Quantum Estimation and Measurement Theory

Problem set 2

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Model of joint position and momentum measurement.- Consider a particle S travelling in one dimension, with which we associate position and momentum operators (dimensionless) $\hat{x}_S$, $\hat{p}_S$, satisfying $[\hat{x}_S, \hat{p}_S] = \mathrm{i}$. Initially the particle is in state $|\psi\rangle_S$. Consider a joint position and momentum measurement where particle S interacts with two "measuring devices" M_1 i M_2 through a unitary evolution:

$$|\Psi\rangle_{SM_1M_2} = U|\psi\rangle_S \otimes |0\rangle_{M_1,M_2}, \quad U = e^{-\mathrm{i}(\hat{x}_S\hat{p}_{M_1} - \hat{p}_S\hat{x}_{M_2})}, \tag{1}$$

where $|0\rangle_{M_1,M_2}$ is the initial state of the measuring devices. After the action of U, position (x_{M_1}) and momentum (p_{M_2}) is measured of respectively systems M_1 and M_2 (these measurements commute!). As a result of measurement we obtain a certain joint probability distribution of measuring position and momentum $J(x,p)$ on state $|\psi\rangle_S$.

a) Using the Heiseneberg picture, evolve measurement operators x_{M_1}, p_{M_2} so that you act with them directly on the input state $|\psi\rangle_S \otimes |0\rangle_{M_1,M_2}$—let us call the evolved operators as $\tilde{x}_{M_1}$, $\tilde{p}_{M_2}$

b) Consider operators $\delta\hat{x} = \tilde{x}_{M_1} - \hat{x}_S$ i $\delta\hat{p} = \tilde{p}_{M_2} - \hat{p}_S$, which can be regarded as operators representing the difference of the operators actually measured and the ideal measurement. Inspecting the structure of $\delta\hat{x}$, $\delta\hat{p}$ what state $|0\rangle_{M_1,M_2}$ you would choose so that the joined measurement be as close as possible to ideal position and momentum measurements and would not distinguish any of them—Hint: calculate, how much the variance of the measurement will be enlarged...

c) [Difficult] Using state $|0\rangle_{M_1,M_2}$ found above, proof that the set of POVM operators corresponding to the above described model of joined measurements $\Pi_{x,p}$ [so that$J(x,p) = \mathrm{Tr}(|\psi\rangle\langle\psi|\Pi_{x,p})$] has the following form:

$$\Pi_{x,p} = \frac{1}{2\pi}|(x,p)\rangle\langle(x,p)|, \quad |(x,p)\rangle = \frac{1}{\pi^{1/4}}\int \mathrm{d}x'\, e^{\frac{-(x'-x)^2}{2}} e^{\mathrm{i}px'}|x'\rangle, \tag{2}$$

where $|(x,p)\rangle$ is the so called coherent state with mean value of position and momentum equal x and p respectively. Therefore, we have a nice interpretation of the joined position and momentum measurements as projections on coherent states:

$$J(x,p) = \frac{1}{2\pi}|\langle\psi|(x,p)\rangle|^2 \tag{3}$$

Remark: in quantum optics, the above probability distribution is called the Hussimi representation.