$, which was a quantum state of light where all the light was present only in the single plane wave mode associated with a wave vector $\vec{k}_0$. Hence, when looking at the expression for the electric field operator (assuming discretized modes)

$$\hat{E}(\vec{r}, t) = \sum_{\vec{k}} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \hat{a}_{\vec{k}} e^{i(\vec{k}\vec{r} - \omega_k t)},$$

then, effectively, when calculating the average intensity on the screen, only the term corresponding to $\vec{k} = \vec{k}_0$ contributed. We observed that interference pattern appears irrespectively of the character of the state $|\psi\rangle$ (be it a coherent, or a Fock state,...) and the mean intensity on the screen is proportional to:

$$\langle \hat{I}(\vec{r}, t) \rangle \propto (1 + \cos[\vec{k}_0(\vec{r}_1 - \vec{r}_2) - \omega_{k_0}(t_1 - t_2)]).$$

Problem 1 Consider now the case when instead of a single pure state of light $|\alpha\rangle_{\vec{k}_0}$ in a well defined plane wave mode we deal with a mixed state:

$$\rho = \int \mathrm{d}\omega L(\omega) \rho_\omega,$$

where $\rho_\omega = |\alpha\rangle_\omega \langle \alpha|$ represents a state of light where we have a coherent state in a plane wave mode corresponding to a wave vector $\vec{k}_\omega = \omega/c\vec{e}_z$, where $\vec{e}_z$ is a unit vector which points in the z direction (direction perpendicular to the screen)—formally we should write the state state $|\psi\rangle_\omega$ as $|0\rangle \otimes |0\rangle \otimes |\alpha\rangle_\omega \otimes |0\rangle \otimes \cdots \otimes |0\rangle$ to represent the fact that all other modes are in vacuum state. This corresponds to the situation, that we have a mixture of coherent states corresponding to plane wave modes of different frequencies, but all propagating in the same direction. In this expression $L(\omega)$ represents the probability of having a given frequency mode occupied. We will take it to be the Lorenzian distribution

$$L(\omega) = \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2},$$

where γ is the width parameter of the distribution. Write explicitly the normalized first order coherence function for this state $g^{(1)}(\vec{r}_1, t_1, \vec{r}_2, t_2)$. To simplify the formulas, you may assume that $\omega_0 \gg \gamma$ (width of the Lorenzian is much smaller than the mean value) which allows us to approximate integrals:

$$\int \mathrm{d}\omega\, \omega L(\omega) e^{i\omega t} \approx \omega_0 \int \mathrm{d}\omega\, L(\omega) e^{i\omega t}.$$

Note that we cannot replace ω with ω_0 in the exponent, as this is a fast oscillating term, and this would be legitimate only if we additionally assumed that $t \ll \gamma$, which we do not want to do. Ignoring the proportionality constants, write an expression for the intensity pattern observed on the screen.

Click to see the answer:

Introduction to Problems 2 and 3 We can look at the Young two-slit interferometric experiment from a slightly different perspective, in which we evolve the state through the two slit barrier rather than evolving back the field operators to act on the state before it hits the barrier. Recall our discussion on the spatio-temporally localized modes long time ago. Let us introduce anihilation operators $\hat{b}(j, k_y, k_z)$ that represent discrete set of modes localized in the x direction, at some point $x = j \times \delta x$, where δx represents the characteristic width of the mode—the exact form of this mode will not be relevant. Note that these modes will still correspond to plane wave modes in the other directions. For simplicity we fix the size of the screen in the x direction to l and also assume that its size correspond to the boundaries of the finite volume in which we quantized the e-m field (so that the plane wave modes to not stretch outside the barrier). As such we will have $L = l/\delta x$ localized modes and index j will go from $-L/2$ to $L/2$ We can approximately express $\hat{a}_{\vec{k}_0}$ in terms of localized mode anihiliation operators as follows (note that this transformation between the anihiliation operators corresponding to different mode set should be unitary to preserve commutaiton relations)

$$\hat{a}_{\vec{k}_0} \approx \frac{1}{\sqrt{L}} \sum_{j=-L/2}^{L/2} e^{ik_{0,x} j\delta x} \hat{b}(j, k_{0,y}, k_{0,z})$$

where $k_{0,i}$ denote appropriate coordinates of vector $\vec{k}_0$

Imagine we have a state in the plane wave mode $\vec{k}_0$ given in general as:

$$|\psi\rangle = f\left(\hat{a}_{\vec{k}_0}, \hat{a}^\dagger_{\vec{k}_0}\right)|0\rangle.$$

When we replace creation and anihliation operators $\hat{a}$ with the corresponding $\hat{b}$ operators we will express the state in terms of x-localized modes.

$$|\psi\rangle = f\Big(\frac{1}{\sqrt{L}}\sum_j e^{ik_{0,x}x}\hat{b}(x, k_{0,y}, k_{0,z}), \frac{1}{\sqrt{L}}\sum_j e^{ik_{0,x}x}\hat{b}(x, k_{0,y}, k_{0,z})\Big)|0\rangle.$$

In this approach the effect of the blocking screen is very intuitive as its effect is the absorbtion of photons in all the modes except the ones for which $x = x_1$ and $x = x_2$. This absorbtion can be regarded as a photon number measurement in all the modes except the ones in x_1 and x_2, where after the measurement all the modes measured are replaced with vacuum states. In this way we will obtain an effective state ρ' describing the state of light *after* the light passed through the barrier, where all other modes except modes x_1 and x_2 are in vacuum state.

As a result, when we are computing the expected intensity on the screen we will express the electric field operator $\hat{E}(\vec{r})$ again as

$$\hat{E}^{(+)}(\vec{r}, t) = K[\hat{E}^{(+)}(\vec{r}_1, t_1) + \hat{E}^{(+)}(\vec{r}_2, t_2)],$$

where now, the electric field operators at points $\vec{r}_i$ will be given in terms of new anihilation operators:

$$\hat{E}^{(+)}(\vec{r_i}, t_i) \propto \hat{b}(x_i, k_{0,y}, k_{0,z})e^{i(k_{0,y}y_i + k_{0,z}z_i) - i\omega t_i} + \text{irrelevant contributions from other orthogonal modes.}$$

(Note that in our approximate treatment we assume that all the modes correspond to the same frequency $\omega = |\vec{k}_0|c = k_0 c$ so that everything is time stationary and we do not have to consider issues related with ω variations). For simplicity we set $y_i = 0$, $z_i = 0$ so finally, restricting just to relevant modes we can write

$$\hat{E}^{(+)}(\vec{r}, t) \propto b_1 e^{-i\omega t_1} + b_2 e^{-i\omega t_2},$$

where b_1, b_2 denote the anihilation operators corresponding to the relevant 'slit modes', and as before $t_i = t - |\vec{r_1} - \vec{r}|/c$. Equivalently we can write:

$$\hat{E}^{(+)}(\vec{r}, t) \propto b_1 e^{is_1 k_0} + b_2 e^{is_2 k_0},$$

where $s_i = |\vec{r_1} - \vec{r}|$ are respective distances from the slits to the point on the screen where intensity is measured.

Problem 2 Assuming the state before the barrier is $|\psi\rangle$ write explicitly what equivalent state ρ' one would have in the 'two slit modes' approach described above in case:

a) $|\psi\rangle = |\alpha\rangle_{\vec{k}_0}$ is a coherent state.

b) $|\psi\rangle = |1\rangle_{\vec{k}_0}$ is a single photon state.

Convince yourself that you will obtain the same predictions regarding the interference pattern as in the approach presented during the lecture.

Additionally, compute the interference pattern in case someone prepared a state that correspond $|\psi'\rangle = |1\rangle_1 \otimes |1\rangle_2$ so two single photon states in localized slit modes 1 and 2. Comment.

Click to see the answer:

b) When we rewrite the single photon state in the new modes we get:

$$|\psi\rangle = a^\dagger_{k_0}|0\rangle = \frac{1}{\sqrt{L}}\sum_j e^{-ik_{0,x}j\delta x}\hat{b}^\dagger_j|0\rangle.$$

Inspecting this state we see that photon hits the barrier with probability $(L-2)/L$ and in this case the remaining modes x_1, x_2 will be in the vacuum state $|0\rangle \otimes |0\rangle$. If, however, the screen does not absorb the photon then (probability $2/L$) we are left we a state which after normalization reads: $|\psi'\rangle = \frac{1}{\sqrt{2}}\left(e^{0ik_{0,x}x_1}|1\rangle \otimes |0\rangle + e^{0ik_{0,x}x_2}|0\rangle \otimes |1\rangle\right)$ So finally we can write the effective state of light in the two modes as:

$$\rho' = \frac{L-2}{L}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2}{L}|\psi'\rangle\langle\psi'|.$$

In case of $|\psi'\rangle = |1\rangle_1 \otimes |1\rangle_2$ there will be no interference fringes.

Problem 3 Please read the introduction before Problem 2. Consider the state in the two-slit localized modes to be

$$|\psi'\rangle = |1\rangle_1 \otimes |1\rangle_2.$$

When discussing first order coherence this state was shown in Problem 2 to yield no interference fringes. Consider now, an experiment in which we detect coincidences, when two photons simultaneously hit the screen at two different positions $\vec{r}'_1$ and $\vec{r}'_2$. Up to a multiplicative factor, compute second order coherence function for the light hitting the screen:

$$G^{(2)}(\vec{r}'_1, \vec{r}'_2, t, t; \vec{r}'_1, \vec{r}'_2, t, t) = \langle\psi'|E^{(-)}(\vec{r}'_1, t)E^{(-)}(\vec{r}'_2, t)E^{(+)}(\vec{r}'_2, t)E^{(+}(\vec{r}'_1, t)|\psi'\rangle$$

where $\vec{r}'_i$ denote points on the screen, and according to the discussion before Problem 2, the electric field operators at the screen are related with the slit modes anihliation operators as:

$$E^{(+)}(\vec{r}'_i, t) \propto \hat{b}_1 e^{ik_0 s_{1i}} + \hat{b}_2 e^{ik_0 s_{2i}},$$

where s_{ij} is the distance from slit i to point on the screen $\vec{r}'_j$ where a detection happens. The computed $G^{(2)}$ function is proportional to the probability of observing two photons detected simultaneously at positions $\vec{r}'_1$ and $\vec{r}'_2$.

Can one interpret the obtained result as some kind of interference fringes?

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