Quantum Estimation and Measurement Theory

Problem set 1

return on 12.10.2018

Problem 1 Let density matrix ρ_{AB}, describing a state of two qubits, written in basis $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$, have the following form:

$$\rho_{AB} = \begin{pmatrix} \frac{5}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{4} \\ -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} \\ \frac{1}{4} & \frac{1}{12} & -\frac{1}{12} & \frac{5}{12} \end{pmatrix} \tag{1}$$

Find reduced density matrix ρ_A, ρ_B. Think of a general and practical method for quick calculation of the reduced density matrices

Problem 2 Reanalyze the Stern-Gerlach experiment as described in Sec. **??**, but this time allow the particle to freely evolve, after the interaction with the magnetic field, for a time t, after which time the measurement of position of the particle is performed—assume the standard free evolution of a particle with mass m under the $H_{\text{free}} = p^2/2m$ hamiltonian. Write the probability distribution for detecting a particle at a given point z, and derive the corresponding POVM operators for this measurement. Discuss the limit $t \to \infty$ and compare it with the direct measurement of momentum that was discussed in the main text.

Problem 3 A general state of a two-dimensional quantum system (a qubitu) can be written as:

$$|\psi_{\theta,\varphi}\rangle = \cos(\theta/2)|0\rangle + \exp(\mathrm{i}\varphi)\sin(\theta/2)|1\rangle,$$

where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ and $|0\rangle$, $|1\rangle$ form an orthonormal basis. As a result, we can imagine a state of a qubit as a point on a sphere—the so called Bloch sphere.

Consider the following set of operators, parameterized with θ, φ:

$$\Pi_{\theta,\varphi} = c|\psi_{\theta,\varphi}\rangle\langle\psi_{\theta,\varphi}|,$$

where c is some normalization constant independent of θ, φ.

a) Is the set of operators $\Pi_{\theta,\varphi}$ a legitimate POVM? If yes, what is the value of the c constant (assume a standard integration measure on the sphere $\mathrm{d}\theta\mathrm{d}\varphi\sin\theta$).

b) Apply the above generalized measurement, and calculate the corresponding probability distribution as a function of (θ, φ) if the state that was measured was $|0\rangle$. This distribution can be treated as representation of information on how well we can identify a given a state on the Bloch sphere if we have one copy at our disposal.

Problem 4 Consider the following unitary operation U representing interaction of the qubitu (S) with "measuring device" (M):

$$U|m\rangle_S \otimes |0\rangle_M = \frac{1}{2}|m\rangle_S \otimes |0\rangle_M(\sqrt{2-p} + (-1)^m\sqrt{p}) + \frac{1}{2}|m\rangle_S \otimes |1\rangle_M(\sqrt{2-p} - (-1)^m\sqrt{p}),$$

where $m = 0, 1$, and parameter $0 \leq p \leq 1$ represents the "strength of the interaction" between S and M.

a) Using Kraus operators, (operators K_i) write down effective evolution of a general state of a qubit S under this interaction, in situation when no particular measurement result is observed in M—we calculate $\rho_S' = \sum_i K_i \rho_S K_i^\dagger$. Interpret the evolution in the language of Bloch sphere transformation, where the general mixed state of a qubit can be parameterized using a three dimensional vector $\vec{n}$: $\rho = 1/2(\mathbb{1} + \vec{\sigma} \cdot \vec{n})$, where $|\vec{n}| \leq 1$, and $\vec{\sigma}$ is a vector consisiting of Pauli matrices

b) Write down measurement operators Π_0, Π_1 acting on system S corresponding to projecting the "measuring device" M on states $|0\rangle_M$, $|1\rangle_M$.

c) Consider a general qubit state $|\psi\rangle$ parameterized using angles θ, φ on the Bloch sphere. Write down probabilities of obtaining measurement results that correspond to measurement operators Π_0, Π_1, and the respective post-measurement states of the qubit S.