Quantum Optics 2019/2020, Problem set 8, 22.12.2020

Problem 1 Show that given a multimode state of light ρ, the probability of detecting given photon numbers n_k in appropriate modes k can be written as

$$p(n_1, \ldots, n_k) = \mathrm{Tr}\left(\rho : \prod_k \frac{\hat{n}_k^{n_k} e^{-\hat{n}_k}}{n_k!} :\right),$$

where $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ is the photon number operator in mode k, and $::$ denotes normal ordering.

Click to see a hint:

Problem 2 Consider plane wave modes $e^{i\vec{k}\vec{r}}$ propagating in the z direction, where $\vec{k} = \vec{e}_z \omega / c$, where ω is the frequency parameter. For simplicity we will consider a discretized spectrum where ω parameter is descritized with steps $\delta\omega$. Let us denote by $\hat{a}_\omega$ anihilation operator related with mode ω. Consider a multimode coherent state

$$|\psi\rangle = e^{\sum_\omega \sqrt{\delta\omega G(\omega)}(\hat{a}_\omega^\dagger \alpha - \hat{a}_\omega \alpha^*)}|0\rangle = \bigotimes_\omega |\alpha \sqrt{\delta\omega\, G(\omega)}\rangle,$$

where $G(\omega)$ is a Gaussian distribution

$$G(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}}.$$

Physically this state corresponds to a classical Gaussian wavepacket. We assume that the discretization step is small enough that, when needed, we will be allowed to replace all the sums $\sum_\omega \delta\omega\, f(\omega)$ with a corresponding integrals $\int \mathrm{d}\omega\, f(\omega)$. Moreover, we assume $\omega_0 \gg \sigma$ so we can regard the Gaussian as well localized around ω_0, and while performing integrals, we can replace slowly varying functions of ω with ω_0.

a) Compute (up to the proportionality constant) the first-order temporal coherence function $G^{(1)}(\vec{r}, t, \vec{r}, t+\tau)$

b) Compute first-order normalized temporal coherence function $g^{(1)}(\vec{r}, t, \vec{r}, t+\tau)$

c) Compare these results with the ones that would be obtained for the state which is an incoherent mixture of coherent states $\rho = \sum_\omega \delta\omega\, G(\omega) |\alpha\rangle_\omega \langle\alpha|$, where $|\alpha\rangle_\omega$ denotes a coherent state in mode ω.

Click to see the answer:

Problem 3 Consider a thermal state in a single plane wave mode. Show that the second order normalized coherence function

$$g^{(2)}(0) := g^{(2)}(\vec{r}, \vec{r}, t, t) = 2.$$

Comment what will change if we consider coherence function for different positions and different times, $g^{(2)}(\vec{r}_1, \vec{r}_2, t_1, t_2)$.

In order to see the bunching effects of the thermal state (so that $g^{(2)}(\tau) < g^2(0)$), one needs to consider a multi-mode thermal state:

$$\rho_T = \frac{1}{Z} e^{-\frac{\hat{H}}{kT}},$$

where $\hat{H} = \sum_j \hbar\omega_j \hat{a}_j^\dagger \hat{a}_j$ is the Hamiltonian of light for discrete modes j each with the corresponding frequency ω_j, and Z is the normalization factor. The state can be written equivalently as a tensor product of single mode thermal states:

$$\rho_T = \bigotimes_j \frac{1}{Z_j} e^{-\frac{\hbar\omega_j}{kT}\hat{a}_j^\dagger \hat{a}_j} = \bigotimes_j \sum_{n_j=0}^{\infty} \frac{\bar{n}_j^{n_j}}{(\bar{n}_j + 1)^{n_j+1}} |n_j\rangle\langle n_j|,$$

where $\bar{n}_j = \frac{1}{e^{\hbar\omega_j/kT} - 1}$ is the mean number of photons in mode j.

Show that for the multimode thermal state:

$$g^{(2)}(\tau) := g^{(2)}(\vec{r}, \vec{r}, t, t+\tau) = 2 + \frac{\sum_{j,l} \bar{n}_j \omega_j \bar{n}_l \omega_l \left(e^{i(\omega_j - \omega_l)\tau} - 1\right)}{\left(\sum_j \bar{n}_j \omega_j\right)^2}.$$

Comment on the behaviour for large τ.

Click to see the answer: