Quantum Estimation and Measurement Theory

Problem set 4

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Problem 1 Consider N i.i.d binary valued random variables $x_i \in \{0,1\}$ $(i = 0, \ldots, N-1)$, where $p(x_i = 0) = p$, $p(x_i = 1) = 1 - p$. Consider the problem of estimating parameter p. (Hint: To simplify further calculations, note that what is really relevant in the observed events is the number a zeros and ones in N realizations and not the order in which they appeared).

a) What does the Cramér-Rao (CR) bound tells us concerning the best achievable precision of estimating p?

b) Is CR bound saturable for finite N? What is the optimal estimator?

c) Does this family of probability distributions belong to the so called exponential family (see Problem 2 in Problem set 3)?

d) Imagine, that in fact $p = \sin^2(\theta/2)$, where $\theta \in [0, \pi]$ and we are actually interested in estimating θ, and not p itself. Derive the CR bound for estimating θ.

e) This time, there is no estimator that saturates the CR bound (check it) for finite N. We can, however, try to use the maximum-likelihood (ML) estimator in order to estimate θ and check whether we can approach the CR bound bound in the limit of large number of experiment repetitions. Proceed as follows:

- Write a program, generating N i.i.d. realizations of random variable x_i, such that $p(x_i = 0) = \sin^2(\theta/2)$, $p(x_i = 1) = \cos^2(\theta/2)$, for some fixed θ (e.g. $\pi/3$, $\pi/2$, $2/3\pi$) and some fixed N (e.g. $N = 10$). Such a sample of N numbers we will call a single realization of the experiment.
- Generate data for k ($k \approx 1000$, or more) experiments
- For each experiment, find the ML estimator $\tilde{\theta}_{\mathrm{ML}}$
- Plot a histogram of obtained values of ML estomator and calculate the spread of the results (standard deviation)— this will be a good approximation of the esimator uncertainty $\Delta\tilde{\theta}$. Compare with the CR bound.
- Repeat above steps for different N, e.g. in the range of 1 to 10000 (of course not for all N but only some representative ones). Generate a plot: estimator uncertainty vs. N and compare it with the CR bound to draw a conclusion concerning the regime where we can claim asymptotic saturation of the CR bound (e.g. you can assume a criterion, that we look for such an N when we are within 1% from the CR bound). Hint: For clarity, it is better to plot $\Delta\tilde{\theta}\sqrt{N}$, rather than $\Delta\tilde{\theta}$, and compare with CR bound for a single realization.