Analysis of interevent times by methods of statistical physics

DOCTORAL DISSERTATION BY

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Abstract

The problem of excessive losses is significant in economic theory and investment practice, as well as for random processes analysis. One of the recently discovered characteristics of losses and profits on financial markets, which was discovered by Bogachev, Bunde, Ludescher, and Tsallis and is not entirely explained yet, is the universality of the distribution of times between losses (or profits) exceeding a given threshold value. In other words, this distribution does not depend on the underlying asset type or the time resolution of data, but rather only depends on the particular threshold height. Interestingly, similar results were obtained, e.g., in geophysical time series (describing the earthquakes exceeding a particular magnitude). In this thesis I present a thorough description of this universality, employing two complementary approaches: (i) an analytical approach, based on the extreme value theory (EVT) and the continuous-time random walk (CTRW) model, and (ii) a numerical approach, based on the Potts model from statistical mechanics. The thesis makes original contributions to the field of knowledge with the following: (i) an analytical model of the aforementioned universality, with a thorough empirical verification for losses and some proposed applications to value-at-risk simulation, profits description, and geophysical data description, and (ii) an agent-based spin model of financial markets with the novel interpretation of the spin variable (as regards financial-market models), reproducing the main empirical stylized facts, such as the shape of a usual and an absolute-value autocorrelation function of the returns as well as the distribution of times between superthreshold losses. These results extend the knowledge and understanding of stochastic processes, agent-based modeling, financial markets and geophysical systems.

Acknowledgments

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Contents

1

Introduction

The first chapter of the present thesis is intended to prepare the reader to comprehend its essential content. First, some initial remarks about the whole field are provided (Sec. 1.1), then the problem under consideration is formulated and placed in a broader scientific context, with several terms that are used later in the thesis being introduced (Sec. 1.2), and finally the framework of the whole thesis is presented (Sec. 1.3).

1.1 Econophysics

The traditional goal of physics until the 20$^{\text{th}}$ century was to explain the basic properties of matter. By this means, physicists managed to describe the natural processes and phenomena occurring on all existing scales, from the quantum level to the level of the whole universe. Social scientists, in turn, were concerned with issues regarding human interactions that, to a certain extent, are independent of the physical environment in which they occur[1].

However, there were always some common characteristics that physicists shared with social scientists, e.g. (after Borrill and Tesfatsion (2011); cf. Jovanovic and Schinckus (2017))

- an interest in understanding the complicated interactions of entities composed of more elementary entities, behaving in accordance with potentially simpler rules;

[1] Based on Borrill and Tesfatsion (2011).

- the need to account for multiple observers with different perspectives on reality, measurements of whom necessarily entail perturbative interactions with persistent (information flow) traces;
- systems of interest that can display what social scientists refer to as "path dependencies", i.e., dependencies on historical conditions.

In the second half of the 20$^{\text{th}}$ century, physicists filled the gap between physics and the social sciences, and today two main research fields that use the methods, methodologies and ideas of physics for the description of social interactions, namely econophysics and sociophysics, do in fact exist, as kinds of hybrid sciences.

As regards econophysics, it is already a fairly well-developed branch of the physical sciences that applies some of the ideas, models and methods of physics to economic and financial phenomena (Schinckus, 2016; Jovanovic and Schinckus, 2017). The methods from statistical physics (Stauffer, 2000), the physics of complex systems (as the economy or its parts can be treated as large complex systems, see Kwapień and Drożdż (2012) and refs. therein), the physics of random walks (Scalas, 2006), or chaos theory (Grech and Mazur, 2004; Grech and Pamuła, 2008; Oświęcimka et al., 2013) have been widely explored by econophysicists, with great success (see also Sec. 1.1.1 below).

The history of modern econophysics began in the last decade of the 20$^{\text{th}}$ century with the works of such physicists as Rosario N. Mantegna (Italy), H. Eugene Stanley (USA) (Mantegna et al., 1995), Didier Sornette (Switzerland) and Jean-Philippe Bouchaud (France) (Sornette et al., 1996). The name of this interdisciplinary field was coined over 20 years ago by Stanley et al. (1996a)[2]. However, the connections between the physical and economic sciences are much older. Economics, despite belonging to the family of the social sciences, always had strong connections with mathematics and some similarities with physics. Suffice it to say that the first statistical model of financial markets, made by the French mathematician, Louis Bachelier, at the turn of the century (Bachelier, 1900), contained an equivalent of the physical model of Brownian motions that was introduced five years later by Albert Einstein and, independently, six years later by Marian Smoluchowski (Einstein, 1905; Smoluchowski, 1906)[3].

When it comes to the present day, according to Schinckus (2016); Jovanovic and Schinckus (2017), econophysics is developing within, roughly speaking, three basic "streams", reflecting the ways of conceptualizing micro-macro interaction:

[2] Compare with Schinckus (2016); Jovanovic and Schinckus (2017).

[3] For more information about the "prehistory" of physical economics and its contributors, such as Vilfredo Pareto or Benoît Mandelbrot, see e.g., Sornette (2014); Roehner (2002).

(i) statistical econophysics (or the original econophysics) that seeks some statistical macropatterns in the macroscale empirical data (cf. Part I of the present thesis); (ii) bottom-up agent-based econophysics reproducing the observed macro-order from simulations based on some assumed microscale mechanics (cf. Part II); and (iii) top-down agent-based econophysics, which determines the parameters of the aforementioned micromechanics from certain macroscale observations.

1.1.1 Main achievements

The dialogue between some distant scientific fields has frequently yielded significant results[4]. Econophysicists, despite the fact their research field has only formally existed for two decades, can also boast of their first successes, as listed below (after Daniel and Sornette (2010)):

- scaling, power laws, "universality" discovered on markets (Mantegna et al., 1995; Cont et al., 1997; Mandelbrot and Stewart, 1998; Gopikrishnan et al., 1999; Stanley, 1999);
- the theory of large price fluctuations (Gabaix et al., 2003);
- agent-based models, induction, evolutionary models (Lux and Marchesi, 1999; Cont and Bouchaud, 2000b; Farmer, 2002; Arthur, 2006);
- the option theory for incomplete markets (Bouchaud and Sornette, 1994; Bouchaud and Potters, 2003);
- interest rate curves (Bouchaud et al., 1999; Santa-Clara and Sornette, 2001);
- minority games (Challet et al., 2013);
- the theory of Zipf law and its economic consequences (Gabaix, 1999, 2011; Saichev et al., 2009);
- the theory of bubbles and crashes (Lux and Sornette, 2002; Sornette, 2003);
- the random matrix theory applied to the covariance of market returns (Laloux et al., 1999);
- the methods and models of dependence between financial assets, and in particular network theory (Malevergne et al., 2003).

[4]For instance, artificial skin is the fruit of collaboration between surgeon and material researcher from MIT. See Burke et al. (1981).

In addition to the above, one of the latest significant achievements includes systemic risk issues (Haldane and May, 2011; Helbing, 2013; Smaga et al., 2016). The list is clearly impressive and testifies to the importance of econophysics as a branch of the physical sciences, though many questions within its field of interest still remain open. One of them, being a subject of this thesis, is presented in the next section (1.2).

1.2 The problem formulation

Financial markets are one of the most important components of the economy, since they constitute a "circulatory system" for all other markets, by providing them with money. Most typical examples of such markets are the New York Stock Exchange (NYSE), the foreign exchange market (Forex) and the Chicago Board of Trade (CBOT). Their role nowadays is rapidly increasing. More and more people are trying their hand at investing in various assets, from classic (but still, risky) stocks to rare and exotic financial assets, such as freight or weather derivatives[5]. Obviously, financial markets are interesting not only due to the possibility of earning money, but also from a scientific point of view.

A fundamental characteristic of the financial markets, and of markets in general, is the activity of the investors operating on them. Obviously, any activity is inherently associated with a risk; however, the bigger the risk, the higher the expected profits from the investment are. Therefore the profits, as well as losses (i.e., negative profits; the more risk we take, the larger both of them can be) are unavoidable features of any well-functioning market. Analysis of profits and losses (so-called *risk analysis*) is a key issue for any investor and many market researchers. **The present thesis is devoted to a phenomenological analysis of excessive losses and profits on financial markets, employing the advanced methods of statistical physics.** Profit and loss are treated here as random variables. The thesis is based mainly on the papers Denys et al. (2013, 2014, 2016a,b), concerning its two essential parts (see Sec. 1.3).

A significant question in the analysis of losses in financial-market time series, closely related to the economic concept of value at risk (see Sec. 2.1.2 of the next chapter), is the description of the time distance between the subsequent losses of a particular magnitude, i.e., extending some given threshold value. **The goal of the present thesis is to present a consistent description of this problem based on – and then also confirmed by – the empirical data.**

[5]Contracts in which payoff depends on the costs of transporting goods or weather conditions.

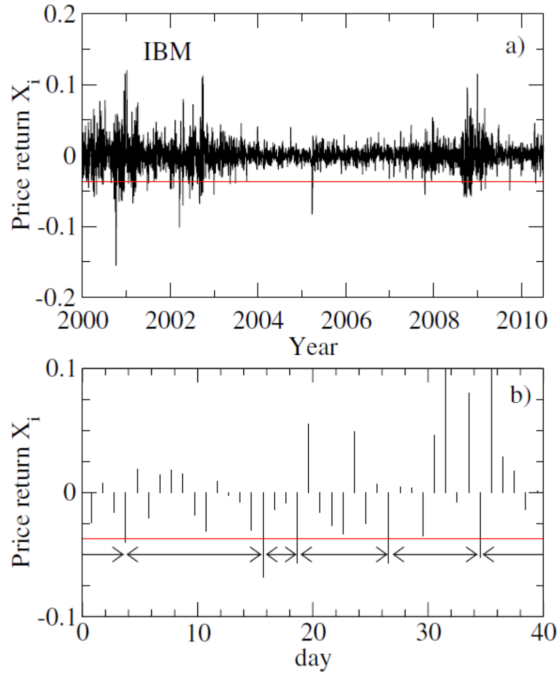

Figure 1.1: The relative IBM daily returns X_i for the periods January 2000–June 2010 (the upper part) and August 27–October 23, 2002 (the lower part). The red line is a threshold $Q = -0.037$ or -3.7%, corresponding to the mean quantile time between losses $R_Q = 70$ (cf. its definition in Sec. 3.2, Chapter 3). The arrows represent the interevent times, defined in Sec. 1.2. Reprinted from Ludescher et al. (2011).

Particularly, the thesis concerns the universality discovered in empirical data by Bogachev, Bunde, Ludescher, and Tsallis (Bogachev and Bunde, 2008, 2009; Ludescher et al., 2011; Ludescher and Bunde, 2014) related to the appearance of excessive profits and losses. They have shown that in a financial time series of market returns[6] the shape of the distribution of times, $\Delta_Q t$, between profits or losses[7] which exceeds some fixed threshold value, Q (cf. Fig. 1.1), depends only on this threshold, but not, e.g., on an underlying asset type or the time resolution of data, as is shown in Fig. 1.2.

[6]That is, relative changes of price of a particular asset.
[7]That is, positive or negative returns.

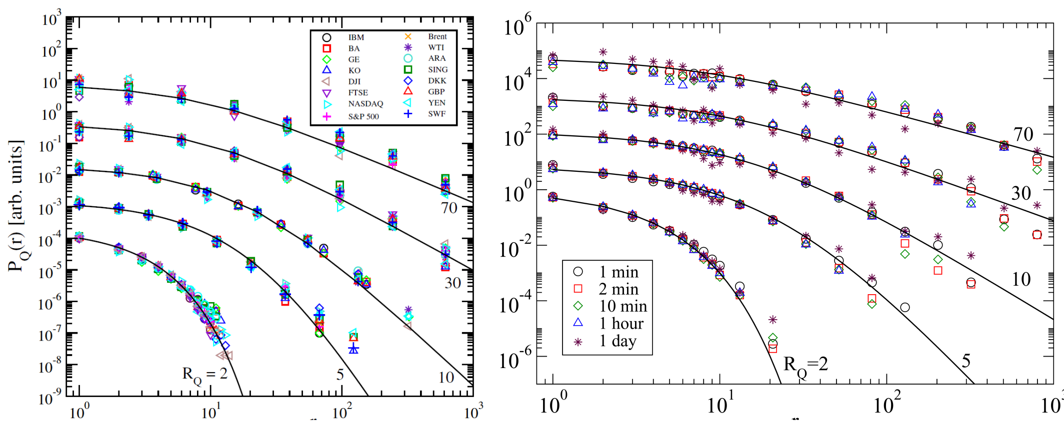

Figure 1.2: The empirical distributions, $\psi_Q(\Delta_Q t)$, of interevent times, $\Delta_Q t$ (colored marks) for the relative daily price returns of sixteen typical financial assets in the period 1962–2010 (on the left) and for time scales from minutes to days for NASDAQ between March 16, 2004 and June 5, 2006 (on the right). The solid black curves are q-exponential fits (cf. Eq. (3) in Ludescher et al. (2011) and Sec. 1.2.1). Reprinted from Ludescher et al. (2011) and Ludescher and Bunde (2014).

This type of empirical-data property, i.e., when the data can be described by one single curve, is called *data collapse*. Actually, in this case the collapse is partial, as it possesses one "degree of freedom", which is the value of the threshold Q or, alternatively, a mean value, R_Q, of the times between losses greater than Q (see Ludescher et al. (2011) and Sec. 3.2 in Chapter 3). The times $\Delta_Q t$ are later on referred to as *interevent times*[8], and their distribution, $\psi_Q(\Delta_Q t)$, is called *interevent-time distribution* or *statistics*.

The universality mentioned above constitutes one of the most recently noticed *stylized facts*, that is, characteristic empirical features of economic data[9]. Thus, this thesis is based on essential empirical findings that were discovered less than ten years ago. Moreover, similar results were also obtained in geophysical data for the times between earthquakes exceeding a particular magnitude on the Richter scale (Corral, 2004, 2003), or biological data for the intervals between the same nucleotides (A-A, C-C, G-G, T-T) in the DNA sequence (Bogachev et al., 2014;

[8]The term "interevent time" can be found in the source literature under such names as "pausing time", "waiting time", "intertransaction time", "intertrade time", or "interoccurrence time" in different versions of the CTRW formalism (Kehr et al., 1981; Bogachev et al., 2007; Perelló et al., 2008; Kasprzak et al., 2010; Gubiec and Kutner, 2010; Sandev et al., 2015).

[9]For more information about stylized facts on financial markets, see e.g., Cont (2001).

Tsallis, 2016). Therefore, **the direct purpose of the thesis is a universalized description of large number of empirical data coming from economic as well as noneconomic sources, and concerning the interevent-time universality.**

The research questions for the present thesis are as follows:

1. How to describe the aforementioned universality analytically/numerically?
2. Is the model of cunning agents (presented initially in Denys et al. (2013) for the numerical description of financial-market characteristics) able to describe the aforementioned stylized fact?
3. What are the differences (if any) between the description of excessive profits and excessive losses?
4. Can we create a formalism that is able to cover nonfinancial data?

My research hypotheses (corresponding to the above questions) are as follows:

1. The aforementioned analytical and (complementary) numerical description of the problem can be provided under the methods of statistical physics.
2. The numerical description can be provided by the model of cunning agents.
3. Excessive profits can be described in the same way as excessive losses.
4. The developed formalism may also be applied, in the spirit of interdisciplinarity, to the description of geophysical interevent-time universality, presented by Corral (2004, 2003).

The general aim of the thesis is to study the universality of the interevent-time distribution with the methods of statistical physics, while the specific purpose is to provide an analytical and numerical description of the phenomenon that would be as fundamental as possible, then to verify this description with the available empirical data, and finally to interpret the obtained results.

Obviously, finding a good analytical or numerical description of the problem, apart from the scientific importance, could have practical application, for example in investing or insurance (to know how much time is going to elapse before the next loss of a particular magnitude), or in disaster management (to know when to expect the next severe earthquake).

1.2.1 The current state of knowledge

Studies on the universal features of varied markets are the mainstream of econophysics and its initial field of interest. The work of Mantegna (1991)[10], based on the Milan Stock Exchange, proved that the statistics of price changes (which are similar to market returns) for contemporary indices and stocks are described, in most time scales, by a Lévy distribution instead of a canonical Gaussian distribution (cf. Feller (2008)). Hereby, power laws with all their consequences (e.g., breaking of the central limit theorem, scaling laws, hierarchies, etc.) were introduced to market studies. One of the most important works of Mantegna et al. (1995) indicated the existence of data collapse in financial-market data, for the S&P500 index[11]; its extension to the Asiatic markets was provided by Wang and Hui (2001). Finally, more exact and systematic analysis was provided by Kiyono et al. (2006). The authors have shown that the central limit theorem is obeyed far away from market crashes[12], but it is broken in time periods containing a crash (and the preceding market bubble[13]). Consequently, a generalization of the central limit theorem, which allows a divergence of a variance of returns, is required.

As far as the description of interevent times between excessive profits or losses is concerned, Bogachev and Bunde (2008) first used the mean interevent discrete time, R_Q (see Sec. 3.2 in Chapter 3) as a control variable that provides a universal description of empirical-data collapse. The same authors (Bogachev and Bunde, 2009) proposed a risk-estimation method for interevent times. Ludescher et al. (2011) provided a semiempirical formula for interevent-time statistics, based on Tsallis q-exponential functions (predictions of this are shown in Fig. 1.2). Ludescher and Bunde (2014) confirmed that interevent times constitute a universal stochastic measurement of market activity on time scales that range from minutes to months (cf. the right part of Fig. 1.2).

As regards the q-exponential formula for interevent-time statistics, the authors have not provided a derivation of their analytical result. According to Tsallis (2016) the parameters of the q-exponential fits have their origin in the microscopic dynamics of the system, similarly to how the parameters of the planets' ellipses in the Solar System depend on their history. However, this comparison appears

[10] Preceded by the pioneering work of Mandelbrot (1963).

[11] Standard & Poor's 500, in the United States, a stock market index that tracks 500 publicly traded domestic companies. It is considered by many investors to be the best overall measurement of American stock market performance (after Encyclopædia Britannica).

[12] That is, sudden dramatic declines of stock prices; cf. Grech and Mazur (2004); Czarnecki et al. (2008).

[13] That is, trade in assets at prices that strongly deviate from the assets' intrinsic values.

to be inexact, as intuitively, the phenomenon of interevent times should depend more on the current state of the market than on its history.

A competing description of interevent times, for using q exponentials, was provided in a systematic way by Perelló et al. (2008) under the assumptions of the stochastic model of a *continuous-time random walk* (or CTRW, see Sec. 2.2 in the next chapter), being a reinterpreted and generalized random-walk valley model. The model, in its canonical version, was created for a description of non-Debye, including the power-law, relaxation of photocurrents in amorphous materials (Scher and Montroll, 1975; Pfister and Scher, 1978; Haus and Kehr, 1987; Weiss, 2005) or organic light emitting diodes (Campbell et al., 1997). Reinterpreted versions of CTRW formalism have been applied to the description of many other phenomena, including processes with memory, e.g., on financial markets – it is a significant trend in econophysics (Scalas, 2006; Kutner and Masoliver, 2017). By the use of the CTRW model, it was shown that the multifractal structure of the interevent times on financial markets (Perelló et al., 2008; Kasprzak et al., 2010) and a single-step memory in the order-book dynamics (Gubiec and Kutner, 2010) are foundational in the analysis of double-auction market activity.

However, the description of the excessive losses provided by Perelló et al. (2008) was made under the assumption of the exponential distribution of the market returns, which is not always true, and without an extensive empirical-data comparison. Therefore, **we still need a convincing and complete explanation of the interevent-time universality, which is as simple as possible, and analytical as well as numerical. The proposition of such an explanation is presented in the present thesis**.

1.3 The thesis framework and scope

The description of the interevent-time universality presented above is provided here through two complementary approaches, corresponding to the two subsequent parts of the thesis.

The analytical approach, based on the continuous-time random walk model (CTRW) and the *extreme value theory* (EVT), and an alternative to the one based on the Tsallis q-exponential function from Ludescher et al. (2011) is discussed in the first part of the thesis. In Chapter 2 some introduction to the CTRW and EVT is given, while in Chapter 3 an analytical model of interevent times, based on these two approaches, is presented. **The analytical model of interevent times, together with its empirical verification and proposed applications, constitute an original contribution to the field of**

knowledge presented in the first part of the thesis. These results are based on the papers Denys et al. (2016a) and Denys et al. (2016b).

The numerical approach, based on agent-based modeling and the Potts model from statistical mechanics is discussed in the second part of the thesis. First, in Chapter 4 the concept of agent-based modeling is outlined and some significant economic and econophysical models are presented. Secondly, in Chapter 5 our own agent-based model of financial markets is described, with particular emphasis on its suitability to reproduce the interevent-time universality depicted in Sec. 1.2. **The model, using a novel spin-value interpretation and reproducing, i.a., the empirical shapes of the autocorrelation function of market returns and absolute returns, as well as the interevent-time universality, constitutes an original contribution to the field of knowledge presented in the second part of the thesis.** The model description is based on the papers Denys et al. (2013) and Denys et al. (2014).

Finally, in Chapter 6 some concluding remarks and a review of the whole work, together with some perspectives for further research, are provided. Additionally, Appendix presents a derivation of the final formula of the analytical model from Chapter 3.

As will be seen, the available empirical data of interevent times are analyzed in the thesis from the viewpoint of stochastic processes (specifically, the CTRW model; see Part I) and agent-based models (that provide slightly better insight into microscopy; see Part II). In other words, **we have two views on the same empirical data.** Thus, the present thesis concerns both statistical econophysics (Part I) as well as bottom-up agent-based econophysics (Part II; cf. Sec. 1.1).

Part I

CTRW approach

2

Selected issues of extreme value theory and continuous-time random walk formalism[1]

This chapter is intended to be a mathematical introduction to the subsequent one (Chapter 3) presenting a key model of this thesis, namely the model of interevent-time superstatistics. It contains the main assumptions and results of the extreme value theory (Sec. 2.1) and the continuous-time random walk model (Sec. 2.2), which are necessary to create the model of superstatistics. This introduction should provide a better understanding of the relevant model and put the problem I address in a broader statistical context.

2.1 Statistics of extremes

A broad range of real systems is characterized by relatively rare extreme events, which may dominate their long-term behavior. Frequently, some central and fluctuating values are insufficient for their satisfactory description. For such cases one may consider a statistics of extreme values. This is a significant field of interest for scientists, e.g., mathematicians, physicists, geophysicists, economists, or sociologists (Scher and Montroll, 1975; Metzler and Klafter, 2000; Sornette, 2002; Zumofen et al., 2012). As far as physics is concerned, the studies on the statistics of extreme events have involved e.g., self-organized fluctuations and critical phenomena (Bak et al., 1987), material fracture, disordered systems at low tem-

[1]Based on Kozłowska and Kutner (2005); Kutner (2016).

peratures, or turbulence (see Moreira et al. (2002) and refs. therein). Once we know the extreme-event statistics, we are able to calculate risk estimators, which are especially useful for the prediction of earthquakes (Sornette and Sornette, 1989), changes in climate conditions or floods (Albeverio et al., 2006), as well as financial crashes (Kozłowska et al., 2016). Recently, the statistics of extremes have also been applied also in network science (Moreira et al., 2002).

Extreme value theory (or *theory of extreme values*, EVT) is a branch of statistics that deals with extreme events. The foundation of this theory lies in the *extremal types theorem* (or *three types theorem*), which states that the statistics of maximal or minimal value from a set of independent and identically distributed random variables has to be exclusively one of three possible distribution types, namely the Gumbel, Fréchet, or Weibull distribution (see below for the details). These distributions, described overall by the *generalized extreme-value distribution* (GEVD), are commonly used in economics (and particularly in risk management, finance, or insurance), hydrology, material sciences, telecommunications, and in many other fields where we deal with extreme events.

A central question of the extreme value theory is to characterize the maximum value $x_n^{max} := max\{x_i\}_{i=1,\dots,n}$ from a set of realizations x_i of some random variable X in a stochastic process[2]. An example of such a maximum value, relevant for this thesis, may be the biggest price change of some financial asset in a fixed time period. In the frame of EVT, the circumscription of x^{max} value is accomplished by providing a probability distribution $P(x^{max} = \Lambda)$ of x^{max}, where Λ indicates some arbitrary threshold.

2.1.1 The maximum limit theorem

The pillar of EVT is the *maximum limit theorem* or the *Fisher–Tippett–Gnedenko theorem* (Fisher and Tippett, 1928; Gnedenko, 1943). For a continuous random variable it takes the form:

Theorem 1 (Maximum limit theorem) *Let $(x_1, x_2, \dots, x_n)$ be a sequence of n independent and identically distributed continuous random variables and $x_n^{max} = max(x_1, x_2, \dots, x_n)$. If there exists a sequence of constants $a_n > 0$ and b_n, $n = 1, 2, \dots$, such that $(x_n^{max} - b_n)/a_n$ has a nondegenerate limit cumulative distribution G as $n \to \infty$, i.e.,*

$$\lim_{n\to\infty} P\left(\frac{x_n^{max} - b_n}{a_n} \leqslant y\right) = G(y), \tag{2.1}$$

[2]We assume here that this variable is continuous.

then

$$G(y) = \exp(-(1+\gamma y)^{-1/\gamma}),\ 1+\gamma y > 0, \tag{2.2}$$

and the parameter γ determines the shape of the tail of the cumulative distribution given above. This distribution, called the generalized extreme-value distribution *(GEVD), belongs to one of the following classes (or types):*

(i) Gumbel distribution (type I, $\gamma = 0$):

$$G(y) = \exp\left(-\exp\left(-y\right)\right),\ y \in \mathbf{R}. \tag{2.3}$$

(ii) Fréchet distribution (type II, $\gamma > 0$):

$$G(y) = \begin{cases} 0, & y \leqslant 0, \\ \exp\left(-y^{-1/\gamma}\right), & y > 0. \end{cases} \tag{2.4}$$

(iii) Weibull distribution (type III, $\gamma < 0$)[3]*:*

$$G(y) = \begin{cases} \exp\left(-\left(-y\right)^{-1/\gamma}\right), & y < 0, \\ 0, & y \geqslant 0. \end{cases} \tag{2.5}$$

Obviously, the functions given above are CDFs (cumulative density functions) of the aforementioned probability distributions. The Weibull and the Fréchet functions, although having very similar forms, provide distributions with essentially different shapes, due to the opposite sign of their shape exponents γ.

Once we know the GEVD and the three possible distributions it comprises, a basic task of EVT is to determine the parameters a, b and the shape parameter γ from the empirical data.

2.1.2 Value at risk and generalized extreme-value distribution

One of the most significant outcomes of the extreme value theory is the calculation of *value at risk* (VaR), a common measure of the risk in a particular investment.

Value at risk, VaR, is generally defined as the maximum loss one can consent in the investment. In practice, this value is specified using a probability p of loss

[3]This is in fact a so-called *inverse Weibull distribution*. The usual Weibull distribution represents the statistics of the minimal value.

not exceeding this (threshold) value. In other words, when we set the confidence level at $1-\alpha$, we can write

$$F(VaR_{1-\alpha}) = \mathcal{P}(x \leqslant VaR_{1-\alpha}) = 1-\alpha, \tag{2.6}$$

where F is a cumulative distribution function of losses.

We can easily determine the CDF of the extreme value now, bearing in mind that we consider only the values of random variable x not exceeding the threshold value $VaR_{1-\alpha}$. Namely, for n large enough, from Theorem 1,

$$G\left(\frac{VaR_{1-\alpha}-b}{a}\right) \approx \mathcal{P}\left(\frac{x_n^{max}-b_n}{a_n} \leqslant \frac{VaR_{1-\alpha}-b}{a}\right) \approx \mathcal{P}(x_n^{max} \leqslant VaR_{1-\alpha}) = \\ = \prod_{i=1}^{n} \mathcal{P}(x_i \leqslant VaR_{1-\alpha}) = [\mathcal{P}(x \leqslant VaR_{1-\alpha})]^n = (1-\alpha)^n, \tag{2.7}$$

where $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Clearly, the use of nth power in the final result is a consequence of the fact that each of the $n \gg 1$ values drawn from the sequence $(x_1, \ldots, x_n)$ has to be less than the set threshold value $VaR_{1-\alpha}$. Substituting (2.2) to the left side of the above equation results in, after some simple transformations, the final formula:

$$VaR_{1-\alpha} \approx b + \frac{a}{\gamma}\left((-n\ln(1-\alpha))^{-\gamma} - 1\right). \tag{2.8}$$

2.2 The continuous-time random walk model[4]

The *continuous-time random walk* (CTRW) model was introduced by Montroll and Weiss (1965) to describe anomalous diffusion, a generalization of an ordinary diffusion process with nonlinear time dependence of variance. Experiments made by Scharfe (1970), Gill (1972) and Pfister (1974) demonstrated the existence of anomalous diffusion in amorphous solids. The well-known *valley model* proposed by Scher and Montroll (1975) describes successful random jumps of a particle between regularly distributed potential wells of different depths using CTRW formalism (cf. Fig. 2.1). Their approach have been extended to the interevent-time description (Perelló et al., 2008; Kasprzak et al., 2010; Kasprzak, 2010) considered in this thesis. In this case, the amount of loss is an analog of the potential well depth. Some other applications of the CTRW model in finance

[4]Based on Weiss (2005); Metzler and Klafter (2000).

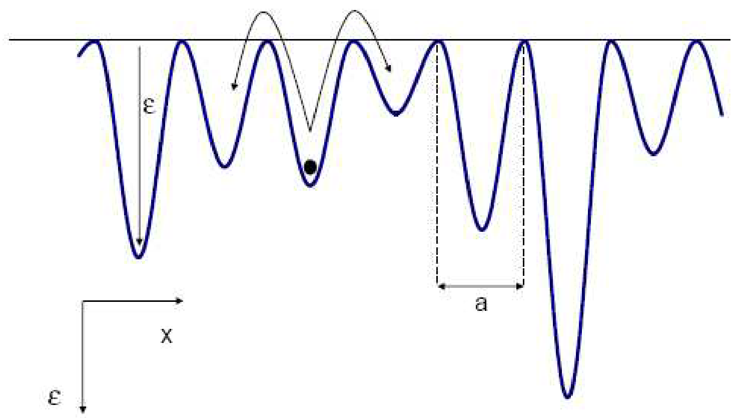

Figure 2.1: The wandering of a particle (represented schematically by black circle) in a valley potential (the schematic navy curve) with no external field. The particle makes jumps over the barrier of the potential (see the curved arrows). Taken from Kasprzak (2010).

were presented e.g., in Scalas et al. (2000); Mainardi et al. (2000); Gubiec (2011); Wiliński (2014). The canonical version of the CTRW model was also applied in description of aging of glasses (Monthus and Bouchaud, 1996; Barkai and Cheng, 2003), hydrology (Boano et al., 2007; Klages et al., 2008), or earthquakes studies (Helmstetter and Sornette, 2002). Despite being 55 years old, the model is still of significant interest (cf. Fig. 2.2 and Kutner and Masoliver (2017)).

The main advantage of the CTRW model is its ability to describe a random walk at any moment of time. Actually, the introduction of the formalism possessing such characteristics is a milestone in the description of random processes. It broaden the scope of possible processes from Gaussian ones (with the central limit theorem preserved) to a wide range of non-Gaussian processes, in particular processes with memory and Lévy processes (Metzler and Klafter, 2000).

The basic assumption of the model is that both the jump lengths x and the times between succeeding jumps t are drawn from a joint probability distribution, $\psi(x,t)$. Hence, the distribution of jumps is

$$p(x) = \int_0^\infty \psi(x,t)\,\mathrm{d}t, \tag{2.9}$$

while the waiting-time distribution

$$\psi(t) = \int_{-\infty}^\infty \psi(x,t)\,\mathrm{d}x. \tag{2.10}$$

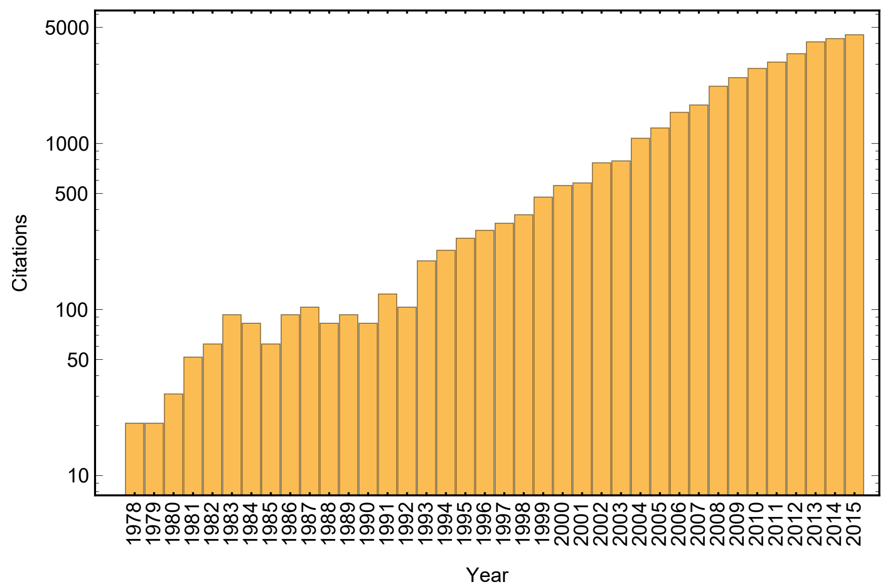

Figure 2.2: CTRW citations from 1978 to 2015 in semi-log scale (based on the information from the Web of Science).

In the following, we assume the random variables x and t are independent, that is, the jump vectors and the time intervals between jumps are uncorrelated. Hence, the joint distribution factorizes,

$$\psi(x,t) = \psi(t)p(x). \tag{2.11}$$

This version of the CTRW model is sometimes called a decoupled CTRW.

Using relation (2.11) one can obtain the probability $\mathcal{P}(x,t)$ of a wandering particle being located in the position x at time t. Although obtaining a general expression for $\mathcal{P}(x,t)$ in a closed form is impossible, we can consider, instead, the Laplace–Fourier transform of it, $\hat{\mathcal{P}}(k,s)$, which is much more useful.

Namely, if we assume that $p_n(x)$ is the PDF (probability density function) of x after n jumps, we may represent $\mathcal{P}(x,t)$ as an infinite series,

$$\mathcal{P}(x,t) = \sum_{n=0}^{\infty} p_n(x) \int_0^t \psi_n(\tau)\Psi(t-\tau)\,\mathrm{d}\tau, \tag{2.12}$$

where $\psi_n(\tau)$ is the probability that the nth jump occurred in time τ, while the CCDF (complementary cumulative density function) $\Psi(t-\tau) = \int_{t-\tau}^{\infty}\psi(t')\,\mathrm{d}t'$ assures us that there was no other jump until $t > \tau$. Remarkably, the CTRW is a non-Markovian model in general; the only exception is a negative exponential case, i.e.,

$$\psi(t) = \frac{1}{T} \exp\left(-\frac{t}{T}\right), \tag{2.13}$$

where T is the mean interevent time. When condition (2.13) is fulfilled, the probability of the next jump does not depend on the particle's wandering history (a process without a memory; cf. Eq. (3.10) from the next chapter).

The Laplace–Fourier transform of Eq. (2.12) is

$$\hat{\mathcal{P}}(k, s) = \frac{1 - \hat{\psi}(s)}{s} \sum_{n=0}^{\infty} \hat{p}^n(k) \psi^n(s) = \frac{1 - \hat{\psi}(s)}{s[1 - \hat{p}(k)\hat{\psi}(s)]}, \tag{2.14}$$

as, from the definition of the characteristic function, $\hat{p}_n(k) = \hat{p}^n(k)$ and $\hat{\psi}_n(s) = \hat{\psi}^n(s)$. This formula may be used to derive the Laplace transform of the moments[5].

* * *

The continuous-time random walk model given above and the highlights of the extreme value theory are used in the next chapter (3) to derive the analytical model of interevent times in financial and nonfinancial time series. This is a basic achievement of this thesis and its first original contribution to the field of knowledge.

[5]However, formula (2.14) is nonstationary, as exactly at $t = 0$ the particle jumps to $x = 0$. To make it stationary we may introduce an individual separate waiting-time distribution for the first jump. For more details see Gubiec (2011) and refs. therein.

3

An analytical model of interevent times[1]

3.1 Introduction

A goal of the present chapter is to provide a consistent analytical description of the problem of interevent times formulated in Chapter 1 and to verify it thoroughly using empirical data. The chapter constitutes the most significant part of the thesis. The model was presented previously in Denys et al. (2016a,b) as an alternative to the approach based on the Tsallis q-exponential functions from Ludescher et al. (2011).

The overall strategy of our approach is to combine in one expression two basic statistics, known from market time-series analysis: (i) the return distribution and (ii) the waiting-time or pause-time distribution being an element of the continuous-time random walk formalism (Pfister and Scher, 1978; Haus and Kehr, 1987; Kutner and Świtała, 2003; Sandev et al., 2015). In the following description a statistical concept of the convolution of distributions to derive the desired formula for a joint statistics is used. In fact, the only assumption required by this derivation is a condition linking the relaxation time and the cumulative distribution function of returns. This yields an analytical formula for the "universal" statistics of interevent times between excessive losses on financial markets.

In Sec. 3.2 a short presentation of the quantities used in our considerations and the empirical data employed to verify our approach is made. In Sec. 3.3 the central equation of the model is derived, then in Sec. 3.4 the formalism is developed

[1]Based on Denys et al. (2016b).

the predictions of which are compared with the corresponding empirical financial data. Additionally, in Sec. 3.5 some applications for the model are proposed. Finally, in Sec. 3.6, conclusions are given. Thus, the first, analytical part of the interevent-times description is presented.

3.2 Data and grounds

For excessive profits and losses the data drawn directly from Bogachev and Bunde (2008); Ludescher et al. (2011); Ludescher and Bunde (2014) were used, where excessive profits are defined as those greater than the positive fixed threshold Q, while excessive losses – as those below the negative threshold $-Q$. Successively, the mean interevent (discrete or step) time R_Q between the losses (or profits) measured in the units of the relevant time-series resolution, is given by

$$\left(\frac{R_Q}{\tau}\right)^{-1} = P(-\varepsilon \leqslant -Q) = \int_{-\infty}^{-Q} D(-\varepsilon)\,\mathrm{d}\varepsilon, \tag{3.1}$$

where τ is an arbitrary calibration time (we extract it from the empirical-data fit, cf. Eq. (3.4) and Fig. 3.1), while $D(\varepsilon)$ is the returns distribution. In the subsequent considerations the "$-$" sign in the above equation is neglected, i.e., for the sake of simplicity, the losses are treated as positive quantities:

$$P(-\varepsilon \leqslant -Q) = P(\varepsilon \geqslant Q) = \int_{Q}^{\infty} D(\varepsilon)\,\mathrm{d}\varepsilon. \tag{3.2}$$

Thus, using Eqs. (3.1) and (3.2), we can write

$$D(\varepsilon) = -\frac{\mathrm{d}\left(\frac{R_Q}{\tau}\right)^{-1}}{\mathrm{d}Q}\Big|_{Q=\varepsilon}. \tag{3.3}$$

Subsequently, the mean interevent discrete time, R_Q, is used as an aggregated basic variable of the model.

3.2.1 The distribution of losses versus empirical data

To find the distribution of returns, $D(\varepsilon)$, the empirical data shown in Fig. 3.1 were used. We found the following functions capable of describing the empirical

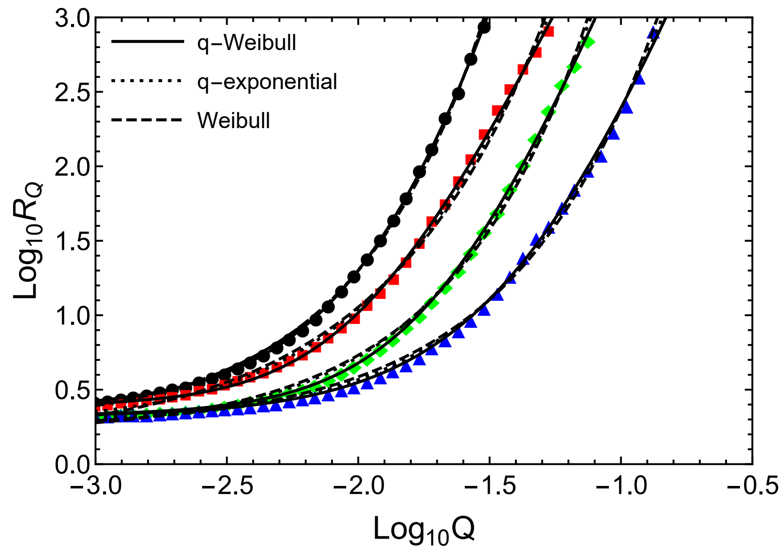

Figure 3.1: Mean interevent discrete time R_Q vs. threshold Q for four typical classes of indices: the USD/GBP exchange rate (black circles), S&P500 index (red squares), IBM stock (green rhomboids), and WTI (or crude oil, blue triangles) between January 2000 and June 2010 (taken from Fig. 2 in Ludescher et al. (2011)). The predictions of q-Weibull distribution (the upper branch of Eq. (4)), q-exponential distribution (the middle branch), and the usual Weibull distribution (the bottom branch) are given (see legends). The q-exponential and Weibull curves only slightly differ from the most accurate q-Weibull one. However, none of them are able to reproduce the weak wavy behavior of the data.

data:

$$\frac{R_Q}{\tau} = \begin{cases} \left(\exp_{q'}^{-(Q/\bar{\varepsilon}')^{\eta}}\right)^{-1}, \\ \exp_q^{Q/\bar{\varepsilon}}, \\ \exp\left((Q/\bar{\varepsilon})^{\eta}\right), \end{cases} \tag{3.4}$$

where τ is an irrelevant calibration time or scale factor of R_Q axis that differs for different branches, $\exp_q^{Q/\bar{\varepsilon}} = (1 + (1-q)Q/\bar{\varepsilon})^{\frac{1}{(1-q)}}$, and $q' = \frac{1}{2-q}$, $q < 2$, $\bar{\varepsilon}' = \bar{\varepsilon} q'^{1/\eta}$, $\bar{\varepsilon}$, $\eta > 0$. The functions given in Eq. (3.4) are q-Weibull (the upper branch), q-exponential (the middle branch), and usual Weibull (the lower branch) cumulative distribution functions. Notably, q-exponential and Weibull functions are only slightly less accurate for modeling the empirical data than the q-Weibull one. The parameters of the fits are listed in Tabs. 3.1-3.3.

Index/Par.	q	η	$\bar{\varepsilon}$	τ
USD/GBP	1.1529 ± 0.0085	1.2670 ± 0.0266	0.0041 ± 0.0000	2.3131 ± 0.0333
S&P500	1.3150 ± 0.0195	1.6202 ± 0.0869	0.0051 ± 0.0001	2.4504 ± 0.0689
IBM	1.2548 ± 0.0106	1.4983 ± 0.0398	0.0086 ± 0.0001	2.1187 ± 0.0267
WTI	1.2088 ± 0.0224	1.2280 ± 0.0637	0.0131 ± 0.0003	2.0885 ± 0.0516

Table 3.1: The values of q, η, $\bar{\varepsilon}$, and τ from the fit of the top branch of Eq. (3.4) to the empirical data from Fig. 3.1.

Index/Par.	q	$\bar{\varepsilon}$	τ
USD/GBP	0.9370 ± 0.0051	0.0040 ± 0.0001	1.9619 ± 0.0302
S&P500	0.8353 ± 0.0114	0.0048 ± 0.0002	1.8354 ± 0.0646
IBM	0.8969 ± 0.0094	0.0086 ± 0.0002	1.7404 ± 0.0414
WTI	0.8639 ± 0.0086	0.0146 ± 0.0004	1.9155 ± 0.0343

Table 3.2: The values of q, $\bar{\varepsilon}$, and τ from the fit of the middle branch of Eq. (3.4) to the empirical data from Fig. 3.1.

Index/Par.	η	$\bar{\varepsilon}$	τ
USD/GBP	0.8756 ± 0.0156	0.0037 ± 0.0003	1.7918 ± 0.0277
S&P500	0.6981 ± 0.0292	0.0035 ± 0.0005	1.3923 ± 0.0569
IBM	0.8246 ± 0.0236	0.0078 ± 0.0007	1.5791 ± 0.0346
WTI	0.7855 ± 0.0182	0.0131 ± 0.0008	1.7150 ± 0.0273

Table 3.3: The values of η, $\bar{\varepsilon}$, and τ from the fit of the bottom branch of Eq. (3.4) to the empirical data from Fig. 3.1.

The CDFs from Eq. (3.4) correspond to the following density functions,

$$D(\varepsilon) = \begin{cases} \frac{\frac{\eta}{\bar{\varepsilon}'}\left(\frac{\varepsilon}{\bar{\varepsilon}'}\right)^{\eta-1}}{1-(1-q')(\varepsilon/\bar{\varepsilon}')^{\eta}} \exp_{q'}^{-(\varepsilon/\bar{\varepsilon}')^{\eta}}, \\ \frac{1}{\bar{\varepsilon}}\left(\exp_q^{\varepsilon/\bar{\varepsilon}}\right)^{-(2-q)}, \\ \frac{\eta}{\bar{\varepsilon}}\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{\eta-1} \exp\left(-\left(\varepsilon/\bar{\varepsilon}\right)^{\eta}\right), \end{cases} \tag{3.5}$$

obtained by using Eq. (3.3) and being, obviously, q-Weibull, q-exponential, and usual Weibull PDFs, going from the top down to the bottom. It is worth mentioning that the q-Weibull distribution in the upper branch tends toward the usual Weibull distribution when $q \to 1$, and the q-exponential distribution in the middle becomes usual exponential then. Incidentally, we found a usual exponential distribution function providing significantly worse fit than the three mentioned here.

Fig. 3.2 shows the three probability distribution functions for the obtained distribution parameters from Tabs. 3.1-3.3. Ultimately, we used a two-point[2] Weibull distribution to study the dependence between successive interevent times, as multivariate q distributions do not exist. Incidentally, in Franke et al. (2004) the usual Weibull distribution is used to describe the statistics of interevent times between subsequent transactions for a given asset.

3.2.2 The justification for using Weibull distribution

In this subsection the question of why the usual single-variable Weibull distribution is almost indistinguishable from q-Weibull and q-exponential ones is explained.

As was shown in Sec. 2.1.1 in the previous chapter (2), the cumulative distribution function of extreme values must be Gumbel, Fréchet, or Weibull CDF. Nevertheless, we verified that only the Weibull one[3] agrees with the empirical data shown in Fig. 3.1. When $\eta < 1$, as in the case of our fits (cf. Tab. 3.3), the Weibull distribution for $\varepsilon/\bar{\varepsilon} \gg 1$ is a stretched-exponentially truncated decreasing power law (Tsallis, 2016).

Notably, our random variable ε is actually an increment of an underlying stochastic process. For the Weibull distribution, the relative mean $\frac{\langle\varepsilon\rangle}{\bar{\varepsilon}} = \frac{1}{\eta}\,\Gamma(1/\eta)$

[2]In the thesis I use interchangeably the forms ("one-") "two-point", "two-particle", or "two-variable".

[3]Sometimes called a type III excessive loss (Tsallis, 1988; Mandelbrot and Stewart, 1998; Ivanov et al., 2004).

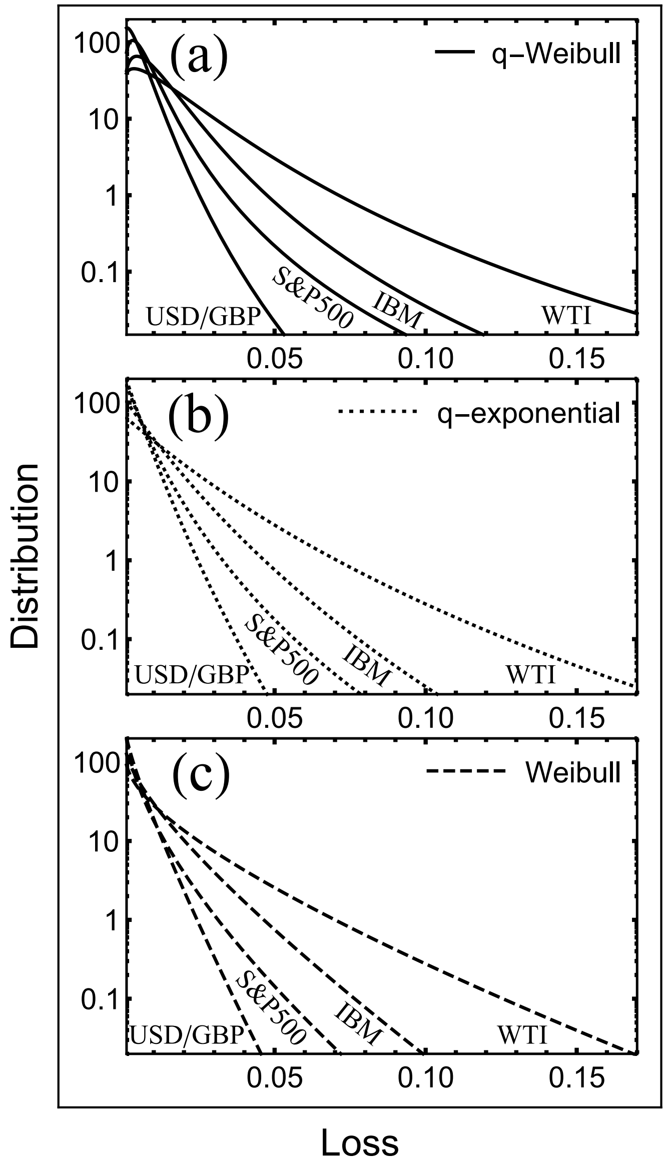

Figure 3.2: The probability distribution functions $D(\varepsilon)$ vs. losses ε corresponding to the CDF-fits from Fig. 3.1. The q-Weibull (a), q-exponential (b), and Weibull (c) distributions differs only slightly, mainly for small losses. Notably, in the interesting range of ε, q exponentials are almost identical to Weibull distribution.

and the relative variance $\frac{\sigma^2}{\langle\varepsilon\rangle^2} = \frac{\langle\varepsilon^2\rangle-\langle\varepsilon\rangle^2}{\langle\varepsilon\rangle^2} = (2\eta\frac{\Gamma(2/\eta)}{\Gamma^2(1/\eta)} - 1)$ depend only on the exponent η, thus, for fixed η they are universal quantities. According to Bertin and Clusel (2006), the Fisher–Tippett–Gnedenko theorem may be extended to the case of strongly dependent random variables.

Now, I will define the conditional mean interevent discrete time, $R_Q(R_{Q_0})$, that is, R_Q value calculated only for the time intervals preceded by an interval equal to R_{Q_0}. When multivariate q functions are not found[4], a bivariate Weibull distribution is used for this purpose. Since

$$\frac{R_Q(R_{Q_0})}{\tau} = \left(\int_Q^\infty D(\varepsilon|Q_0)\,\mathrm{d}\varepsilon\right)^{-1} \tag{3.6}$$

and the conditional distribution $D(\varepsilon|Q_0)$ is given by

$$D(\varepsilon|Q_0) \overset{\text{def.}}{=} \frac{\int_{Q_0}^\infty D(\varepsilon,\varepsilon_0)\,\mathrm{d}\varepsilon_0}{\int_{Q_0}^\infty D(\varepsilon_0)\,\mathrm{d}\varepsilon_0} = \frac{R_{Q_0}}{\tau}\int_{Q_0}^\infty D(\varepsilon,\varepsilon_0)\,\mathrm{d}\varepsilon_0 \tag{3.7}$$

(we used single-variate and bivariate distributions), from Eq. (5.1) in Lee (1979) and Eqs. (3.6) and (3.7) we obtain

$$\frac{R_Q(R_{Q_0})}{\tau} = \left(\frac{R_{Q_0}}{\tau}\right)^{-1} \times \exp\left(\left(\left(\ln\left(\frac{R_Q}{\tau}\right)\right)^{1/\gamma} + \left(\ln\left(\frac{R_{Q_0}}{\tau}\right)\right)^{1/\gamma}\right)^{\gamma}\right), \tag{3.8}$$

where parameter γ can be taken from the empirical-data fit, as in Fig. 3.3. Notably, the nonunitary value of γ exponent informs us about a possible dependence between interevent times. It is evident that all the well-fitted curves in Fig. 3.3, except the Brent quotes for $R_Q = 30$, reveal such a dependence (cf. Tab. 3.4). Since we could not obtain this result using q functions, we found the usual Weibull distribution better to use for the purpose of describing interevent times and, furthermore, for the description of returns on financial markets.

3.3 The main formula and superscaling

Now, I derive a closed form of the distribution $\psi_Q^\pm(\Delta_Q t)$ of interevent times $\Delta_Q t$. I use a complex-statistics (or superstatistics) form of this distribution,

$$\psi_Q^\pm(\Delta_Q t) = \frac{\int_Q^\infty \psi_Q^\pm(\Delta_Q t|\varepsilon)D(\varepsilon)\,\mathrm{d}\varepsilon}{\int_Q^\infty D(\varepsilon)\,\mathrm{d}\varepsilon} = -\frac{\int_Q^\infty \psi_Q^\pm(\Delta_Q t|\varepsilon)\,\mathrm{d}\left(\int_\varepsilon^\infty D(\varepsilon')\,\mathrm{d}\varepsilon'\right)}{\int_Q^\infty D(\varepsilon)\,\mathrm{d}\varepsilon}. \tag{3.9}$$

[4]For more details see Eq. (5.1) in Montroll and Weiss (1965).

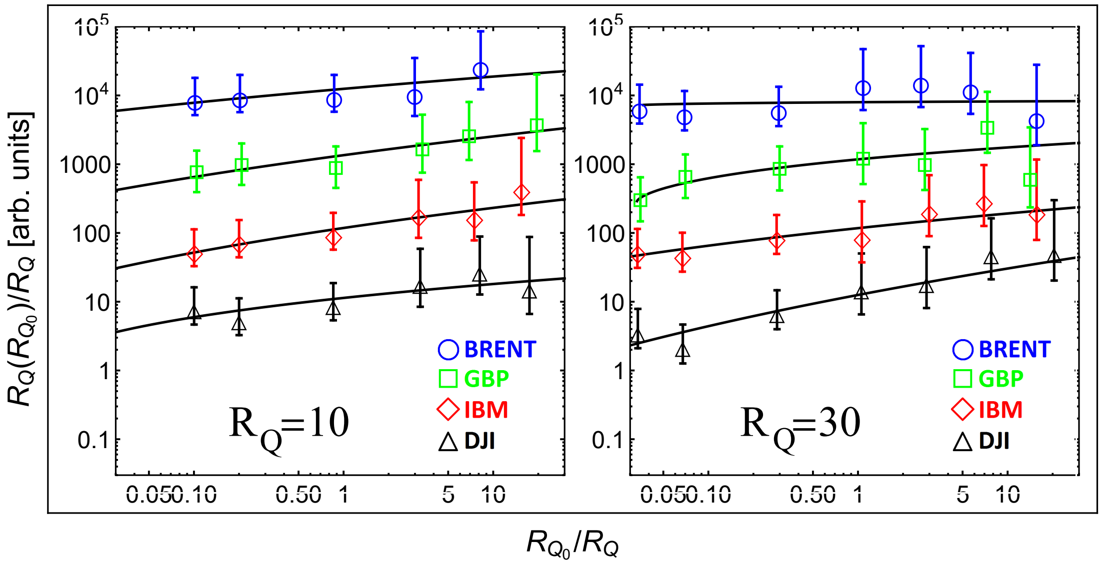

Figure 3.3: The relative conditional mean interevent discrete time $R_Q(R_{Q_0})/R_Q$ vs. R_{Q_0}/R_Q for four varied indices (see legends) and $R_Q = 10$ (on the left) and 30 (on the right). Empirical data drawn from Fig. 7 in Bogachev and Bunde (2008). Solid lines are fits of Eq. (3.8).

I assume that the conditional distribution, $\psi_Q^{\pm}(\Delta_Q t|\varepsilon)$[5], takes an exponential form,

$$\psi_Q^{\pm}(\Delta_Q t|\varepsilon) = \frac{1}{\tau_Q^{\pm}(\varepsilon)} \exp\left(-\frac{\Delta_Q t}{\tau_Q^{\pm}(\varepsilon)}\right), \tag{3.10}$$

with some relaxation time $\tau_Q^{\pm}(\varepsilon)$ defined as the mean time-distance from the last loss extending the threshold Q to the next such loss of a magnitude equal to ε (cf. Eq. (2.13) from the previous chapter). The sign "$\pm$" in the superscript of the above equations includes the cases of monotonically increasing (for $+$) or monotonically decreasing (for $-$) relaxation time, i.e., when larger losses are less frequent (an expanding hierarchy of the interevent times) or when volatility clustering occurs, respectively. In practice, both of these effects may occur. Moreover, in $\tau_Q^{\pm}(\varepsilon)$ we place the necessary dependence between ε and $\Delta_Q t$, averaged over all possible preceding losses extending the threshold Q. The exponential form of the conditional distribution (3.10) does not exclude a statistical dependence between successful interevent (continuous) times.

[5]Here, the condition means that the subsequent loss is exactly ε.

	$R_Q = 10$		$R_Q = 30$	
Index/Par.	γ	τ	γ	τ
DJIA	1.30	0.1	1.50	0.01
IBM	1.40	0.01	1.30	0.01
GBP/USD	1.37	0.001	1.27	1.0
Brent	1.25	0.0001	1.02	1.10

Table 3.4: The values of γ and τ from the fit of Eq. (3.8) to the empirical data from Fig. 3.3 (with an accuracy of about 10%).

Under the following assumption, linking the point and the cumulative quantities,

$$\left(\frac{\tau_Q^{\pm}(0)}{\tau_Q^{\pm}(\varepsilon)}\right)^{\pm\alpha_Q^{\pm}} = \int_{\varepsilon}^{\infty} D(\varepsilon')\,\mathrm{d}\varepsilon' = \left(\frac{R_{Q=\varepsilon}}{\tau}\right)^{-1}, \ \alpha_Q^{\pm} > 0, \tag{3.11}$$

from Eqs. (3.9) and (3.10) we obtain the final formula,

$$\psi_Q^{\pm}(\Delta_Q t) = \frac{1}{\tau_Q^{\pm}(Q)} \frac{\alpha_Q^{\pm}}{\left(\frac{\Delta_Q t}{\tau_Q^{\pm}(Q)}\right)^{1\pm\alpha_Q^{\pm}}} \Gamma^{\pm}\left(1 \pm \alpha_Q^{\pm}, \frac{\Delta_Q t}{\tau_Q^{\pm}(Q)}\right), \tag{3.12}$$

where $\Gamma^{\pm}\left(1 \pm \alpha_Q^{\pm}, \frac{\Delta_Q t}{\tau_Q^{\pm}(Q)}\right)$ denotes the lower (for "+") and upper (for "−") incomplete gamma functions, respectively, and no specific form of $D(\varepsilon)$ and $\tau_Q^{\pm}(\varepsilon)$ is assumed (see Appendix for details). Incidentally, the implicit assumption of the derivation given above is that $\varepsilon \geqslant Q$ is the maximal loss in the considered time period of the conditional distribution, Eq. (3.10). That justifies the use of the cumulative quantities instead of the usual ones after the first "=" sign in Eq. (3.11).

Actually, an analogous formula was obtained in Kasprzak (2010) (cf. Eq. (2.60) therein), where an exponential form of the distribution of losses, $D(\varepsilon)$, was assumed. But as it was mentioned in Sec. 3.2.1, exponential distribution is not sufficient to describe the whole dependence R_Q vs. Q. **Therefore, the achievement of this thesis that constitutes an original contribution to the field of knowledge, is the derivation of the formula given above without assuming any specific distribution of losses.** We found, however, the Weibull distribution to be the most appropriate for its known multivariate forms (see Sec. 3.2.2). **Another original achievement of the thesis is the overall systematic comparison of the predictions of the formula given above with**

empirical data (see Sec. 3.4) and the suggestion of some possible applications of the model that go beyond economics (see Sec. 3.5), which confirms the usefulness and versatility of our approach.

The conjecture (3.11), that we need to derive formula (3.12), links the relaxation time $\tau_Q^{\pm}(\varepsilon)$ and the distribution of losses. Actually, the existence of a dependence between these two is intuitively comprehensible. The mean time between two losses of a particular height should depend somehow on how often such losses occur, i.e., on their distribution function. However, one has also to remember the nontrivial Q-dependence in the relaxation time. That is to say, the threshold Q "cuts off" all the losses below it, thus they do not affect the value of $\tau_Q^{\pm}(\varepsilon)$ (unlike the losses larger than Q).

To satisfy Eq. (3.11) one should find a suitable integration exponent $\alpha_Q^{\pm}$, which allows us to transform Eq. (3.9) into a useful form (3.12). This exponent makes the left-hand side of Eq. (3.11) a Q-independent quantity and, consequently, the integration in Eq. (3.9) feasible (see Appendix for a detailed derivation). From Eq. (3.4) one obtains

$$\frac{\tau_Q^{\pm}(\varepsilon)}{\tau_Q^{\pm}(0)} = \begin{cases} \left(\exp_{q'}^{-(\varepsilon/\bar{\varepsilon}')^{\eta}}\right)^{\mp 1/\alpha_Q^{\pm}}, & \text{for } q\text{-Weibull pdf,} \\ \left(\exp_q^{\varepsilon/\bar{\varepsilon}}\right)^{\pm 1/\alpha_Q^{\pm}}, & \text{for } q\text{-exp pdf,} \\ \left(\exp\left((\varepsilon/\bar{\varepsilon})^{\eta}\right)\right)^{\pm 1/\alpha_Q^{\pm}}, & \text{for Weibull pdf.} \end{cases} \tag{3.13}$$

If we assume that the relaxation time $\tau_Q^{\pm}(\varepsilon)$ is a stretched exponential (the form typical for relaxation phenomena in disordered systems), namely

$$\frac{\tau_Q^{\pm}(\varepsilon)}{\tau_Q^{\pm}(0)} = \exp\left(\pm(B_Q^{\pm}\varepsilon)^{\eta}\right), \tag{3.14}$$

then, from Eq. (3.13), we obtain

$$\alpha_Q^{\pm} = \begin{cases} \frac{1}{(B_Q^{\pm}\bar{\varepsilon}')^{\eta}}, & \text{for } q\text{-Weibull pdf,} \\ \frac{1}{B_Q^{\pm}\bar{\varepsilon}}, & \text{for } q\text{-exp pdf,} \\ \frac{1}{(B_Q^{\pm}\bar{\varepsilon})^{\eta}}, & \text{for Weibull pdf,} \end{cases} \tag{3.15}$$

where in the upper and the middle branch $|1-q'| \ll 1$ and $|1-q| \ll 1$, respectively (cf. Tabs. 3.1 and 3.2)[6].

[6]Taking (for a more consistent approach) $\exp_q$ instead of the usual exp would strongly complicate the view.

The stretched-exponential relaxation time assumed in Eq. (3.14) is a straightforward generalization of the simple exponential one that is used in the canonical version of the CTRW valley model of anomalous photocurrent relaxation in amorphous films (Scher and Montroll, 1975; Pfister and Scher, 1978; Weiss, 2005; Schulz and Barkai, 2015) with B_Q playing the role of inverse temperature (cf. the description of its scaling with Q in Sec. 3.3.1) and the mean valley depth $\bar{\varepsilon}$. In Eq. (3.13), in the upper and lower branches, we assumed the same exponent η as in Eqs. (3.4) and (3.5) to reduce the number of free parameters (the law of parsimony) and derive our final formula (3.12). In the canonical version of the CTRW exponent $\eta = 1$, and we use the usual exponential distribution in accordance with the Hoff–Arrhenius law of the thermally activated over-barrier transitions and with the Vogel–Tammann–Fulcher law of diffusion and transport in glasses. Thus, $\bar{\varepsilon}$, $\bar{\varepsilon}'$, η and $B^{\pm}_Q$ form the $\alpha^{\pm}_Q$ dynamic exponent, as shown in Eq. (3.15). This exponent defines the long-time behavior of the superstatistics $\psi^{\pm}_Q(\Delta_Q t)$ and, together with relaxation time, govern its full evolution (cf. Eq. (3.12)).

Below, I give the asymptotic forms of Eq. (3.12). For the "+" case they are as follows. For $\frac{\Delta_Q t}{\tau^+_Q(Q)} \gg 1$ the superstatistics takes a power-law form,

$$\psi^+_Q(\Delta_Q t) \approx \frac{1}{\tau^+_Q(Q)} \frac{\alpha^+_Q}{\left(\frac{\Delta_Q t}{\tau^+_Q(Q)}\right)^{1+\alpha^+_Q}} \Gamma^+(1+\alpha^+_Q). \tag{3.16}$$

For $\frac{\Delta_Q t}{\tau^+_Q(Q)} \ll 1$ it takes an exponential form,

$$\psi^+_Q(\Delta_Q t) \approx \frac{1}{\tau^+_Q(Q)} \frac{\alpha^+_Q}{1+\alpha^+_Q} \exp\left(-\frac{1+\alpha^+_Q}{2+\alpha^+_Q} \frac{\Delta_Q t}{\tau^+_Q(Q)}\right). \tag{3.17}$$

For $\alpha^+_Q \gg 1$ it becomes α^+_Q-independent exponential,

$$\psi^+_Q(\Delta_Q t) \approx \frac{1}{\tau^+_Q(Q)} \exp\left(-\Delta_Q t/\tau^+_Q(Q)\right), \tag{3.18}$$

consistently with Eq. (3.17).

For the "−" case, when $\frac{\Delta_Q t}{\tau^-_Q(Q)} \gg 1$, Eq. (3.12) reduces to a power law truncated by the upper incomplete gamma function $\Gamma^-\left(1-\alpha^-_Q, \frac{\Delta_Q t}{\tau^-_Q(Q)}\right)$. For $\frac{\Delta_Q t}{\tau^-_Q(Q)} \ll 1$ and

$\alpha_Q^- < 1$ it becomes a pure short-time power law, namely

$$\psi_Q^-(\Delta_Q t) \approx \frac{1}{\tau_Q^-(Q)} \frac{\alpha_Q^-}{\left(\frac{\Delta_Q t}{\tau_Q^-(Q)}\right)^{1-\alpha_Q^-}} \Gamma^-(1-\alpha_Q^-). \tag{3.19}$$

3.3.1 Superscaling

In this subsection a scaling hypothesis for $\ln R_Q$ or Q as possible scaling variables (related by a one-to-one correspondence (3.4)) is formulated. I provide the evidence for the sole R_Q- or, alternatively, Q-dependence of our central formula (3.12), making it a universal one. Ludescher et al. (2011) have shown this dependence empirically in their description of interevent times by q-exponential functions. To prove it analytically here, the following hypothesis,

$$B_Q^{\pm} = Q^{\zeta} \times \begin{cases} B_{\pm}^{1/\eta}/\bar{\varepsilon}'^{1+\zeta}, & \text{for } q\text{-Weibull pdf,} \\ B_{\pm}/\bar{\varepsilon}^{1+\zeta}, & \text{for } q\text{-exp pdf,} \\ B_{\pm}^{1/\eta}/\bar{\varepsilon}^{1+\zeta}, & \text{for Weibull pdf,} \end{cases} \tag{3.20}$$

is formulated, where $B_{\pm} > 0$ and $\zeta > 0$ are Q-independent control parameters.

Consequently, from Eqs. (3.4) and (3.15) we obtain the scaling of scaling exponent $\alpha_Q^{\pm}$ (or the superscaling), namely

$$\frac{1}{\alpha_Q^{\pm}} = B_{\pm} \times \begin{cases} \left(-\ln_{q'}\left(\frac{R_Q}{\tau}\right)^{-1}\right)^{\zeta}, & \text{for } q\text{-Weibull pdf,} \\ \ln_q^{\zeta}\left(\frac{R_Q}{\tau}\right), & \text{for } q\text{-exp pdf,} \\ \ln^{\zeta}\left(\frac{R_Q}{\tau}\right), & \text{for Weibull pdf,} \end{cases} \tag{3.21}$$

with the universal superscaling exponent ζ. From Eq. (3.11) a useful relation

$$\ln\left(\frac{\tau_Q^{\pm}(Q)}{\tau_Q^{\pm}(0)}\right) = \pm\frac{1}{\alpha_Q^{\pm}} \ln\left(\frac{R_Q}{\tau}\right) \tag{3.22}$$

is obtained. Hereby, I subordinated all quantities we use, i.e., $B_Q^{\pm}$, $\frac{1}{\alpha_Q^{\pm}}$, and $\frac{\tau_Q^{\pm}(Q)}{\tau_Q^{\pm}(0)}$, to the single control variable R_Q/τ. In the next section (3.4) I compare these expressions with the empirical data (e.g., for IBM). Additionally, I consider an R_Q-dependence of $\tau_Q^{\pm}(0)$.

	Fig. 3.4(a)		Fig. 3.4(b)		Fig. 3.4(c)		Fig. 3.4(d)	
R_Q	α_Q	$\tau_Q(Q)$	α_Q	$\tau_Q(Q)$	α_Q	$\tau_Q(Q)$	α_Q	$\tau_Q(Q)$
2	5.0	1.15	14.0	1.27	7.8	1.256	12.2	1.34
5	3.8	3.12	2.4	2.79	3.0	2.85	3.0	2.85
10	2.0	4.58	1.9	4.6	2.0	4.39	2.20	5.09
30	·	·	1.08	5.55	1.2	5.67	1.1	5.51
70	·	·	0.55	5.39	0.5	3.5	0.5	4.08

Table 3.5: The values of α_Q and $\tau_Q(Q)$ for the empirical-data fits of Eq. (3.12) shown in Fig. 3.4 (with an accuracy of about 5%).

R_Q	Q	α_Q	$\tau_Q(Q)$
2	0.0014	1000	1.4286
5	0.0093	3.0	3.33
10	0.0164	1.9	5.0
30	0.0289	0.95	4.55
70	0.0393	0.47	3.85

Table 3.6: The values of α_Q and $\tau_Q(Q)$ for the IBM empirical-data fit of Eq. (3.12) shown in Fig. 3.5 (with an accuracy of about 1%).

3.4 Empirical verification of the model

In this section I compare the predictions of our model with the empirical data from Bogachev and Bunde (2008); Ludescher et al. (2011); Ludescher and Bunde (2014). I consider only the "+" case herein; the "−" case is considered in Sec. 3.5. The "±" sign is omitted herein for the simplicity.

Fig. 3.4 shows the desirable data collapse for the fixed values of the control variable R_Q. Tab. 3.5 presents the values of the relevant parameters for this figure. Fig. 3.5 shows the individual fit, for example, for the typical IBM company with parameters given in Tab. 3.6. For $R_Q > 5$ and $\Delta_Q t > 30$ the power-law relaxation of $\psi_Q(\Delta_Q t)$ is clearly seen. Moreover, the inset plot for $R_Q = 2$ reveals the exponential decay of $\psi_Q(\Delta_Q t)$ for this value of R_Q, as expected for large α_Q (cf. Tab. 3.6 and Eq. (3.18)).

We used (a quantile) R_Q value as the control variable, since it is more convenient to use here than the mean interevent time $\langle \Delta_Q t \rangle$. Consequently, from Eqs. (3.9) and (3.10), the lower branch of Eq. (3.5) (a Weibull distribution case), and

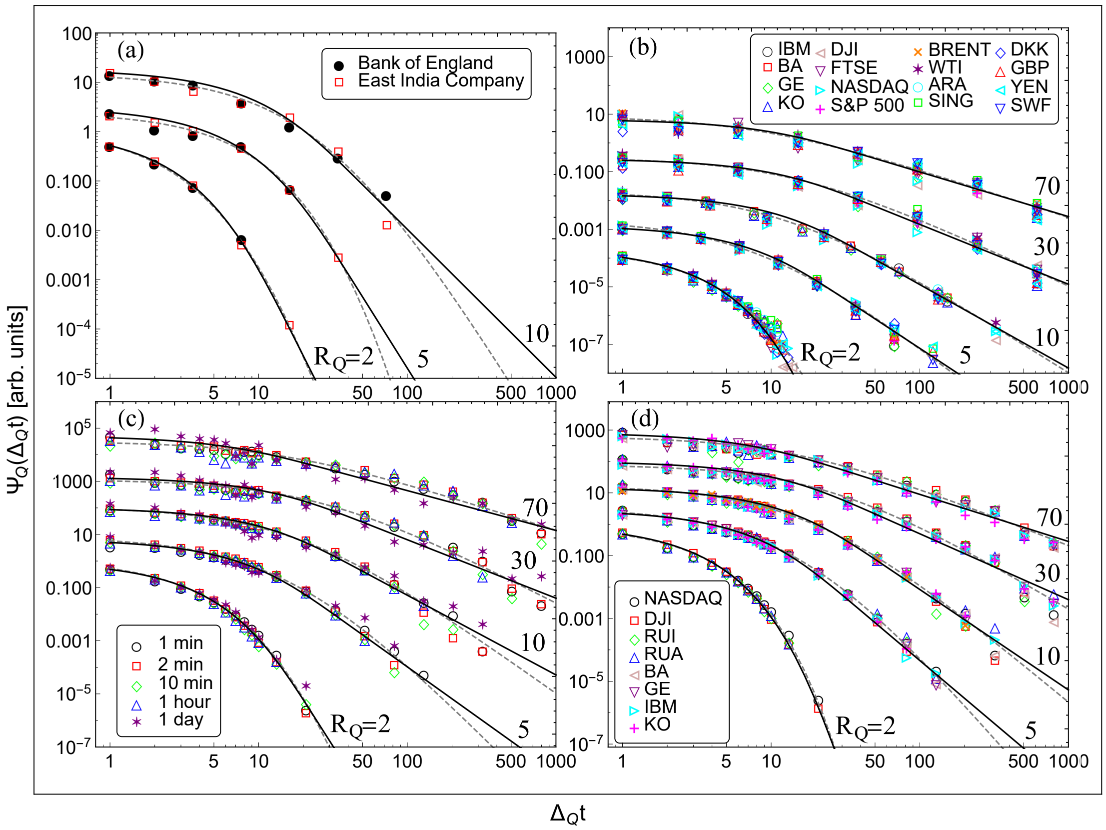

Figure 3.4: The empirical statistics of interevent times (colored marks) for: (a) the monthly returns of the Bank of England and the East India Company, 1709–1823, (b) the relative daily price returns for varied quotes, 1962–2010, (c) NASDAQ between March 16, 2004 and June 5, 2006, different time ranges, (d) the detrended minute-by-minute typical examples of financial data for different R_Q values (in the corresponding units) indicated on the plots. The black solid curves are fits of our theoretical superstatistics, $\psi_Q(\Delta_Q t)$ vs. $\Delta_Q t$, Eq. (3.12). The gray dashed curves are the relevant q-exponential fits, Eq. (3) in Ludescher et al. (2011). Empirical data taken from Ludescher et al. (2011); Ludescher and Bunde (2014).

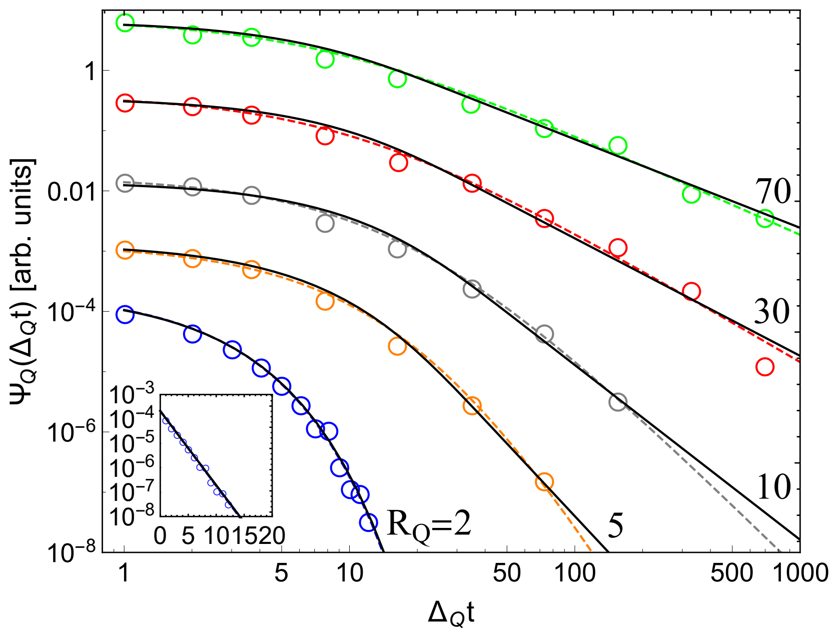

Figure 3.5: Empirical interevent-time statistics for the daily returns of IBM, 1962–2010 (empty circles), in log-log scale for $R_Q = 2, 5, 10, 30, 70$ (in units of days). The black solid curves are fits of our theoretical superstatistics, Eq. (3.12). The gray dashed curves are the relevant q-exponential fits, Eq. (3) in Ludescher et al. (2011). The inset is the plot for $R_Q = 2$ in a semilog scale, to show the exponential decay in this case. Empirical data taken from Ludescher et al. (2011).

Eq. (3.14) we obtain

$$\langle (\Delta_Q t)^m \rangle = \int_0^{\infty} (\Delta_Q t)^m \, \psi_Q(\Delta_Q t) \, \mathrm{d}(\Delta_Q t) = (\tau_Q(Q))^m G_{Q,m}, \;\; m = 0, 1, 2, \ldots, \tag{3.23}$$

where the key factor of the mth moment of variable $\Delta_Q t$ is

$$G_{Q,m} = \frac{m!}{1 - m/\alpha_Q}. \tag{3.24}$$

The value of the mth moment is finite only for $\alpha_Q > m$, otherwise it is infinite[7]. In contrast, quantity R_Q is always finite for its quantile (not momentum) origin.

[7]See, e.g., Kutner (1999b); Kutner and Regulski (1999); Kutner and Świtała (2003); Bel and Barkai (2005, 2006) for more information concerning infinite moments in random-walk models.

PDF	B	ζ
q-Weibull	0.27194 ± 0.06850	1.0037 ± 0.1459
q-exp	0.1572 ± 0.0586	1.6912 ± 0.2626
Weibull	0.1028 ± 0.0446	2.2590 ± 0.3393

Table 3.7: The universal parameters B and ζ from the fit of the dependence B_Q vs. Q given by Eq. (3.21) to the empirical data presented in Fig. 3.6(a) for IBM.

Parameters	L	R
a_s	0.435	-0.019
b_s	0.79	5.161

Table 3.8: The parameters $a_s, b_s; s = L, R$ of the linear regressions $\tau_Q(Q)$ vs. R_Q for IBM, shown in Fig. 3.6(b) (with the accuracy about 1%).

For instance, for IBM the first moment, $\langle \Delta_Q t \rangle$, is finite only when $R_Q \leqslant 10$ (cf. Tab. 3.6) and in like manner for the other indices (cf. Tab. 3.5).

Fig. 3.6 presents the empirical-data confirmation of the superscaling hypothesis formulated in Sec. 3.3.1, for example, for IBM. In Fig. 3.6(a) the predictions of Eq. (3.21) are fitted to the $1/\alpha_Q$ values obtained from the third column of Tab. 3.6. The resulting values of the fit parameters B and ζ are given in Tab. 3.7 (the values of the corresponding calibration parameter τ are given in Tabs. 3.1-3.3). The inset plot presents the agreement between Eq. (3.20) and the empirical data transformed by Eq. (3.15). Fig. 3.6(b) presents relationship $\tau_Q(Q)$ vs. $R_Q(Q)$ from Tab. 3.6 in the form of two straight lines (or a broken line). Tab. 3.8 gives the parameters of this regression. Therefore, we can treat $\tau_Q(Q)$ as a composition of two linear functions of R_Q. The inset plot in the lower part of Fig. 3.6 compares the theoretical and empirical values of $\tau_Q(0)$ obtained using Eq. (3.22).

In this way, I proved such an R_Q-dependence of the superstatistics $\psi_Q(\Delta_Q t)$ that describes the data collapse presented in Figs. 3.4 and 3.5. As I mentioned in Sec. 3.2.2, the usual Weibull, as well as q-Weibull and q-exponential distributions are able to describe single-variable empirical results. Therefore, both the extreme value theory approach and the nonextensive q-function viewpoint (the result of long-term dependence) represented by these functions, may be closely related. Indeed, they may be “two sides of the same coin”, linked by means of a certain fluctuation–dissipation relation. However, as mentioned in Sec. 3.2.1, the bivariate q distributions do not exist, hence it is better to use the usual Weibull distribution here, which explains the observed dependence between the interevent times shown in Sec. 3.2.2.

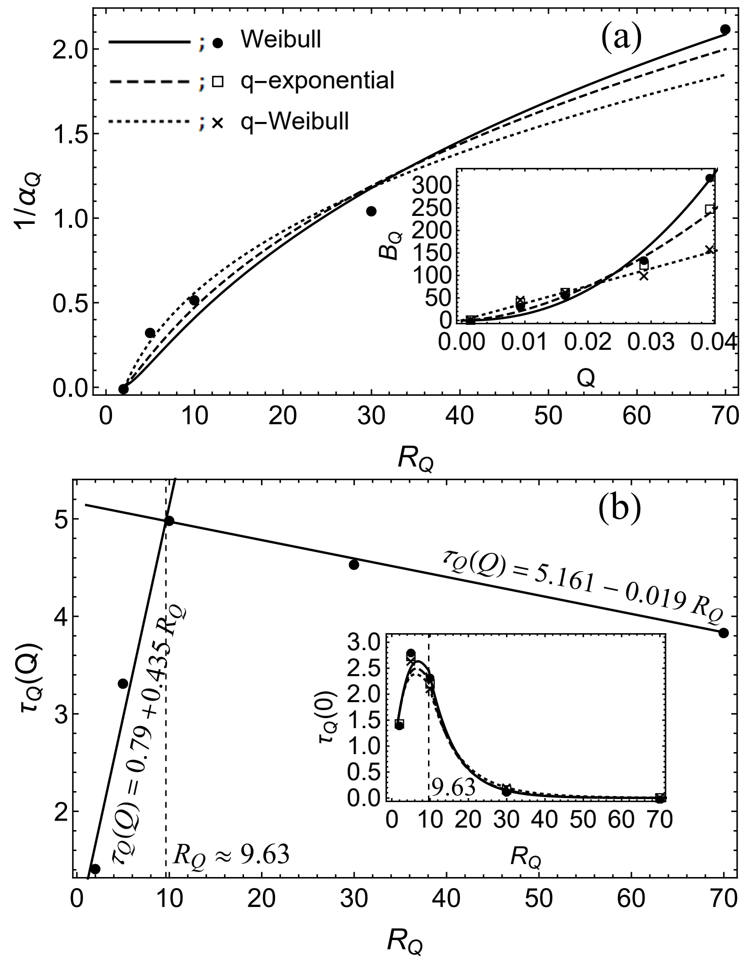

Figure 3.6: The dependence between $1/\alpha_Q$ (a) and $\tau_Q(Q)$ (b) vs. R_Q for IBM. The empirical data for the main plots (black circles) are taken from Tab. 3.6. Theoretical predictions (solid curves) are obtained by (a) the fit of Eq. (3.21) to the empirical data (see Tab. 3.7 for the parameters) and (b) the linear regressions $\tau_Q(Q) = a_s R_Q + b_s$, where $s = L$ and $s = R$ are the left-hand and right-hand side straight lines, respectively; $R_Q \geqslant -b_R/a_R$ and $\tau_{Q=0}(0) = a_L\tau + b_L$ as $R_{Q=0} = \tau$ (see Tab. 3.8 for the corresponding values of the parameters). The inset plots show: (a) dependence B_Q vs. Q and (b) $\tau_Q(0)$ vs. R_Q (b) for Weibull PDF (black circles and solid lines), q-exponential PDF (empty squares and dashed lines) and q-Weibull PDF (crosses and dotted lines) with the parameters of these PDFs taken from Tabs. 3.1-3.3 for IBM. The indirect empirical data for these plots (circles, squares, and crosses) are obtained from (a) Eq. (3.15) and (b) Eq. (3.22) with α_Q and $\tau_Q(Q)$ taken from Tab. 3.6. The theoretical predictions for the inset plots are given by solid curves, with B and ζ taken from Tab. 3.7 for (a) and a_s, b_s for (b) (needed for $\tau_Q(Q)$ values) taken from Tab. 3.8.

3.5 Exemplary applications of our model

The approach presented above may be applied to a broad spectrum of threshold phenomena, not only in finance, but also in other fields, such as geophysics, to describe some seismic data (see Sec. 3.5.3). As is shown in this section, these applications are both practically and theoretically significant.

3.5.1 An application to risk estimation

The importance of the interevent-time distribution $\psi_Q^{\pm}(\Delta_Q t)$ in financial engineering is that it enables risk calculation. The crucial quantity here is the risk function $W_Q^{\pm}(t;\Delta t)$, defined as a conditional probability that the next loss exceeding the threshold Q will occur within the time interval t if the last such loss occurred t time units ago. Bogachev et al. (2007) have shown that the relation between $W_Q^{\pm}(t;\Delta t)$ and $\psi_Q^{\pm}(\Delta_Q t)$ is given by

$$W_Q^{\pm}(t;\Delta t) = \frac{\int_t^{t+\Delta t} \psi_Q^{\pm}(\Delta_Q t)\,\mathrm{d}\Delta_Q t}{\int_t^{\infty} \psi_Q^{\pm}(\Delta_Q t)\,\mathrm{d}\Delta_Q t}. \tag{3.25}$$

Substituting $\psi_Q^{\pm}(\Delta_Q t)$ from Eq. (3.12) and calculating the above given integrals, we obtain

$$W_Q^{\pm}(t;\Delta t) = 1 - \frac{\frac{\Gamma^{\pm}(1\pm\alpha_Q^{\pm},(t+\Delta t)/\tau_Q^{\pm}(Q))}{((t+\Delta t)/\tau_Q^{\pm}(Q))^{\pm\alpha_Q^{\pm}}} \pm \exp\left(-\frac{t+\Delta t}{\tau_Q^{\pm}(Q)}\right)}{\frac{\Gamma^{\pm}(1\pm\alpha_Q^{\pm},t/\tau_Q^{\pm}(Q))}{(t/\tau_Q^{\pm}(Q))^{\pm\alpha_Q^{\pm}}} \pm \exp\left(-\frac{t}{\tau_Q^{\pm}(Q)}\right)}. \tag{3.26}$$

The above given formula includes both the "+" and the "−" case; however, only the "+"-sign case is considered herein, as it covers all the previous empirical data. For this case, the asymptotic forms of Eq. (3.26) are

$$W_Q^{+}(t;\Delta t) \approx \alpha_Q^{+}\frac{\Delta t}{t},\ \min\left(\frac{t}{\tau_Q^{+}(Q)},\ \frac{t}{\Delta t}\right) \gg 1 \tag{3.27}$$

and

$$W_Q^{+}(t;\Delta t) \approx \frac{\alpha_Q^{+}}{1+\alpha_Q^{+}}\frac{\Delta t}{\tau_Q^{+}(Q)},\ \frac{t+\Delta t}{\tau_Q^{+}(Q)} \ll 1, \tag{3.28}$$

that is, a universal Zipf law for long times and a time-independent value for short times, respectively.

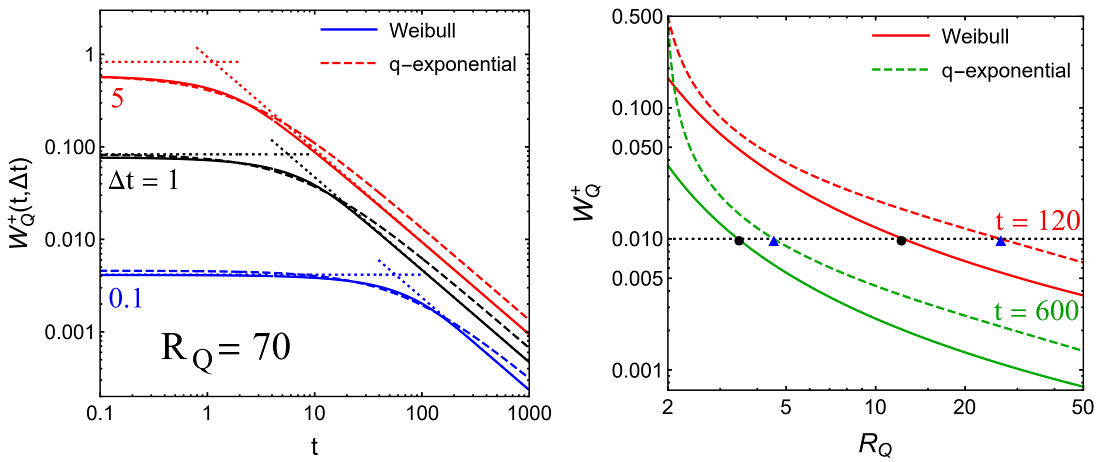

Figure 3.7: Comparisons of two risk functions: Weibull-based, given by Eq. (3.26) (solid curves) and q-exponential-based, given by Eq. (6a) from Ludescher et al. (2011) (dashed curves). On the left: $W_Q^{\pm}(t;\Delta t)$ vs. t for $R_Q = 70; \Delta t = 0.1, 1,$ and 5 (blue, black, and red line, respectively). Dotted straight asymptotes are given by Eqs. (3.27) and (3.28). On the right: $W_Q^{\pm}(t;\Delta t)$ vs. R_Q for $\Delta t = 1$ and $t = 120, 600$ (red and green line, respectively). The dotted horizontal line stands for probability $p = 0.01$. Remarkably, for a fixed W_Q^+ a longer t implies a smaller R_Q value (cf. the position of small circles and triangles).

Fig. 3.7 presents the behavior of the risk function. The monotonic decreasing of the risk function versus time t at fixed Δt and R_Q is shown in the left part, while versus R_Q at fixed Δt and t is shown in the right part. Additionally, predictions of an alternative risk function, proposed earlier by Ludescher et al. (2011), are shown. It can be seen that the above given asymptotic behaviors are preserved by both functions.

The risk function obtained above may be used for some numerical computations. Here, it is shown how to use it for a value-at-risk (VaR) simulation. The concept of VaR is presented in Sec. 2.1.2, Chapter 2. The algorithm of the numerical sampling on the basis of a given risk function was circumscribed by Bogachev and Bunde (2009); Ludescher et al. (2011). The version of this algorithm given below is taken from Denys et al. (2016b).

First, we calculate the zero-order threshold value Q from Eqs. (3.1), (3.2), and (3.4). We set $P(\varepsilon \geqslant Q) = p$ to obtain

$$Q = \begin{cases} \bar{\varepsilon}' \left(-\ln_{q'} p\right)^{1/\eta}, \\ \bar{\varepsilon} \ln_q \left(\frac{1}{p}\right), \\ \bar{\varepsilon} \left(-\ln p\right)^{1/\eta}, \end{cases} \tag{3.29}$$

where $\ln_q(\ldots)$ is a q logarithm (the inverse of a q-exponential function). Then we set initially $|VaR| = Q$.

Next, we draw time t from the distribution $\psi_Q^+(t = \Delta_Q t)$, Eq. (3.12). Using t and the current Q we calculate $W_Q^+(t; \Delta t)$ from Eq. (3.26) and check if it falls into the range $p \pm \Delta p$, $\Delta p \ll p$. If that is true, we set $|VaR| = Q$ and draw the next value of t from $\psi_Q^+(t = \Delta_Q t)$ for this Q, etc. When the value of risk function $W_Q^+(t; \Delta t)$ is outside the range $p \pm \Delta p$, we multiply Q by factor $1 \pm \gamma$ where $\gamma \ll 1$, "+" for $W_Q^+(t; \Delta t) > p + \Delta p$ and "−" for $W_Q^+(t; \Delta t) < p - \Delta p$. We repeat this step $n_+ + n_-$ times until $Q' = Q(1+\gamma)^{n_+}(1-\gamma)^{n_-}$ and the drawn time t (once for a given time step) give the value of the risk function within the range $p \pm \Delta p$. Then the new $|VaR| = Q'$, and we use it as an initial value for the next step.

By this means we obtain the $|VaR|$ series in time, that is, at $(t + \Delta t)$-s. An exemplary outcome of the algorithm presented above is shown in Fig. 3.8. Remarkably, the $|VaR|$ series obtained in the frame of our approach contains values all below the initial $|VaR|$, Eq. (3.29). Thus, our approach appears to reduce the estimated $|VaR|$ value. The algorithm from Bogachev and Bunde (2009); Ludescher et al. (2011) did not possess this property (cf. e.g., Fig. 6 from Ludescher et al. (2011)).

3.5.2 The application to profit analysis

The empirical-data collapse presented in Sec. 3.4 relates to the empirical data of losses, confirming the usefulness of our model. Since our derivation is symmetric (i.e., stable against the change of the loss sign), the question arises of how it applies to the analysis of profits. In this variant of the model we consider excessive profits instead of excessive losses, only by changing the interpretation of variable Q to be the threshold for profits instead of losses. Therefore, the final result is analogous to those for the losses, i.e., it is also given by Eq. (3.12). An application of these superstatistics to profit analysis is presented in Fig. 3.9.

Nevertheless, the symmetry between profits and losses turns out to be only functional, not literal. The reason is that the control parameters α_Q and $\tau_Q(Q)$

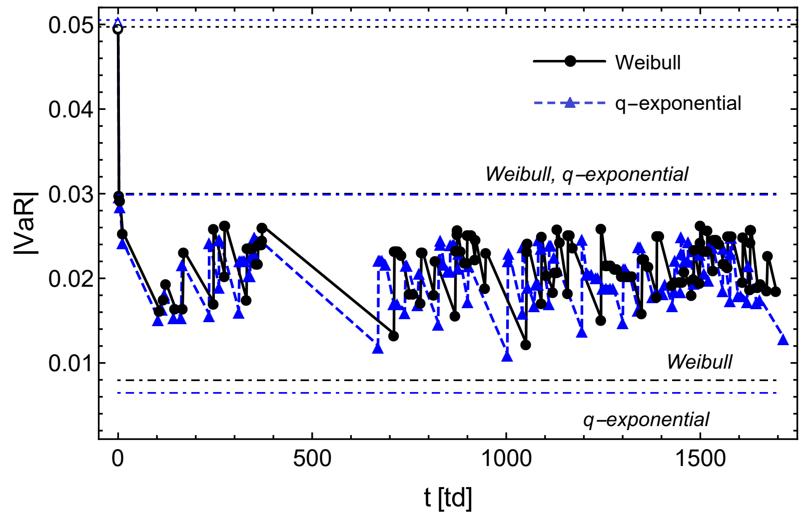

Figure 3.8: Weibull-based (black circles and solid segments) and q-exponential-based (blue triangles and dashed segments) simulation of $|VaR|$, obtained using Eq. (3.26) for $\Delta t = 0.1$, $1 - p = 0.99$ and the time range as in Fig. 6 in Ludescher et al. (2011) (IBM quotations between 2002 and 2008). The horizontal dotted lines and the single empty circle and triangle for $|VaR| \approx 0.05$ are the zero-order $|VaR|$-s for Weibull (black) and q-exponential (blue) case, i.e., with no dependence between interevent times assumed. Two pairs of dashed-dotted horizontal lines stand for empirical long-range spreads for Weibull (black) and q-exponential (blue) PDF.

appear to be different than in the case of losses. We cannot say anything definite here though, because the statistical errors in the empirical data that we used are too large. For instance, we obtained (cf. Tabs. 3.5 and 3.6 for excessive losses) $1.70 \leqslant \alpha_Q^+ \leqslant 3.10$ and $0.10 \leqslant \tau_Q^+(Q) \leqslant 0.25$ for $R_Q = 10$, $0.90 \leqslant \alpha_Q^+ \leqslant 1.50$ and $0.12 \leqslant \tau_Q^+(Q) \leqslant 0.35$ for $R_Q = 30$, and finally $0.60 \leqslant \alpha_Q^+ \leqslant 1.10$ and $0.08 \leqslant \tau_Q^+(Q) \leqslant 0.36$ for $R_Q = 70$, i.e., the fairly extended ranges. Therefore, we also cannot verify the universality addressed in this chapter in the case of the excessive profits.

3.5.3 A geophysical application

Our model concerns the times between events on financial markets. However, nothing stands in the way of extending the range of its applicability to some

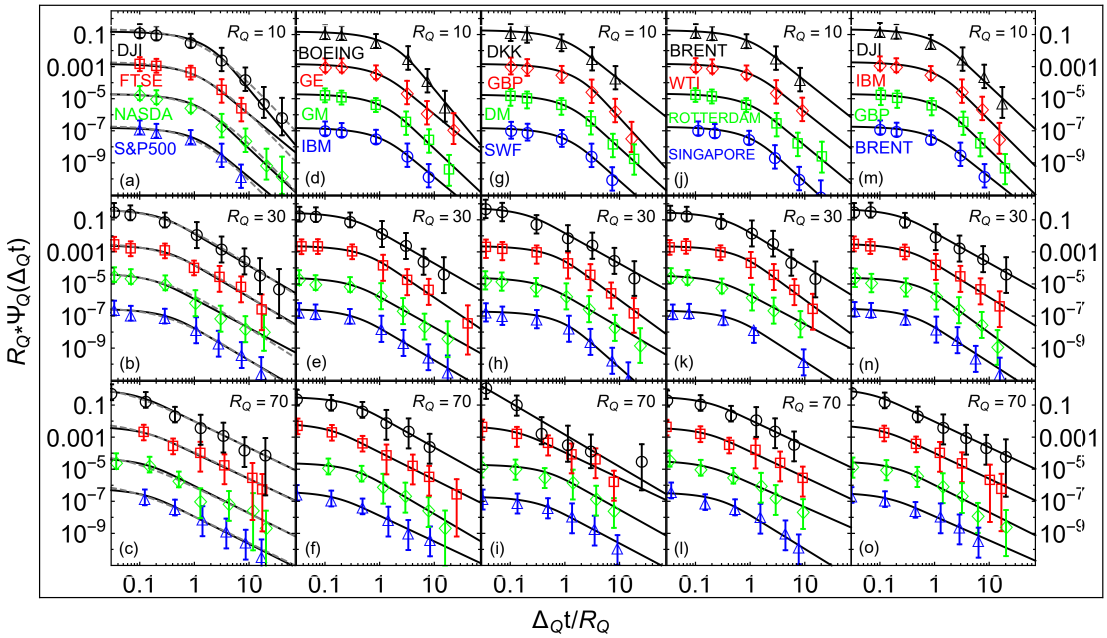

Figure 3.9: The interevent-time statistics for profits obtained for daily returns of various assets from various markets, e.g., the stock exchange, forex, or the resource market. The black solid curves are fits of our theoretical superstatistics, Eq. (3.12). The gray dashed curves on (a), (b), and (c) are the relevant exemplary q-exponential fits, Eq. (3) in Ludescher et al. (2011), shown for a comparison. Empirical data (discrete marks with bars) taken from Bogachev and Bunde (2008).

nonfinancial noises, such as geophysical ones, known for their similarity to financial data (Sornette, 2002).

The widely used Gutenberg–Richter law determines the frequency of earthquakes with a magnitude greater than the threshold Q in the form of an exponential function (Corral, 2004). The exponential distribution as the underlying statistics (or a Weibull distribution with exponent $\eta = 1$, cf. Denys et al. (2016b)) may be used likewise to derive our superstatistics, Eq. (3.12). The Gutenberg–Richter law is, actually, the reversed lower branch of Eq. (3.4) with $\eta = 1$, i.e., the exponential case.

Therefore, taking into account (i) the large volatility clustering (avalanches of earthquakes or aftershock sequences), and (ii) the small volatility clustering (weak aftershock sequences), we proposed the following superposition of two interevent-

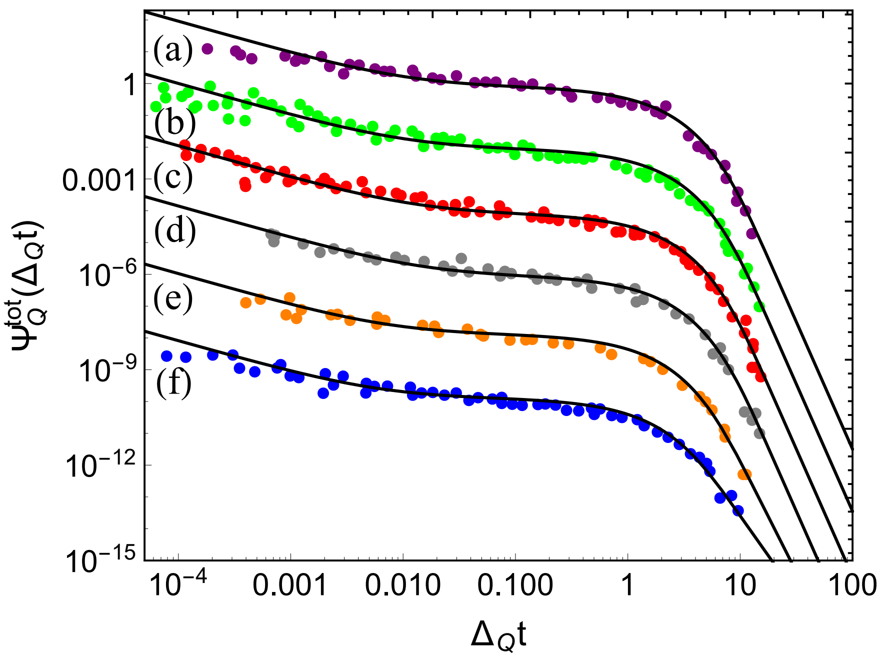

Figure 3.10: The rescaled statistics of interevent times (from 2 min to about 1.5 years) between earthquakes exceeding the threshold from 1.5 to 7.5 on the Richter scale (small circles) for (a) the NEIC worldwide catalog for regions with $L \geqslant 180°$, 1973–2002, (b) NEIC with $L \leqslant 90°$, 1973–2002, (c) Southern California 1984–2001, 1988–1991, and 1995–1998, (d) Northern California 1998–2002, (e) Japan 1995–1998 and New Zealand 1996–2001, (f) Spain 1993–1997, New Madrid 1975–2002, and Great Britain 1991–2001. The data are spread vertically for a better view. The black solid curves are fits of our superposition formula (3.30). Empirical data taken from Corral (2004).

time superstatistics,

$$\psi_Q^{tot}(\Delta_Q t) = w_Q^- \psi_Q^-(\Delta_Q t) + w_Q^+ \psi_Q^+(\Delta_Q t),\ w_Q^- + w_Q^+ = 1,\ w_Q^- \geqslant 0,\ w_Q^+ \geqslant 0. \quad (3.30)$$

Fig. 3.10 shows that the above given formula describes the empirical data of earthquakes on different time scales well. Moreover, the data collapse for different regions and threshold values is clearly visible. Tab. 3.9 presents the values of the parameters of the fits for Fig. 3.10.

Both parts of the superposition (3.30) are necessary to describe the empirical data; however, w_Q^- and α_Q^- cannot be determined precisely as α_Q^- is too small. What is more, we did not observe a Q-dependence of exponent $\alpha_Q^\pm$ and $\tau_Q^\pm(Q)$.

Region	w_Q^-/w_Q^+	$\alpha_Q^-; \alpha_Q^+$	$\tau_Q^-(Q); \tau_Q^+(Q)$
a	90/9	0.001; 7.0	1.0; 1.0
b	1000/1	0.00001; 7.0	1.0; 1.0
c	11000/9	0.00001; 7.0	1.0; 1.0
d	1400/1	0.00001; 7.0	1.0; 1.0
e	20/12	0.005; 6.0	1.0; 0.8
f	80/11	0.001; 4.0	1.0; 0.7

Table 3.9: Values of the fit parameters $w_Q^\pm, \alpha_Q^\pm, \tau_Q^\pm(Q)$ from Eq. (3.30) for Fig. 3.10 (with an accuracy of about 5%).

Actually, each region (a)–(f) in Fig. 3.10 may be characterized by a single value of $\alpha_Q^\pm$ and $\tau_Q^\pm(Q)$ (cf. Tab. 3.9). Thus, from Eq. (3.14) and the lower branches of Eqs. (3.15) and (3.20) for the decreasing ζ, we obtain

$$\alpha_Q^\pm = (B_\pm)^{-1}; \ \frac{\tau_Q^\pm(Q)}{\tau_Q^\pm(0)} = \exp\left(\pm B_\pm \frac{Q}{\bar{\varepsilon}}\right) \tag{3.31}$$

with a constraint

$$\tau_Q^\pm(0) \propto \exp\left(\mp B_\pm \frac{Q}{\bar{\varepsilon}}\right). \tag{3.32}$$

Formula (3.30) reproduces the seismic empirical-data scalings in Fig. 3.10 well, since it contains two power laws: the term ψ_Q^- is responsible for the Omori law for short interevent times (Fujiwara, 2013; Corral, 2004), while the term ψ_Q^+ is responsible for a power law driven by Pareto-Lévy exponents greater than 2 for long interevent times.

Hereby, I have shown three exemplary applications of our model. The range of other possible ones is much larger; one may consider, e.g., some astrophysical or biological (genomic) noises that we have not examined yet (cf. Tsallis (2016)).

3.6 The conclusions of Part I

In the first part of the thesis I analyzed empirical data by means of a stochastic continuous-time random walk (CTRW) process and the extreme value theory (EVT). In this way I derived a key conjecture (3.11) that enabled us to obtain a general form of the interevent-time statistics in a closed analytical form (3.12), independent of the particular form of losses (or profits) distribution. The obtained formula is most characteristic due to its factorized form: the first universal factor is a power law, and the second dedicated factor strongly depends on the character of the stochastic process. For a process where larger losses (or profits) turn out to be less likely, it stabilizes the power law asymptotically. In the reverse situation, it acts as truncating factor, stabilizing an entire distribution only for short interevent times.

It needs to be highlighted that the obtained formula (3.12) can be derived using, for instance, a Weibull distribution and two q distributions (i.e., an exponential one and a Weibull one) as an underlying return distribution (q exponentials were used earlier by Ludescher et al. (2011) directly as the interevent-time distribution functions – cf. Sec. 1.2.1 in Chapter 1). These distributions describe the output empirical data of returns shown in Fig. 3.1 particularly well. It should be noted that condition (3.11) is a kind of a single-particle filter that permits a whole class of models instead of the single (particular) one.

The approach mentioned above provided the foundations for making the next step, i.e., constructing a two-particle filter, based on copulas, in other words, joint cumulative distribution functions. Thereby, the conditional quantile mean interevent-time range (3.8), based on the Weibull copula, was built.

Unfortunately, copulas for q Weibull and q exponential distributions are as yet unknown (which does not mean that they do not exist). Therefore, a construction of the corresponding two-particle distribution is impossible in these cases, though it is not required for obtaining the appropriate final result, and on the "external", one-particle level we cannot distinguish the right from the alternative approaches: the Weibull one and the q-function one. Notwithstanding that, formula (3.8) obtained using a two-variable Weibull distribution, describes the empirical data well – cf. Fig. 3.3. Ipso facto an approach was proposed that allows the selection of physical models required for the description of empirical data and, thereby, to better understand the phenomena occurring on varied financial markets and on different time scales.

I proved that the extreme value theory, together with a dependence between the events, are sufficient to describe the relevant threshold phenomena. Moreover, I found that the scaling shape exponent of our superstatistics scales itself (a "superscaling"). However, the profound physical meaning of this scaling requires some further studies.

Our approach opens extremely significant practical applications concerning risk dynamics. It may be used to calculate a dynamic risk function and, subsequently, to simulate a time series of value at risk (VaR), a quantity commonly used in financial risk analysis for the estimation of risk investment. This opens up opportunities to apply our model in economic theory and teaching, as well as the possibility of some practical investment or insurance applications.

I demonstrated a functional (not literal) balance between excessive profits and losses, that is, the interevent statistics for profits may be described by the same superstatistics formula (3.12) but with different parameter values. The simple extension of our approach, i.e., employing a superposition of two superstatistics, also provides a satisfactory description of empirical seismic data collapse, giving our work an interdisciplinary character. In the future, this may allow the occurrence of an earthquake to be modeled and maybe even predicted. On top of that, as we shall see in Part II, the results of our analytical model may be reproduced by means of an agent-based approach.

The model presented in this chapter is an analog of the canonical one-dimensional CTRW valley model with the valley depth being the magnitude of the loss (or profit). Our $\psi_Q(\Delta_Q t)$ statistics, Eq. (3.12), stands for the distribution of times between jumps and the PDF for finding a given particle at time lag $\Delta_Q t$ in the adjacent valley with a depth $\varepsilon > Q$. This analogy with the CTRW valley model leads to the assumed form of integration exponent $\alpha^{\pm}_Q$, Eq. (3.15), and subsequently, to the above mentioned superscaling. The superscaling, in turn, is the way in which we obtain required the universality of formula (3.12)[8].

Furthermore, we extended the canonical CTRW view, assuming a Weibull relaxation time instead of a canonical exponential one, but obtaining a decreasing power-law of $\psi_Q(\Delta_Q t)$ for $\Delta_Q t \gg 1$ as before. Although we are able to construct $\psi_Q(\Delta_Q t)$ for excessive losses and excessive profits separately, to develop a full CTRW formalism for financial markets we would have to consider both losses and profits simultaneously, which still remains a challenge.

The formalism I presented in this chapter reduces a broad range of empirical data to the use of a single control variable, pointing out the universality contained therein. This universality may lead us, finally, to find some deeper laws of market behavior. It is our hope that this work will constitute a strong contribution to the studies on the universal properties of financial markets.

After a presentation of the analytical description of times between excessive events, in Part II a numerical agent-based approach to this matter is presented. Thereby, the description of the phenomenon considered in this thesis becomes more complete and comprehensive.

[8]However, the meaning of the superscaling exponent ζ from Eq. (3.20) still remains unknown.

Part II

Simulations in the frame of agent-based models

4

Canonical agent-based models of financial markets

> *We wish to emphasize that the aim of simulations is not to provide better 'curve fitting' to experimental data than does analytic theory. The goal is to create an understanding of physical properties and processes which is as complete as possible, making use of the perfect control of 'experimental' conditions in the 'computer experiment' and of the possibility to examine every aspect of system configurations in detail.*
>
> – Landau and Binder (2014), p. 5-6

4.1 The concept of agent-based models

In this part of the thesis a complementary point of view for interevent-time universality is provided, by means of so-called *agent-based models* that open the possibility of describing the phenomena and processes occurring on financial markets at ab initio level.

Market modeling is one of the main challenges of modern economics, especially after the world wide financial crisis of 2007-08. This is also a canonical branch of econophysics (Cont, 2001; Bouchaud et al., 2008; Sornette, 2014; Schinckus, 2016). The main purpose of market models is to reveal the laws and underlying processes of market behavior, especially during bull- or bear-market periods. Such models enable us to make some predictions about markets, or at least to specify some signatures or warnings of upcoming changes that are signaled by the market data.

One of the central parts of economic and econophysical market modeling is the approach based on the *agent-based models* (ABMs), also called *computational economic models*. Nowadays, ABMs are widely exploited in economics (Farmer and Foley, 2009), sociology (Macy and Willer, 2002) and the environmental sciences (Billari, 2006). The concept of agent-based modeling is based on the connection between the micro- and macroscale activities of complex systems. The microscale is represented by a single *agent*, i.e., the basic element of ABM. It can be a single voter, an investor, a whole company, or even a whole country. The macroscale, or the considered complex system, can be a nation (e.g., in election modeling, Sznajd-Weron and Sznajd (2000)), a crowd (in evacuation models, Schadschneider et al. (2011)), a group of social networking service users (see, e.g., Bollen et al. (2011)), and last but not least, a market, which is the subject of the considerations presented in this part of the thesis.

Clearly, the possible range of ABM applications is wide, inspiring, and promising. A whole separate issue, which could have an impact on the final results, is the topology of a network (preferably realistic) assumed in a particular ABM (Newman, 2003; Rahmandad and Sterman, 2008; Alfarano and Milaković, 2009; Dorogovtsev, 2010; Kwapień and Drożdż, 2012; Dorogovtsev and Mendes, 2013; Raducha and Gubiec, 2016). Remarkably, the ABM approach constitutes a bridge between social science and physics, as we can consider agents in the model as a collection of particles in a physical system, with a wide range of statistical physics methods at our disposal. Besides simulations of varied social processes and some prediction making, ABMs frequently allow us to "look inside the black box" and understand the laws behind social and economic phenomena, such as the formation and bursting of stock market bubbles (see, e.g., Sec. 4.2).

This chapter presents several examples of financial-market agent-based models starting from the late eighties until today (cf. Samanidou et al. (2007); Cristelli et al. (2011); Sornette (2014)). Similarly to Chapter 2 from Part I, this chapter is also intended to be a preparation for reading the subsequent one (Chapter 5), where our second model of interevent times, i.e., the numerical model of cunning agents, is presented. The main purpose of this chapter is to outline the idea of agent-based models and place the description of the cunning-agents model in a broader historical context.

In Secs. 4.2 and 4.3 the pioneering ABM works are presented, then, in Secs. 4.4 and 4.5, the well-known Lux–Marchesi and Cont–Bouchaud models are shown. In Secs. 4.6 and 4.7 the econophysical idea of the Ising-model market application is discussed and two relevant models are presented. In Sec. 4.8 an alternative attempt to reproduce the interevent-time distribution by an ABM (for that presented in Chapter 5) is shown. Finally, in Sec. 4.9, conclusions are given.

4.2 The Kim–Markowitz multiagent model[1]

The model by Kim and Markowitz (1989) is considered to be the first modern multiagent model of financial markets (Samanidou et al., 2007). Performing their microsimulation study, the authors were motivated by the stock market crash of 19th October 1987 (so-called Black Monday) when the Dow Jones Industrial Average index decreased by more than 20% over a single day. Because the occurrence of this exceptional crash could not be caused only by exogenous information, Kim and Markowitz concentrated on some noninformational factors in their research. Their main concept was to exploit portfolio insurance strategies on the market.

In the Kim–Markowitz (KM) model we consider two groups of investors, (i) the *rebalancers* and (ii) the *portfolio insurers*, each trading in two assets, i.e., stocks and cash. The total interest rate on the market is equal to 0. The wealth w of each agent at time step t is given by the formula

$$w(t) = q(t)p(t) + c(t), \tag{4.1}$$

where $q(t)$ is the number of stocks, $p(t)$ is their price at time step t, while $c(t)$ is the amount of cash that the agent holds at t. The main purpose of the investment in the case of rebalancers is to maintain a balanced portfolio with an equal amount of stocks and cash, i.e.,

$$q(t)p(t) = c(t) = 0.5w(t). \tag{4.2}$$

Rebalancers stabilize the market: when prices increase, they raise their supply or reduce their demand, and vice versa in the case of a price decrease.

The second group of investors, the portfolio insurers, focuses on keeping some minimal wealth f, called the *floor*, at a specified date of insurance expiration, according to the *constant proportion portfolio insurance* (CPPI) method by Black and Jones (1987). The method is based on keeping a proportion of the asset value (called the *cushion*) constant and the difference s between the current value of the portfolio w and the floor f. This strategy can therefore be expressed as

$$q(t)p(t) = ks(t), \tag{4.3}$$

where $s(t) = w(t) - f$, and the multiplier k is chosen greater than 1, which allows portfolio insurers to benefit from a possible price increase.

[1] Based on Samanidou et al. (2007).

When the prices decrease, the wealth of the portfolio insurers also decreases, but not lower than the bottom value f. In the case of a bear market, the fraction of risky assets in their portfolio is close to zero and they possess only riskless cash of a value equal to f (to good approximation), which is the minimal value of their portfolio (providing that the trading frequency is high enough). This kind of investment strategy has an effect similar to put options. The portfolio insurance strategy, contrary to the strategy of rebalancers, leads to destabilization of the financial market, as portfolio insurers buy stocks during a bull market and sell them during a bear market. The magnitude of this destabilizing effect is proportional to the value of the multiple k.

The evolution of the stock price and the trading volume in the model is a result of demand and supply changes. From time to time, investors have an opportunity to change their portfolio (i.e., when a given investor is drawn). They compute the price forecast $p^i_{est}(t)$ for their assets according to following rules:

1. If only buy orders (asks) exist, the estimated price is 101% of the highest ask price.

2. If only sell orders (bids) exist, the estimate is 99% of the lowest bid price.

3. If both asks and bids exist, the estimate is an average between the highest ask and the lowest bid from the previous period.

4. If there are no orders on the market, the estimate is equal to the previous trading price.

For rebalancers, if the estimated ratio between stocks and assets is higher (smaller) than the desirable 0.5, they place, respectively, a sale (buy) order with the price $p^i_{bid}(t) = 0.99p^i_{est}(t)$ for the sale (or $p^i_{ask}(t) = 1.01p^i_{est}(t)$ for the purchase). Similarly, portfolio insurers place a sale (buy) order with the above values of the price when the estimated ratio between stocks and cushion is higher (smaller) than the desirable value k (however, some deviations from the target value, within a precisely defined range, were tolerated, cf. Samanidou et al. (2007)). These orders are matched with each other in the model order book, i.e., if there is a counter-offer that matches the already placed one, the orders are executed immediately with the price of the counter-offer. If the matching offer does not exist, the order is put on a list to be filled later, when the relevant counter-offer comes. If some orders are still open at the end of the trading day, the agents who placed them can reevaluate their portfolio structure and place a new order. When this is completed and all the agents had a chance to trade, the trading day finishes. Agents who

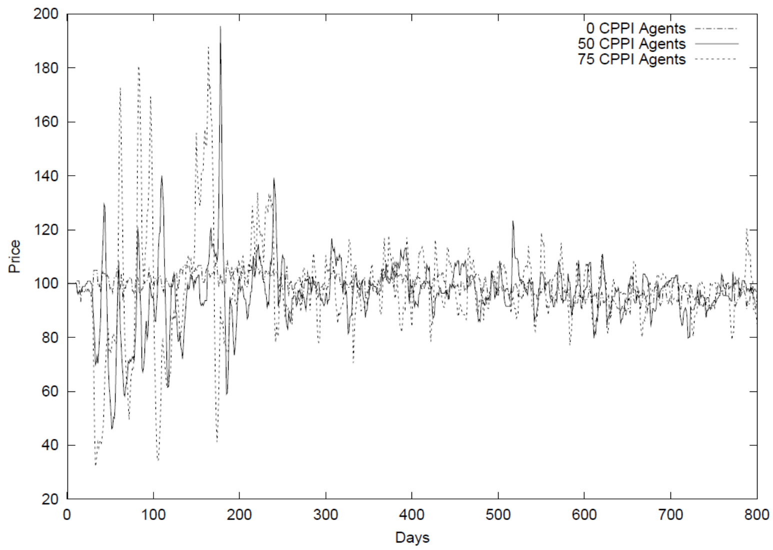

Figure 4.1: Daily price changes in the Kim–Markowitz model with different numbers of CPPI agents: 0 (dashed-dotted line), 50 (solid line), and 75 (dashed line) among a total number of 150 agents. The oscillation amplitude increases with the increasing CPPI agents fraction. Taken from Samanidou et al. (2007).

have lost all their wealth (their cash plus the value of stocks) are eliminated from further trading.

The original simulations by Kim and Markowitz started with rebalancers having more or less stocks than in the state of equilibrium. The influence of an exogenous market was incorporated by random flows of cash for each investor imitating deposits and withdrawals, wherein the average amount of the deposit was higher than the average withdrawal amount.

The simulation made by Kim and Markowitz demonstrated that portfolio insurance strategies destabilize the financial markets (see Fig. 4.1). It was the most important result of their approach and also an important voice in the academic discussion on the causes of the 1987 crash. Although the model rather does not reproduce the empirical scaling laws (Samanidou, 2000), it was not created to do that. In fact, it is a representative example of developing an approach dedicated to describing a specific problem, and in this role works satisfactorily. Furthermore, the model was, in those days, pioneering and should be treated as such in all kinds of assessments and rankings.

4.3 The Levy–Levy–Solomon financial-market model[2]

Unlike the Kim–Markowitz model, the example considered next, the fruit of collaboration between economists and physicists from Hebrew University, is much simpler in terms of the assumed microscopy of agents activity. Such an approach is characteristic for econophysical agent-based models rather than economic ones. The model was initially presented in Levy et al. (1994) with some later improvements (see Samanidou et al. (2007) for refs.).

4.3.1 The model description

The Levy–Levy–Solomon (LLS) model consists of a set of n interacting investors. At the beginning of the simulation, each investor has the same initial wealth, W, divided individually into shares and bonds (no other forms of wealth, especially cash, are allowed). If we denote the share of stocks in the ith investor's portfolio as X_i, we can decompose his wealth $W_i(t+1)$ in the following way,

$$W_i(t+1) = \underbrace{X_i W_i(t)}_{\text{sum of shares}} + \underbrace{(1-X_i)W_i(t)}_{\text{sum of bonds}}, \tag{4.4}$$

where t is time. The initial share of stocks is the same for each investor. The supply of shares in the model, N_A, is assumed constant. The bonds earn a fixed interest rate r, while the stocks return $H(t)$ is a composition of the price $p(t)$ changes and the dividend payment $D(t)$, namely

$$H(t) = \frac{p(t) - p(t-1) + D(t)}{p(t-1)}. \tag{4.5}$$

The utility function U, specifying the preferences of investors[3], is assumed to be logarithmic in the basic version of the model, i.e., $U(W(t+1)) = \ln W(t+1)$, and therefore the law of diminishing marginal utility is fulfilled therein, and the investors are risk-averse. Moreover, the utility function in this form results in a wealth-independent optimal ratio of stocks in the investor's portfolio, and therefore the share of stocks is constant.

Investors in the model form groups defined by the memory length k, i.e., investors from group G form the expected value of their returns in the future based on their k last total stock returns, namely, $H(t-k+1), H(t-k+2), \ldots, H(t)$.

[2]Based on Samanidou et al. (2007).

[3]For more details about utility function and expected utility function see Varian (1999).

Each of the remembered k returns is expected to reappear with a probability equal to $1/k$ in the next period. The corresponding expected utility function takes the form

$$EU(X_i^G) = \frac{1}{k}\left\{\sum_{j=t}^{t-k+1} \ln\left[(1 - X_i^G)W_i(t)(1 + r) + X_i^G W_i(t)(1 + H(j))\right]\right\}, \quad (4.6)$$

where X_i^G is the share of stocks for the ith investor, belonging to group G. Therefore the maximization formula for this function is

$$f(X_i^G) = \frac{\partial EU(X_i^G)}{X_i^G} = \sum_{j=t}^{t-k+1} \frac{1}{X_i^G + \frac{1+r}{H(j)-r}} = 0. \quad (4.7)$$

The limits for the share of stocks in the investor's portfolio, imposed by the authors, are 0.01 (lower) and 0.99 (upper). Moreover, neither short-selling nor loans taking is allowed. So far, the model seems to be perfectly deterministic. However, the authors incorporated some randomness into it, that is, after the optimum share X_i^G calculation for the ith investor, a noise term from the normal distribution, ϵ_i, is added to the result. This term stands for an individual demand or supply of each investor and represents, e.g., the investor's own beliefs or emotions. In Levy et al. (1994) this term has a physical interpretation as a result of the market "temperature". The stock price and, therefore, the new return value in each time step is determined from the aggregated demand of all the investors. Finally, the oldest return in the investors' memory vanishes and this new one is added there.

4.3.2 Results

Some selected results of the model are shown in Figs. 4.2 and 4.3. In the case of only one group of investors, the price in the model changes periodically with strong growths and drops, where the length of a cycle depends on the parameter k (Fig. 4.2). The growths mentioned above are caused by a positive feedback effect, when the price increases encourage investors to buy more stocks (as in their memory span some new positive returns are appearing and the possible negative old ones are vanishing) which causes further price increases, etc. When the share of stocks in the investors' portfolios is close to the maximal value, further price growth becomes impossible, and because the price is high, the dividend yield D/p is relatively small, so the price growth stops. In this case, after k time steps with small values of return, when the previous higher values disappear from the memory

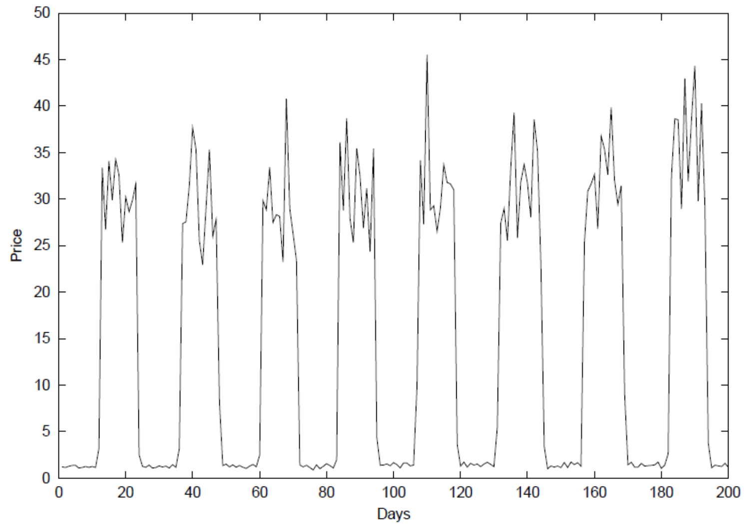

Figure 4.2: Sample price fluctuations in the LLS model with a single group of traders and a logarithmic utility function. The cyclical behavior of the series can be clearly seen. Taken from Samanidou et al. (2007).

span of the investors, even a slight decrease of the price (possible because of the presence of the noise term ϵ_i in the model) can be an incentive to the emergence of a crash caused by negative feedback, in the same way as the rapid growth was previously caused by the positive feedback effect. If that occurs, the share of stocks in the investors' portfolios drops to the minimal value, the price is then low and, consequently, the dividend yield is relatively high, which is an incentive for new rapid growth, and thus the cycle closes.

When the simulation includes two groups of investors with different memory spans k, the aforementioned periodicity is still possible, but for some values of memory spans some other dynamic patterns occur, which is associated with the interaction of both groups investing on the market. In the presence of three or more investor groups some other irregularities appear, shown in Fig. 4.3 (for three groups with $k = 10, 141, 256$ and $n = 100$ investors)[4]. The long-run results of the model are sensitive to the initial conditions, i.e., the drawn initial "history" of the agents. Depending on the particular drawing, either the group with $k = 256$ or $k = 141$ may dominate on the market.

[4]However, the behavior becomes periodical again as n increases (Hellthaler, 1996).

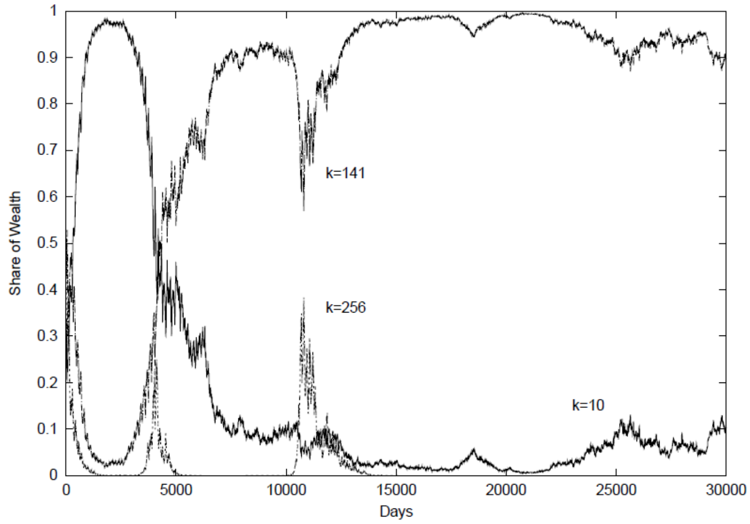

Figure 4.3: The development of wealth distribution in a simulation of the LLS model for three groups with $k = 10, 141$, and 256. In this case the group with $k = 141$ dominates the market. Taken from Samanidou et al. (2007).

Despite the variety of possible results it gives, the LLS model does not reproduce power-law return statistics (they are simply Gaussian) and the shape of the autocorrelation function of absolute returns (which quickly decays to zero as the autocorrelation of raw returns (Zschischang, 2000)). However, some other related models by these authors reproduce the desirable scaling laws (Levy and Solomon, 1996). As the time unit in the model is not precisely defined – it could be a day as well as a month – the Gaussian statistics of returns that the model yields can be correct even if we assume that this concerns the case of a larger time scale, although the underlying mechanism of this result is different than in real markets (Samanidou et al., 2007).

4.4 The Lux–Marchesi model of speculations[5]

4.4.1 The model description

The next model, presented for the first time in Lux and Marchesi (1999), is based on the concept of mutual exchange and interaction between different groups of

[5]Based on Samanidou et al. (2007).

investors and the process of price adjustments with a demand-supply imbalance. In the Lux–Marchesi (LM) model, we consider an artificial market with N agents (investors), n_c of which are the *noise traders* (or *chartists*), while n_f are the *fundamentalists*, holding $n_c + n_f = N$ (cf. Kim–Markowitz's *portfolio insurers* and *rebalancers* from Sec. 4.2). A group of n_+ noise traders are optimists, while n_- of them are pessimists, holding $n_+ + n_- = n_c$. There are two values of price, the market price, p, and the fundamental value, p_f.

Chartists' opinion-changing

Noise traders change their opinions (or moods) from pessimistic to optimistic and vice versa with the probabilities $\pi_{+-}\Delta t$ and $\pi_{-+}\Delta t$, respectively, where Δt is considered a (small) time interval and

$$\begin{array}{rcl} \pi_{+-} & = & v_1 \frac{n_c}{N} \exp(U_1), \\ \pi_{-+} & = & v_1 \frac{n_c}{N} \exp(-U_1), \\ U_1 & = & \alpha_1 x + \frac{\alpha_2}{v_1} \frac{\mathrm{d}p}{\mathrm{d}t} \frac{1}{p}, \end{array} \tag{4.8}$$

where $x = \frac{n_+ - n_-}{n_c}$ is an average chartist's opinion and $\frac{\mathrm{d}p}{\mathrm{d}t}\frac{1}{p}$ is a current price trend. It can be seen that π_{ab} is the probability of transition between states a and b, herein, an optimistic and a pessimistic state. The coefficients v_1, π_1, and π_2 are some adjustment parameters specifying, respectively, the frequency of opinion changes, the importance of overall opinion, and the importance of the current trend in the agents' opinion-shifts (parameter α_2 is normalized by v_1, because the trend is considered over the *average* time interval, cf. Samanidou et al. (2007)). The term n_c/N is a fraction of the noise traders among all the investors.

Chartist-fundamentalist conversions

As above, changing groups between noise traders and fundamentalists is based on transition probabilities, namely

$$\pi_{+f} = v_2 \frac{n_+}{N} \exp(U_{2,1}), \ \ \pi_{f+} = v_2 \frac{n_f}{N} \exp(-U_{+,f}), \tag{4.9}$$

$$\pi_{-f} = v_2 \frac{n_-}{N} \exp(U_{2,2}), \ \ \pi_{f-} = v_2 \frac{n_f}{N} \exp(-U_{-,f}), \tag{4.10}$$

where

$$U_{+,f} = \alpha_3 \left\{ \frac{r + \frac{1}{v_2}\frac{\mathrm{d}p}{\mathrm{d}t}}{p} - R - s \cdot \left| \frac{p_f - p}{p} \right| \right\}, \tag{4.11}$$

$$U_{-,f} = \alpha_3 \left\{ R - \frac{r + \frac{1}{v_2}\frac{\mathrm{d}p}{\mathrm{d}t}}{p} - s \cdot \left| \frac{p_f - p}{p} \right| \right\}. \tag{4.12}$$

Quantities $U_{+,f}$ and $U_{-,f}$ are the differences of profits made by chartists (optimistic or pessimistic, respectively) and fundamentalists, with coefficient α_3 standing for an investor's sensitivity to profit changes. The first terms in $U_{+,f}$ and $U_{-,f}$ represent the profit of noise traders from the optimistic and pessimistic groups, respectively, while the second term is profit earned by the fundamentalists. The parameter v_2 stands for the speed of an investor's trading strategy verification. Excess profit (in comparison with some alternative investments with the rate of return per asset equal to R) made by chartists consists of a nominal dividend, r, and the revenue from the price change $\frac{\mathrm{d}p}{\mathrm{d}t}$, divided by the market price p to give an average gain from one asset. The order of the terms – the term described above and the R term – in the excess profit calculation depends on the mood of the chartists, i.e., whether they gain from asset price increases (for optimistic ones) or falls (for the pessimistic) as it is clearly seen in the above formulas. On the other hand, the second part of the U functions or the fundamentalists' profit consists of the absolute deviation between the actual and fundamental price, p and p_f, representing their potential revenue from arbitrage strategy. The factor $s < 1$ stands for the uncertainty of the fundamentalists' profit size, as it occurs only in the unsettled future, in contrast to the immediate noise traders' profit. Moreover, the fundamentalists' gain does not contain the dividend r, because we assume they anticipate it as the source of alternative investments, namely $r/p_f = R$, and the corresponding terms cancel out.

Price changes

The price change in the LM model is an endogenous market response to an imbalance of demand and supply. The difference between demand and supply in the model, called the *excess demand*, ED, is defined as a sum of the component excess demands ED_c and ED_f for the chartists and fundamentalists, respectively, where

$$ED_c = (n_+ - n_-)t_c \quad \text{and} \quad ED_f = n_f \cdot \gamma \frac{p_f - p}{p}. \tag{4.13}$$

For the chartists, ED_c is proportional to the difference between the number of optimists and pessimists, as they provide demand and supply, respectively; t_c denotes an average trading volume per transaction. For fundamentalists, in turn, ED_f is proportional to deviations between actual and fundamental price, with the coefficient γ standing for the strength of the response.

Like the conversions between the groups of investors, price adjustments are also defined using some transition probabilities. Specifically, if a percentage of a price increases (↑) or decreases (↓) during a certain time, Δt, fulfills $\Delta p = \pm 0.001p$, the corresponding transition probabilities are

$$\pi_{\uparrow p} = \max[0, \beta(ED + \mu)] \quad \text{or} \quad \pi_{\downarrow p} = -\min[\beta(ED + \mu), 0], \tag{4.14}$$

respectively, with parameter β quantifying the speed of the price adjustment[6] and parameter μ establishing some additional stimulus for an increase (or decrease) of the price, even if the overall excess demand ED is negative (or positive).

Price changes in the model influence investors' decisions, namely, when the price goes up, pessimistic noise traders are likely to move into the optimistic group and inversely, when the price goes down, optimists tend to change their mood to the pessimistic one. Price changes also affect the length of $p_f - p$ interval, specifying the strength of fundamentalists' beliefs, or alternatively the chartists' force. In turn, the changes mentioned above affect the size of the excess demand and, consequently, the price, and the circle closes.

As for the fundamental price, to eliminate the possibility that some of the stylized facts result from exogenous factors, Lux and Marchesi assumed that the fundamental price p_f follows an exponential Brownian motion (Ross, 2014), i.e., its logarithm follows a Brownian motion, $\ln p_f(t) - \ln p_f(t-1) = \varepsilon_t$, where $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon)$ and $\mathcal{N}$ denotes a Gaussian distribution.

Some additional assumptions

To ensure a Poisson type dynamics with asynchronous investors' strategies and opinions updating in the model, one has to make some additional assumptions. For instance, in order to prevent artificial synchronizations during the simulation, appropriately small increments of time should be chosen[7]. Similarly, to prevent the system from falling into "absorbing state", i.e., either $n_c = 0, n_f = N$, or $n_c = N, n_f = 0$, some bounds for the minimal/maximal number of chartists and fundamentalists should be set (cf. Sec. 5.2.2 from the next chapter).

4.4.2 Results

Volatility clustering, power-law distributions of returns, or the long-term autocorrelation of absolute returns are the real-market characteristics covered by the LM

[6] Actually, one obtains $\frac{\mathrm{d}p/\,\mathrm{d}t}{p} = \beta \cdot ED$.

[7] For more information see Samanidou et al. (2007) and refs. therein.

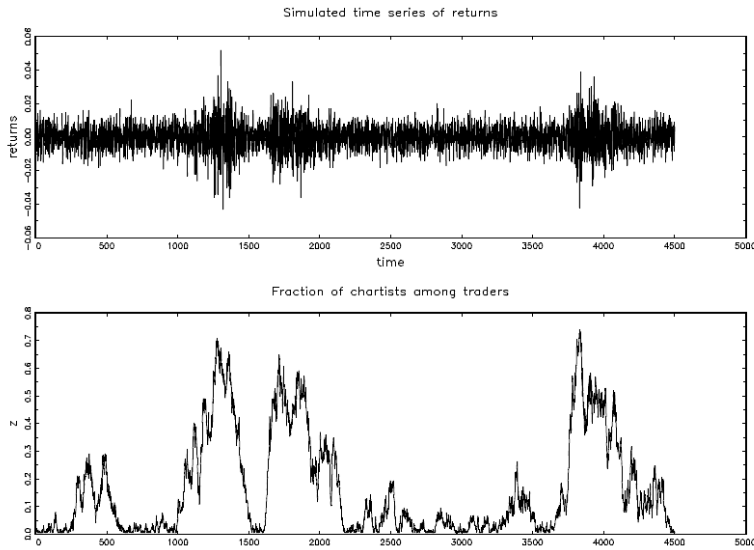

Figure 4.4: Returns (on the top) and the corresponding ratio of chartists on the market (at the bottom) in an example simulation of the LM model. The mutual dependence of these two is clearly visible. Taken from Samanidou et al. (2007).

model. Strong interconnection between the chartists number and the amplitude of price changes in the model (cf. Fig. 4.4) indicates that they are actually responsible for large fluctuations on the market (cf. a similar influence of portfolio insurers on the market in the Kim–Markowitz model, Sec. 4.2).

It is noteworthy that the LM model has a feature of self-stability, since the number of noise traders self-reduces after a period of large fluctuations. This is a result of frequent transitions to the fundamentalist group in such conditions, as their potential profits are high then.

The Lux–Marchesi model was thoroughly investigated by a number of researchers in a large number of publications, proving its usefulness and capabilities (cf. Samanidou et al. (2007) and refs. therein). Although the model is indeed comprehensive and able to reproduce or even explain many real-market characteristics, the large number of independent parameters used by the authors remains its major disadvantage in further studies.

4.5 The Cont–Bouchaud percolation model[8]

In contrast to the previous model by Lux and Marchesi, the one presented next is rather simple and has a small number of free parameters. The main concept used in the Cont–Bouchaud (CB) model (Cont and Bouchaud, 2000a) is the percolation phenomenon known from physics, chemistry, material science, and also mathematics. Recently, percolation models have been widely used in econophysical financial-market modeling (Stauffer, 2001), however, percolation theory was created to explain a chemical phenomenon of polymer gelation, occuring, e.g., when an egg is cooked (Flory, 1941). The father of the theory was the 1974 Nobel laureate in chemistry Paul Flory. Thereafter the theory was applied in many different issues, from strictly technical ones, such as coal-dust filters (Broadbent and Hammersley, 1957) to extremely abstract ones, such as the origin of life (Kauffman, 1993).

In the percolation-model approach, we consider a large lattice with each of the sites either occupied with some probability, p, or empty, with probability $1 - p$. Each group of occupied neighbors in the lattice form a *cluster*. The mass of a cluster, s, is defined as the number of occupied sites which belong to it. If the mass of a cluster increases with a positive power with the lattice size, we call it an *infinite cluster*. As the probability p increases, at some level p_c between 0 and 1 the first infinite cluster appears. For a large lattice that is isotropic enough, when $p > p_c$, there is exactly one infinite cluster, filling the fraction p_∞ of this lattice, while for $p < p_c$ the infinite cluster does not appear. For $p = p_c$, an incipience of the infinite cluster in the form of a fractal occurs. When p falls, approaching to p_c value from above, the fraction p_∞ disappears proportionally to $(p - p_c)^\beta$, where the critical exponent β $(0 < \beta < 1)$ is increasing with the dimension d of the lattice.

Taking into account the percolation model methodology, Cont and Bouchaud considered the lattices[9] (or markets) where each occupied site represents the trader, while the clusters represent groups of traders who make collective decisions. At each time step t, a cluster trades with the probability $2a$, otherwise it sleeps with the probability $1-2a$. The trading can be either buying or selling (usually each case with the probability a), where the amount of trade is proportional to the size s of a cluster. The market price p is proportional to the exponential function of the difference between the overall supply and demand on the market.

[8]Based on Samanidou et al. (2007).

[9]In further studies, the assumed dimension d of the lattice were between 2 and 7, cf. Stauffer (2001).

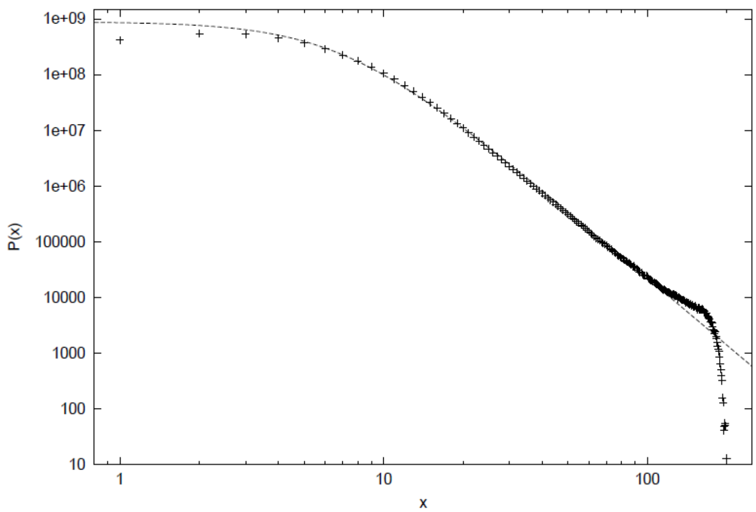

Figure 4.5: The averaged distribution of price changes in the Cont–Bouchaud model for simulations with 301×301 square lattice (black pluses). The dashed curve is the Student-t distribution (see de Souza and Tsallis (1997)). The cut-off for large values of price change results from the finite size effect. Taken from Samanidou et al. (2007).

As for the concentration value p, it can be constant, varying between 0 and 1, or between 0 and p_c.

The model in practice generates either a power-law or Gaussian-tailed distributions of price changes, depending on whether the activity parameter a is small or close to the largest possible value $1/2$, respectively. The exponent of the aforementioned power-law for $d = 2$ or 3 (which seems to be a realistic case) is, for the cumulative distribution, close to desired three (cf. Fig. 4.5). Other stylized facts that CB model and its modifications reproduce are, i.a., volatility clustering, positive trading volume and returns correlations, asymmetry between the sharp peaks of price and its flat valleys, and oscillations of price close to logperiodic ones[10]. The model was successfully applied in the studies on triple correlations between the Tokyo, Frankfurt and New York markets (Schulze, 2002) as well as on the effects of a small Tobin tax on all transactions (Ehrenstein, 2002).

[10] See Stauffer (2001) for more details. For the information about logperiodicity on financial markets see, e.g., Drożdż et al. (2003).

The approach by Cont and Bouchaud[11], i.e., to take a physical model (in this case the percolation model) and reuse it for a certain economic issue by making some reinterpretation of its quantities, is widely used in econophysics, as opposed to the classical economic approach, where in creating a model we start with strictly economic, real-life basis and assumptions. Nevertheless, although the CB model and its further improvements provide many interesting results resembling some of the empirical ones, these results are mostly obtained by making very specific assumptions. This calls into question the utility of the model and its potential to provide an explanation of empirical findings that is "independent of microscopic details", as Stanley et al. (1996b) postulate in their econophysics manifesto (cf. Samanidou et al. (2007)).

4.6 The Iori model

4.6.1 Ising-based network models – introduction[12]

The Ising model, commonly used in varied branches of physics over the last century, was introduced by the German physicist, Wilhelm Lenz, in 1920, as a model of ferromagnetism, and named after his student Ernst Ising who solved it for the one-dimensional case (Ising, 1925). The Norwegian-born American physicist and chemist, Lars Onsager, found the analytical two-dimensional solution in zero field (Onsager, 1944). Higher-dimensional cases, and also the case of nonzero field in two dimensions are still unsolved analytically, yet we have certain numerical methods to make some approximate calculations (cf. Münkel et al. (1993)). The model is, basically, some finite number of elements with some finite number of possible states interacting with each other. In the original version these elements were magnetic moments, or spins. Spins are arranged in a kind of graph, or network, with links between some of them.

In the basic version of the Ising model only two possible states of each spin are assumed, denoted by $+1$ or -1 (for ferromagnetic material they represent one of the two possible directions in the material that the single spin can point in, e.g., "up" or "down"). Spins only interact with their nearest neighbors that have a tendency to self-arrangement, i.e., to form a structure where all of them are point in the same direction. On the other hand, they are also punched out of this state, due to the nonzero temperature of the crystal. This tension between the

[11] The conclusions in this paragraph are written on the basis of Samanidou et al. (2007).
[12] With the help of Sornette (2014).

ordering and disordering forces in the model yields a nontrivial system behavior, with phase transition and bifurcation for two or more dimensions.

Incidentally, if the network is not regular, but node degrees vary according to some given distribution, the usual coordination number of the network has to be replaced by the *mean coordination number* to maintain the results mentioned above. The derivation of the result remains the same; however, the critical temperature changes. This approach is called the *mean field approximation* (Weiss, 1907). If we want the results to be more precise, we can extend this view by making some small amendments[13]. In this way, the Ising model on a regular network can be replaced by a model with an irregular network.

Aside from the physics of magnetic materials, the applications of the Ising model and its generalizations in physics of ill-condensed matter (Mézard et al., 1987) and neurobiology (Hopfield, 1982) are well-known. The application of Ising-based models in social science was initiated by the German physicist, Wolfgang Weidlich (Weidlich, 1971). His pioneering work started the development of quantitative sociology in the 1970s.

In its socioeconomic version, the Ising model describes, in a simple way, the competition between the ordering social interactions and the disordering individual conclusions, opinions and emotions of each separate unit. The link between physical and socioeconomical modeling is constituted here by economic *discrete choice models* (Bierlaire, 1998). Discrete choice models consider entities making decisions by choosing between a set of alternatives (e.g., between two political parties to vote for or some financial assets to buy or sell). Combining this approach with the Ising model, we obtain a model where each spin represents a single entity: an investor, a voter, a company, etc. and his/her current state (or alternatively the change of this state, see Chapter 5) is somehow related with the decision he actually makes.

Furthermore, some psychological justification for socioeconomical reinterpretation of the Ising model, with respect to microscopic interactions, is provided by the *information integration theory* (IIT) of Anderson (1962). The theory determines how the information from multiple sources influences the final decision of a given entity and constitutes some alternative for the traditional economic approach based on a concept of *utility maximization*, as it allows the decisions of the entity to be irrational, unlike in the traditional approach, which assumes fully rational behavior. This quality is suitable for the behavior of investors on a financial market as they frequently act in conditions of stress and with the neces-

[13]See also Aleksiejuk et al. (2002); Dorogovtsev et al. (2002, 2008) for information about the dependence between assumed network topology and the Ising model results.

sity of making a quick decision, which is certainly not favorable for any rational movements.

In the spirit of IIT, the states of the neighbors of a given spin in the Ising model influence its current state in the same way as the investor's final decision is a derivative of his colleagues' advice (related somehow to their own investment decisions; cf. Anderson and Shanteau (1970)). In the Ising model, the influence of others on a given spin is additive, due to the cumulative character of the energy, and this is actually also the basic case considered in the theory of information integration, that is, if we denote the jth stimulus by s_j and its importance (a weight) by w_j, the overall response R (that can be used then for making a decision) is given by[14]

$$R = C + \sum_j w_j s_j, \tag{4.15}$$

where C is a constant that stands for a response-bias. Moreover, the concept of weighting information deriving from different sources (as, e.g., less or more reliable or important for the entity) present in the IIT is also used by econophysicists in their financial Ising-like models (Zhou and Sornette, 2007; Lipski and Kutner, 2013).

The above presented analogy between the Ising model and the financial markets is quite obvious and produces behavior such as spontaneous symmetry breaking or phase transitions in the artificial social systems under consideration, together with a great number of statistical physics methods that one can adapt[15].

4.6.2 The model assumptions and results[16]

The first Ising-based market model presented here is the Iori model, which is well-known in econophysics (Iori, 1999). In the model we consider a square lattice $L \times L$, where each node $i = 1, \ldots, L^2$ stands for an agent, or investor (cf. the lattice of traders in the Cont–Bouchaud model, Sec. 4.5). The links between the nodes are the social (business) links between investors. Each agent interacts only with its four nearest neighbors on the lattice, wherein periodical boundary conditions apply (the agents at the upper edge of the lattice influence those at the lower edge, and similarly for the left and the right edge). In other words, the interaction matrix in the model has a torus-like topology.

[14] Compare with Eqs. (4.17), (4.26), and (5.1).

[15] For more information about financial application of the Ising model see, e.g., Dvořák (2012); Sornette (2014).

[16] Based on Iori (1999); Denys (2011).

At the beginning of the simulation, every agent has the same initial capital, consisting of two values: the cash $M_i(0)$ and $N_i(0)$ stock units. At each time step t a given agent, i, undertakes one of three possible actions, represented by a three-state spin variable $S_i(t)$: buying one unit of the stock ($S_i(t) = +1$), selling it (-1) or remaining inactive (0). The decision the agent makes is limited by his capital (i.e., the agent may not be able to buy or sell stock because of a lack of cash or stocks in his portfolio) and by a condition that he can buy or sell only one unit of stock in each time step.

Aside from the traders, the role of *market maker* also features in the model. Its job is the execution of orders and price adjustment. The market maker also starts the trading with an initial amount of money and stocks, varying in time and limiting its transaction capabilities, although its investment decision at a given time step t is not limited to one unit of stock, as with other investors.

Decision-making by agents

Each agent, i, makes its decision at time step t basing on three factors:

1. The aggregated opinion of its four nearest neighbors interacting with it, where each component is represented by the term $S_j(\tilde{t})$ (j is the number of agents with whom the ith agent interacts). Variable $\tilde{t}$ means the time between the beginning and the end of a given time period t, in other words the presence of the variable $\tilde{t}$ tells us that changes occur instantaneously, one after another, not all at the end of the time period t.

2. Its own individual opinion about market conditions, represented by factor $\nu_i(t)$, which is a stochastic variable from the uniform distribution on the interval $[-1, 1]$.

3. The overall signal $\epsilon(t)$, which is the same for all agents, representing exogenous market information and analogous to an external field in the classical version of the Ising model.

Thus, the decision making formula is:

$$\begin{array}{lll} S_i(\tilde{t}) = 1 & \text{if} & Y_i(\tilde{t}) \geqslant \xi_i(t), \\ S_i(\tilde{t}) = 0 & \text{if} & -\xi_i(t) < Y_i(\tilde{t}) < \xi_i(t), \\ S_i(\tilde{t}) = -1 & \text{if} & Y_i(\tilde{t}) \leqslant -\xi_i(t), \end{array} \tag{4.16}$$

where $\xi_i(t)$ is the activation threshold, specific for each agent i, and $Y_i(\tilde{t})$ is an aggregated signal for this agent, given by

$$Y_i(\tilde{t}) = \sum_{\langle i,j \rangle} J_{ij} S_j(\tilde{t}) + A\nu_i(t) + B\epsilon(t). \tag{4.17}$$

In the above formula J_{ij} is an interaction matrix (specifying the influence of the jth neighbor's decision S_j on the ith agent; $\langle i, j \rangle$ means that the sum is made over the nearest neighbors), while coefficients A and B measure the strength of the agent's individual opinion and the exogenous information, respectively (cf. Eqs. (5.1) and (5.2) from the next chapter).

The activation threshold $\xi_i(t)$ forms a kind of trading friction, and can be interpreted, e.g., as some transaction cost (in a broad sense). The signal $Y_i(\tilde{t})$ has to exceed this threshold if the agent trades. Values $\xi_i(t)$ are initiated by random variables from the Gaussian distribution, with the standard deviation $\sigma_\xi(0)$ and the mean equal to 0. From one trading period to another these values evolve for each agent proportionally to the stock price movements. The author of the model assumed that the interaction matrix is symmetrical.

The volume and the price

When the process of decision making comes to an end, the traders together with the market maker submit their orders and trading at a single price takes place. The market maker looks at the aggregated demand, $D(t)$, and supply, $Z(t)$, for stocks at time t,

$$D(t) = \sum_{i:S_i(t)>0} S_i(t), \qquad Z(t) = -\sum_{i:S_i(t)<0} S_i(t), \tag{4.18}$$

together with the volume $V(t) = Z(t) + D(t)$, and then adjusts the price P according to the formula

$$P(t+1) = P(t) \left(\frac{D(t)}{Z(t)} \right)^{\alpha}, \tag{4.19}$$

where

$$\alpha = a \frac{V(t)}{L^2}, \tag{4.20}$$

with some proportionality coefficient a and L^2 being the number of agents and, simultaneously, the maximal number of stocks that could be traded at each time step.

The logarithmic return $r(t)$ and the market volatility $\sigma(t)$ are defined as follows:

$$r(t) = \ln \frac{P(t+1)}{P(t)}, \; \sigma(t) = |r(t)|. \tag{4.21}$$

The influence of the price change on the threshold value in the subsequent time step is assumed as

$$\xi_i(t+1) = \frac{P(t+1)}{P(t)} \xi_i(t). \tag{4.22}$$

Evidently, full symmetry between buying and selling activity is preserved.

Results

The author of the model reproduced the main stylized facts gathered from real financial markets, as volatility clustering of returns and positive correlation between volatility and trading volume (cf. Fig. 4.6), the power-law decay of autocorrelation function of volatility (i.e., the absolute returns), or fat-tailed return statistics. We must, however, note that these results were achieved for specific parameters of the simulation only.

4.7 The Bornholdt spin model[17]

The Bornholdt model was presented for the first time in Bornholdt (2001), and then developed in Kaizoji et al. (2002). The model is useful to describe the price changes of an asset in short intervals, e.g., during a single day.

4.7.1 Market participants in the model

Fundamentalists

Similarly as in the Kim–Markowitz (Sec. 4.2) and Lux–Marchesi (Sec. 4.4) models, in the Bornholdt model there are two types of market participant (or agent), namely, we consider m *fundamentalists* and n *interacting traders* (or *noise traders*). Fundamentalists only respond to price changes. They know the fundamental value of stock, $p^*(t)$, and if the actual price, $p(t)$, is below that value, they buy stocks, regarding them as undervalued. In the opposite case, i.e., if $p(t) > p^*(t)$, the

[17] Based on Kaizoji et al. (2002); Denys (2011).

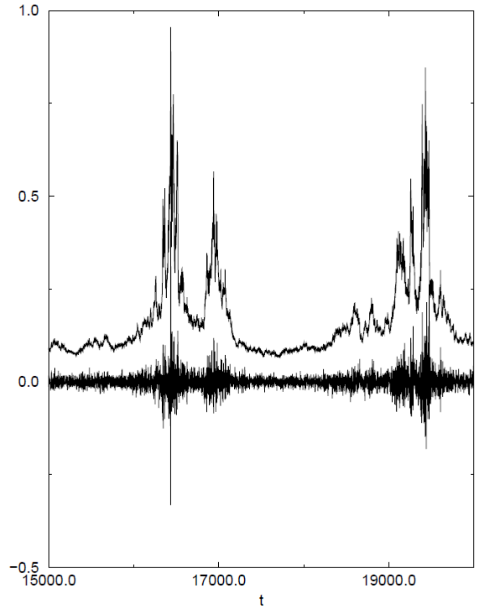

Figure 4.6: Returns and trading volume from the Iori model simulation. A correlation between both can be clearly seen. Taken from Iori (1999).

fundamentalists sell stocks as overvalued and therefore risky assets. Thus, their collective buy/sell order is given by

$$x^F(t) = am(\ln p^*(t) - \ln p(t)), \tag{4.23}$$

where factor a determines the strength of the fundamentalists' reaction to a given interval between the fundamental and the market price of an asset.

Interacting traders

With regard to the interacting traders, each of them, i, is assigned to the spin variable $s_i, i = 1, \dots, n$. The spin variable can take one of two values, depending on whether the trader buys ($s_i = +1$) or sells (-1) the stock at the moment.

We consider the following strategy of the ith agent:

$$\begin{aligned} s_i(t+1) &= +1 \quad \text{with probability} \quad p = \tfrac{1}{1+\exp(-2\beta h_i(t))} \quad \text{or} \\ s_i(t+1) &= -1 \quad \text{with probability} \quad 1-p, \end{aligned} \tag{4.24}$$

where $h_i(t)$ is the local field of the spin model, specifying the strategy of ith trader, while parameter β is an analog of an inverse temperature.

The decision that an interacting trader makes depends on two crucial factors: (i) the *local* and (ii) the *global information*. The local information is constituted by the behavior of the neighboring (i.e., acquaintance) agents. We assume that each trader is influenced only by its nearest neighbors in an accordingly defined neighborhood. On the other hand, the global information depends on whether the trader belongs to the *majority*, or the *minority* group of buyers/sellers at a given time step, and on the size of those groups. The asymmetry in the size of the majority and minority groups is defined by an absolute value of the *magnetization*, $|M(t)|$, where

$$M(t) = \frac{1}{n} \sum_{i=1}^{n} s_i(t). \tag{4.25}$$

Obviously, the purpose of interacting traders is to increase their capital by trading. They know that if they want to do this, they have to be in the majority group, though this is not sufficient for an enlargement of the majority group in the next period. On the other hand, traders in the majority group expect that the larger the value $|M(t)|$ is, the harder the further enlargement of this group will be. Because of this, interacting traders from the majority group move to the minority group from time to time, to avoid capital loss after a market crash, which is more likely when the majority group is large. In other words, the larger the majority group is, the more traders from that group tend to avoid risk. On top of that, the traders from the minority group can also move to the majority group in order to increase their capital, and the larger the majority group is, the more likely they are to take this risk.

To sum up, the larger the value $|M(t)|$ is, the larger the probability for interacting traders to change the group is, regardless of whether it is the majority or the minority group. In accordance with Bornholdt (2001), the local field $h_i(t)$, taking into account the interactions depicted above, can be given by

$$h_i(t) = \sum_{j=1}^{m} J_{ij} S_j(t) - \alpha S_j(t) |M(t)|, \tag{4.26}$$

where $\alpha > 0$. The first term corresponds to local information, with the nearest neighbors' influence $J_{ij} = J$, and $J_{ij} = 0$ for other couples.

In the model we assume that excessive demand for stocks for interacting traders can be approximated according to the formula

$$x^I(t) = bnM(t), \tag{4.27}$$

where b is the number of stocks the agent buys or sells at each time step.

4.7.2 Market price and volume. Results.

The authors of the model assumed the existence of a clearing system on the market, i.e., there are market makers participating in the market game and adjusting the market price to their clearing value. The transaction is made when a certain buy order and a certain sell order equalize. Therefore, the equality of supply and demand can be written as

$$x^F(t) + x^I(t) = am[\ln p^*(t) - \ln p(t)] + bnM(t) = 0. \tag{4.28}$$

Hence, we can calculate the stock price:

$$\ln p(t) = \ln p^*(t) + \lambda M(t), \quad \lambda = \frac{bn}{am}, \tag{4.29}$$

and the volume:

$$V(t) = bn\frac{1 + |M(t)|}{2}. \tag{4.30}$$

According to Eq. (4.29) the situation of the market may be one of the following:

- If $M(t) = 0$, the market price $p(t)$ is equal to the fundamental value $p^*(t)$.
- If $M(t) > 0$, the market price $p(t)$ exceeds the fundamental value $p^*(t)$ (a bull market case).
- If $M(t) < 0$, the market price $p(t)$ is below the fundamental value $p^*(t)$ (a bear market case).

Using Eq. (4.29) we can also calculate the logarithmic rate of return, $r(t)$, in the Bornholdt model:

$$r(t) = \ln p(t) - \ln p(t-1) = (\ln p^*(t) - \ln p^*(t-1)) + \lambda(M(t) - M(t-1)). \tag{4.31}$$

Let us assume that only fundamentalists participate in the market game. Obviously, the market price $p(t)$ then is always equal to the fundamental price $p^*(t)$, and therefore the *efficient-market hypothesis* works[18]. The efficient market model by Malkiel and Fama (1970) predicts then that the fundamental value $p^*(t)$ follows a random walk. Since the continuous limit of a random walk is the Wiener process, the probability density of the logarithmic return $r(t) = \ln p(t) - \ln p(t-1)$ is represented by a Gaussian distribution. Unfortunately, in real markets we

[18]According to the efficient market hypothesis, the value of a given asset in each moment fully reflects all available associated information, cf. Fama (1970).

observe significant deviations from this distribution. The Bornholdt model, however, anticipates these deviations, due to the existence of both fundamentalists and interacting traders on the market. Thus in this model, interacting traders are responsible for the non-Gaussian behavior of the market, including the possibility of bull or bear markets occurring. Incidentally, the authors of the model assumed for simplicity that the fundamental price is constant in time.

The Bornholdt model reproduces many major real-market stylized facts, e.g., power-law return distributions, volatility clustering (cf. Fig. 4.7), a positive correlation between volatility and volume, and self-similarity between volatilities on various time scales. However, the shape of the absolute-return autocorrelaton function is not a desired power law.

4.8 The Gontis et al. model of interevent times[19]

Although the models circumscribed above are all interesting and valuable for investigating the properties of financial markets, none of them was used to model the interevent-time statistics, presented in Chapter 1 and then carefully examined in Part I of the present thesis. In 2014 for this purpose we used our agent-based model (Denys et al. (2014); see Sec. 5.3.3 in the next chapter for details) and it was, to the best of our knowledge, the first successful attempt in this regard. However, then also Gontis et al. (2016) presented that their model (Gontis and Kononovicius, 2014) reproduces the stylized fact of universal interevent-time distribution on the markets. This section presents their approach as another possible agent-based description of this universality.

4.8.1 The model description

The aim of Gontis and his collaborators was to create a model capable of reproducing financial-market stylized facts, in particular the characteristics of volatility. The authors defined volatility as an absolute-return fluctuation in a given time scale. The model is substantially a composition of two concepts: (i) the ARCH-like model approach, and (ii) the agent-based modeling approach.

A logarithmic return $r_\delta(t)$ in the model is given by

$$r_\delta(t) = \sigma(t)\omega(t), \tag{4.32}$$

[19]Based on Gontis et al. (2016).

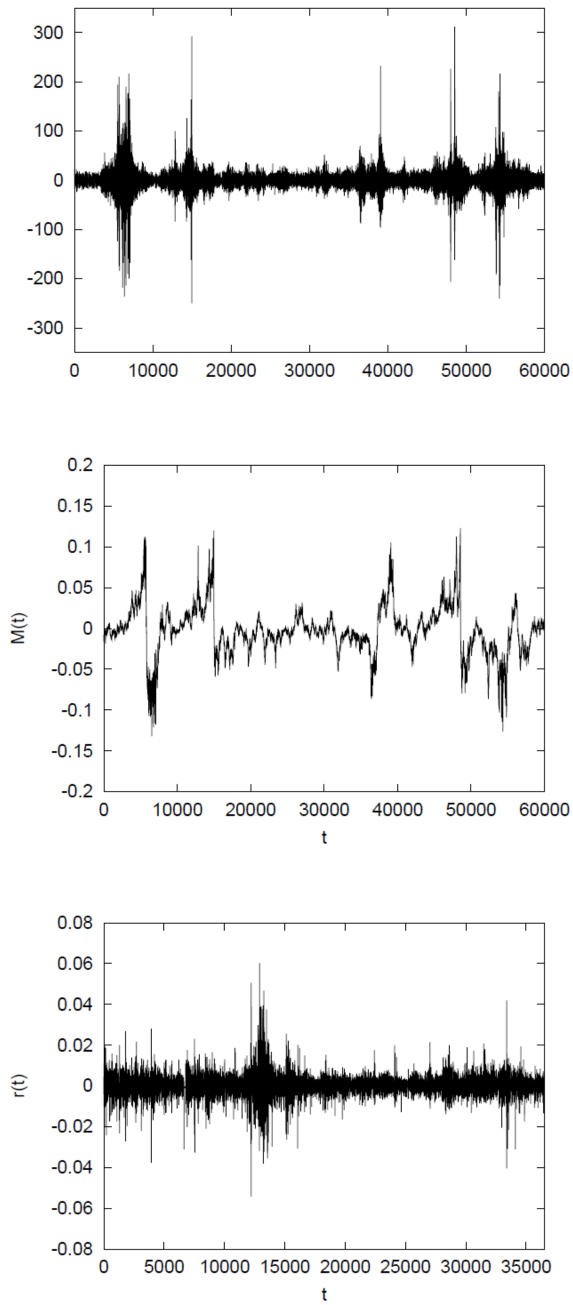

Figure 4.7: On the top: logarithmic returns in the Bornholdt model. In the middle: the corresponding magnetization $M(t)$. On the bottom: for the purposes of comparison, daily logreturns for DJI between 1896 and 1996. Taken from Kaizoji et al. (2002).

where $\omega(t)$ stands for an exogenous noise, assumed for simplicity as Gaussian, while $\sigma(t)$ is an (endogenous) volatility, derived from the underlying agent-based model, i.e.,

$$\sigma(t) = b_0(1 + a_0|p(t)|), \tag{4.33}$$

where $p(t)$ is a logarithmic ratio between the market price $P(t)$ and the fundamental value P_f from the ABM, a_0 is an endogenous dynamics impact factor, and b_0 stands for the normalization.

The agent-based model used by the authors is a three-state model, where the state value, namely f, o, or p, stands for the trading strategy of each agent, i.e., *fundamental*, *optimistic*, and *pessimistic*, respectively. Fundamental traders, as usual, look at the fundamental value, P_f, and assume that the actual price will be equal to it. Optimistic and pessimistic traders are those who always buy or sell stocks, respectively. Together, they form a group of chartists with the *chartist* strategy (denoted by c)[20]. In other words,

$$ED_f = n_f[\ln P_f - \ln P(t)], \;\; ED_c = r_0(n_o - n_p) = r_0 n_c \xi, \tag{4.34}$$

where ED_f and ED_c are excess demands for fundamentalists and chartists, respectively (cf. Sec. 4.4.1), n_i is the number of participants in market group i, r_0 is some impact coefficient for chartists, while $\xi = \frac{n_o - n_p}{n_c}$ is the average market mood. Hence, we obtain

$$p(t) = \ln \frac{P(t)}{P_f} = r_0 \frac{n_c}{n_f} \xi = r_0 \frac{1 - n_f}{n_f} \xi. \tag{4.35}$$

The transitions from the i to j trading group are made according to transition probabilities,

$$\mu_{ij} = \sigma_{ij} + h_{ij} n_j, \tag{4.36}$$

where h_{ij} measures the influence of peers, while σ_{ij} is an "individual" part. In the simplified version of the model there is no distinction between optimistic and pessimistic chartists, i.e., $\sigma_{op} = \sigma_{po} = \sigma_{cc}$ and $\sigma_{pf} = \sigma_{of} = \sigma_{cf}$. We assume that the chartists exchange with themselves with a frequency H times larger than the one for the fundamentalists, i.e., $h_{op} = Hh_{fc} = Hh$, and fundamentalists do not distinguish between optimistic and pessimistic chartists, i.e., $\sigma_{fp} = \sigma_{fo} = \sigma_{fc}/2$ and $h_{fp} = h_{fo} = h$. Additionally, the authors assumed that $H \gg 1$, $\sigma_{cc} \gg \sigma_{cf}$, and $\sigma_{cc} \gg \sigma_{fc}$ to incorporate the long-term vs. short-term strategies of the fundamentalists vs. chartists, respectively.

[20] Compare with some similar investor groups from Secs. 4.2, 4.4, and 4.7.

Using the assumptions above, one can describe the behavior of the system with the following stochastic differential equations:

$$\mathrm{d}n_f = \frac{(1-n_f)\varepsilon_{cf} - n_f\varepsilon_{fc}}{\tau(n_f)}\,\mathrm{d}t_s + \sqrt{\frac{2n_f(1-n_f)}{\tau(n_f)}}\,\mathrm{d}W_f, \tag{4.37a}$$

$$\mathrm{d}\xi = -\frac{2H\varepsilon_{cc}\xi}{\tau(n_f)}\,\mathrm{d}t_s + \sqrt{\frac{2H(1-\xi^2)}{\tau(n_f)}}\,\mathrm{d}W_\xi, \tag{4.37b}$$

where we used the scaled parameters $\varepsilon_{cf} = \sigma_{cf}/h, \varepsilon_{fc} = \sigma_{fc}/h, \varepsilon_{cc} = \sigma_{cc}/(Hh)$, and $t_s = ht$, the interevent time $\tau(n_f)$ and the (independent) Wiener processes W_f, W_ξ. The interevent time is assumed as

$$\tau(n_f) = \left(1 + a_\tau \left|\frac{1-n_f}{n_f}\right|\right)^{-\alpha}, \tag{4.38}$$

where a_τ and α are certain parameters.

The whole model, defined by Eqs. (4.32), (4.33), (4.35), (4.37a-b) and (4.38), is similar to the nonlinear stochastic GARCH(1,1) model (Bollerslev, 1986); however, using ABM instead of a stochastic model provides a better connection between the model parameters and the real-world market parameters. Additionally, the authors introduced some extensions to the above picture, linking endogenous and exogenous changes (e.g., the signs of chartist-mood change and exogenous noise ω) to better reproduce some stylized facts. Moreover, parameter b_0 was changing day by day, according to the empirical daily pattern, i.e., $b_0(t) = b_0 \exp\left[-(t \mod 1 - 0.5)^2/w^2\right] + 0.5$, where w specifies the amplitude of b_0 changes.

4.8.2 Results

Aside from reconstructing the main stylized facts of power-law market-return statistics and spectral density on varied time scales from minutes to days (cf. Gontis and Kononovicius (2014); Gontis et al. (2016)), the authors also claim to reproduce the shape of the interevent-time statistics on NYSE and Forex for various values of the threshold parameter q and various time scales (cf. Fig. 4.8). Apart from the unconditional distributions, the conditional ones are also covered by the model (cf. Gontis et al. (2016)).

The work by Gontis and his collaborators is an original attempt to reconstruct interevent-time market statistics. Nonetheless, in their study the authors used a

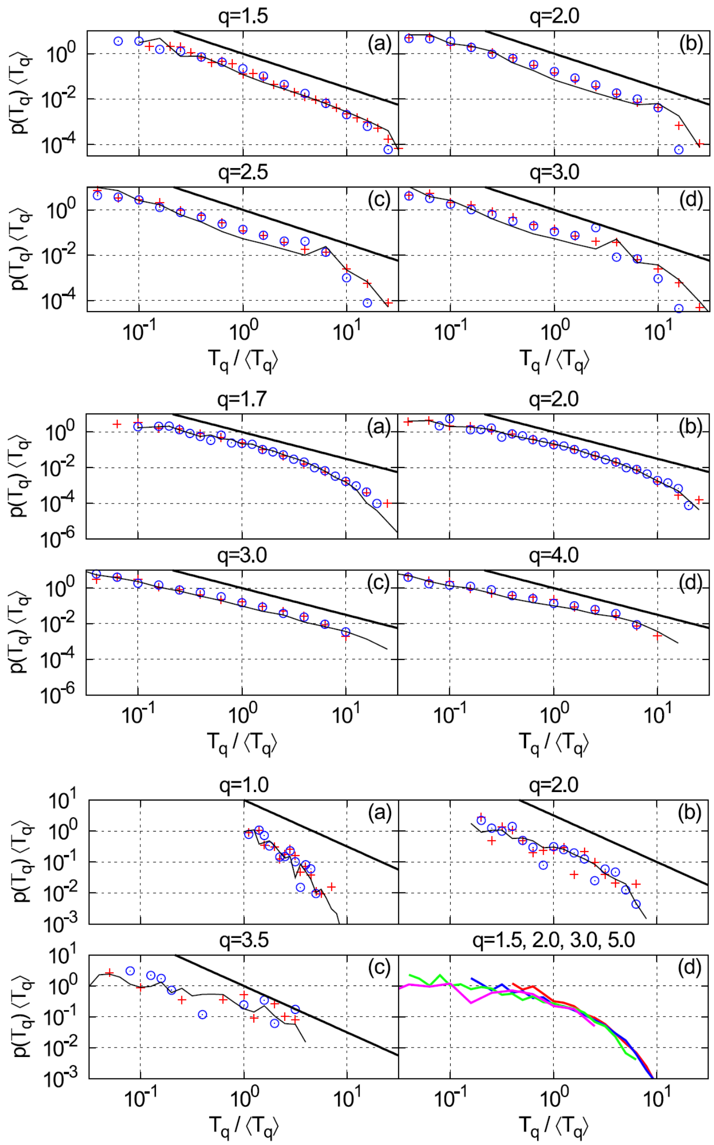

Figure 4.8: Comparison of the theoretical (derived from the Gontis model, black solid lines) and empirical interevent-time distributions for intra-day (upper four plots), daily (the middle ones) and monthly (the bottom ones) normalized returns for NYSE stocks (blue circles) and Forex USD/EUR quotes (red pluses) and for different values of the threshold parameter q written in the heading of each plot. The returns are measured in units of their standard deviation (specific for a given asset). The last plot, (d) in the bottom set for the monthly returns, presents the comparison of four theoretical (model) statistics for $q = 1.5, 2.0, 3.0, 5.0$. For such a large a time scale, the distribution of returns approaches to an exponential function (cf. Fig. 3.4(a) in Chapter 3). Taken from Gontis et al. (2016).

scaled statistics that is supposed to be universal, independent of the threshold value, q (Yamasaki et al., 2005). Such an approach seems to be controversial, since the mean value of the interevent time, used therein as a scaling factor, may not exist at all (cf. Eqs. (3.23) and (3.24) in Chapter 3 together with the discussion therein). Actually, it can be understood directly from the paper, as the obtained statistics of interevent times has a power-law tail with slope 3/2. For such an exponent, neither the mean nor the higher moments exist (cf. Eqs. (3.16), (3.23), and (3.24) in the previous chapter), which calls into question the appropriateness of the method used.

Therefore the approach that we used in this matter in Denys et al. (2016a,b) (see Chapter 3 of the present thesis), namely analysing the mean *quantile* interevent time R_Q instead of the ordinary momentum mean and also the distinction between the interevent-time statistics with disparate values of R_Q, seems to be appropriate and more coherent than the approach outlined in this section. When examining the usefulness of our agent-based model for reproducing interevent-time statistics (see Denys et al. (2014) and Sec. 5.3.3 with Fig. 5.12 in the next chapter) it is crucial that we do not rescale it, and thus allow the possibility of an infinite mean or higher moments of the underlying return statistics.

4.9 Discussion

In the present chapter some selected agent-based models of financial markets were discussed. Starting from the pioneering models of Kim and Markowitz (1989); Levy et al. (1994) and the even more well-known works of Lux and Marchesi (1999); Cont and Bouchaud (2000a), we went on to talk about some Ising-like econophysical models by Iori (1999); Bornholdt (2001); Kaizoji et al. (2002), to finish on Gontis et al. (2016) work that was another attempt, after Denys et al. (2014), to reproduce interevent-time distribution by ABM. The purpose of the review made in this chapter was to familiarize the readers with the idea of agent-based modeling in finance and to prepare them for reading the next chapter (5), dedicated to our agent-based model proposed in Denys et al. (2013).

All the authors of the presented models aimed to reproduce some characteristics of financial markets. Frequently they divided the model investors into two basic groups, the first which stabilizes the market (*rebalancers* or *fundamentalists*) and the second which destabilizes it (*portfolio insurers*, *noise traders*, *interacting traders*, or *chartists*; cf. Secs. 4.2, 4.4, 4.7, and 4.8). The latter group is usually indicated as the cause of fat-tailed distributions of returns on the markets.

It is evident that the vast majority of main financial-market stylized facts have already been covered by the presented models, e.g., the time series of returns with volatility clustering and the aforementioned fat-tailed statistics, the short-range correlation of the normal (raw) values of returns together with the long-range correlation of their absolute values, the positive correlation between returns and trading volume, and more.

These results are quite promising, however, a model capable of describing, at least superficially, the whole market and enabling some straightforward predictions, in the way that Einstein's relativity theory predicted the motion of Mercury, still does not exist. What is worse, it might never be created, when we consider the nature of the market, in particular its derivation from human beings, with all their complex mental structure, the cultural and social interactions between them and, moreover, the variety of possible forms of the market as such, which is only limited by the limits of human creativity. In this regard, the market is not a single concept, but rather a collective term, with only few permanent features, and a multiplicity of possible alterations. However, one should always bear in mind how many "impossibilities" from the past are possible nowadays, just to mention airplanes and spaceflights. As someone once said, only those who attempt the absurd will achieve the impossible.

Furthermore, in this chapter I mentioned and have shown, at least roughly, the difference between economic and econophysical ABMs. Generally speaking, economists start from more actual, economic foundations to finish with economic results and conclusions, while econophysicists, in contrast, rather take a *physical* model (as the percolation model in the case of Cont–Bouchaud work from Sec. 4.5) and, by drawing certain analogies and reinterpreting some quantities, convert it to a model of economic reality. Furthermore, a general tendency among physicists, as opposed to economists, is that they create models to be as simple as possible, usually using fewer variables, but also neglecting some factors they consider less important. In other words, economic models are often closer to reality but farther from understanding, while the econophysical ones – inversely.

Both philosophies seem to have their advantages and disadvantages. Economic concreteness and accuracy may be useful, but may also lead to the situation where the model becomes too complicated, due to the large number of parameters it uses; there is also a well-known fact that the more parameters we have, the easier the fit to the empirical data is, even if the model itself does not correspond to reality. On the other hand, econophysical models, although usually simpler and more transparent, may not correspond to reality either, as they are built on analogies with the physical world that, actually, might be incomplete and sometimes simply naive. Therefore both economists and physicists have to be very careful developing their models, because it is easy to create something that, although it reproduces the empirical findings, in fact misses the reality.

Despite all the critical comments above, as was shown with the example of the works of Kim–Markowitz, Cont–Bouchaud, or Lux–Marchesi, agent-based financial-market modeling can tell us a lot about the real causes of some observed market data properties. The main reasons for this applicability are (i) their nature, i.e., being a complex system *in silico*, and (ii) a closer relation of their parameters to real-world scenarios and real human behavior, as compared to pure stochastic models (Gontis et al., 2016). The former property is all the more important since many properties of complex systems, not excluding financial markets, are considered to come from so-called emergent phenomena, such as when a system of individuals behaves differently than it would if the behavior resulted from the simple addition of the behavior of each individual[21].

* * *

After the summary of agent-based models of financial markets made in this chapter, in the next chapter (5), the model that we proposed in 2013, and successfully examined for reproducing interevent-time statistics a year later, is presented.

[21]For more information see, e.g., Kwapień and Drożdż (2012) (a holistic view of emergent phenomena and physics) and Axelrod (1997a) or *PNAS* **99** suppl 3 (2002) (ABM considerations).

5

The financial-market model of cunning agents[1]

5.1 Introduction

As was shown in Chapter 4, agent-based modeling inspired by the Ising and Potts models play a key role in studies on financial markets. Works based on the Potts model and some of its modifications are significantly fruitful (Lux and Marchesi, 1999; Iori, 1999; Kaizoji et al., 2002; Sornette, 2004; Zhou and Sornette, 2007; Denys, 2011). Although we still do not possess a convincing overall model of financial markets and, crucially, even the causes of some market properties are not entirely understood yet, the promising results so far have encouraged us to make further studies.

The model presented in this chapter is based on the threshold model of financial markets by Sieczka and Hołyst (2008). Our model is in fact essentially a reinterpretation that aims to make it more suited for the description of real markets. The reasons why we chose the Sieczka–Hołyst (SH) model are as follows (cf. Denys et al. (2013)):

(i) It contains a threshold mechanism, which is well-established in psychology

(ii) It considers the interaction between agents as well as the uncertainty in their activity caused by informational noise.

(iii) It assumes the strength of the interaction, which strongly depends on the macroscopic state of the system.

[1] Based on Denys et al. (2013, 2014).

(iv) The SH model is sensitive to concrete types of emotion which each agent subordinates during the stock market evolution (Czaplicka et al., 2010; Kwapień and Drożdż, 2012).

Nevertheless, the SH model contains a fundamental inconsistency, namely buying (selling) the stocks by the agents does not lead to a price increase (decrease). There is even the extreme possibility that all the agents can buy (sell) the stocks at a given time step, but the corresponding price does not change at all. This forced us to make our reinterpretation of the SH model.

In such an approach, the price can change if and only if the spin values are changed. This protects the model from the paradox mentioned above. Specifically, the investment decision in our model is defined as the *change* of the spin variable of a given agent. In previous works, the spin variable was interpreted directly as an investment decision (e.g., the purchase of a certain number of stocks) that repeats at the successive time steps when this variable remains unchanged. The changes of this variable, in turn, could be interpreted as the beginning another decision being executed at the successive time steps (e.g., selling stocks). That is, previously a communicated opinion was identified solely with the (temporary) investment decision, not the (durable) market state of an investor (cf. Zhou and Sornette (2007)).

The present approach (which cannot be simply reduced to the previous one) appears to be more intuitive and reasonable, as the actual state of a spin is here identified with the actual market state of the corresponding agent, while the action (investment decision) is connected with the change of the state, as expected. Actually, such a property characterizes some epidemic models where the contagion is a change in the state of an agent, from susceptible to infected (Pastor-Satorras and Vespignani, 2001; Moreno et al., 2002), or the models of cultural patterns where the pattern adoption also requires a change of state (Axelrod, 1997b; Raducha and Gubiec, 2016). Nevertheless, to the best of our knowledge, this approach is novel in the frame of Ising-based econophysical modeling (cf. Sec. 5 in Sornette (2014) and refs. therein) and therefore **the investment decision defined as the change of a spin variable, not the plain value of this variable, constitutes the next significant contribution to the field of knowledge presented in this thesis.**

What is more, the irrational component in our model is provided by the (discrete) Weierstrass–Mandelbrot noise (Kutner, 1999a) as opposed to the (continuous) Gaussian one that is frequently used for this purpose (Zhou and Sornette, 2007; Sieczka and Hołyst, 2008). This allowed us to model some stronger emotions of agents occurring together with gentler ones. The details of our reinterpretation and the assumptions of the model are given in Sec. 5.2.

The question arises of whether the approach sketched above allows us to reproduce the main stylized facts from financial markets (see Chapter 1), in particular those as significant as the interevent-time statistics, carefully examined in Chapter 3. In Sec. 5.3 it is shown that the numerical simulations of our model give results consistent with a wide range of empirical financial-market data. Finally, in Sec. 5.4, a summary is given.

5.2 The model description

We consider N agents (or traders) on a square lattice $n \times n$, $N = n^2$, represented by a three-state spin variable $s_i = 0, \pm 1$, $i = 1, \ldots, N$. Most directly, s_i is interpreted as the current market state of the ith agent, $+1$ when it has bought some stocks (that is, it has the long position open), -1 when it has sold some stocks (that is, it has the short position open), and 0 if it is neutral. One can also consider this value of s_i as advice that the ith agent gives to its nearest neighbors at a given time step, when they ask it whether to buy stocks or sell them. Value $s_i = +1$ is the advice to buy, $s_i = -1$ is the advice to sell, while $s_i = 0$ is simply a lack of advice or the advice to wait or stay inactive. Taking into consideration the above comments, one may conclude that the agents are imitating themselves, in an affirmative sense, that is, by giving mutual examples to themselves.

On the other hand, when, for instance, ith agent advises the others to buy stocks ($s_i = +1$), it cannot buy them by itself, as the value of its spin variable cannot increase, but can only decrease or remain unchanged. And vice versa for $s_i = -1$, that is, the agent who advises others to sell is only able to buy them now or at least remain inactive. This effect ensures a negative coupling between subsequent values of a given spin variable. However, for $s_i = 0$ the agent can both buy or sell stocks, since it occupies a neutral position. For the reason given above, the model was called *the model of cunning agents* (Denys et al., 2014).

In each time period t, we draw a spin a predetermined number of times. The drawn spin, $s_i(t), i = 1, 2, \ldots, N$, is updated according to the social impact rule

$$s_i(t) = \mathrm{sgn}_{\lambda|M(t-1)|} \left[\sum_{j=1}^{N} J_{ij} s_j(t) + \epsilon_i(t) \right], \tag{5.1}$$

where

$$\mathrm{sgn}_q(x) = \begin{cases} +1 & \text{if} \quad x \geqslant q, \\ 0 & \text{if} \quad -q \leqslant x < q, \\ -1 & \text{if} \quad x < -q. \end{cases} \tag{5.2}$$

and the constant λ is positive. The exchange coefficient $J_{ij} = J > 0$ if j is one of the four nearest neighbors of i and $J_{ij} = 0$ otherwise, to ensure the (nonnegative) nearest-neighbor interaction. The noise term $\epsilon_i(t)$ is added to include some individual opinions or emotions of agents that accompany their decisions.

The magnetization, $M(t)$, is given conventionally by the following average,

$$M(t) = \frac{1}{N}\sum_{i=1}^{N} s_i(t), \tag{5.3}$$

and indicates the overall market state.

The threshold value $\lambda|M(t)|$ in Eq. (5.1) is a crucial one with regard to the model mechanics. If the impact of neighbors plus the opinion/emotion of the ith agent is lower than this value, the agent switches to neutral $s_i = 0$. A large value of magnetization, corresponding to the large dominance of one of the two – short or long – positions occupied by the agents, at fixed λ yields a large threshold value and a high probability of switching to neutral position. In other words, the suspicion of the agents is high in this case. This is understandable, since such a strong imbalance on the market is rather considered as dubious and unsafe.

The spin value s_i is updating according to Eq. (5.1) immediately, i.e., its neighbors "see" its new value just after it was drawn. We introduce the notion of a *round* as N spin drawings, enabling each trader on average one chance to change its state and forming a single time step. The round is an equivalent of a 1MCS/spin from dynamical Monte Carlo simulations[2].

Notably, Eq. (5.1) has a completely different interpretation from all the counterparts used earlier (cf., e.g., Eqs. (4.16)-(4.17), from the Iori model or (4.26) from the Bornholdt model in the previous chapter). Namely, this formula concerns the *state* of an agent, not its *activity*. Regarding the activity of an agent, it is defined as a *change* of its spin, that is, $d_i(t) = s_i(t) - s_i(t-1)$ is an agent demand for $d_i > 0$ or supply (a negative demand) for $d_i < 0$. Therefore the agent's activity consists of two subsequent different values of the spin variable s_i. The agent declares a demand when s_i increases during a given round, or alternatively offers a supply when s_i decreases. This change, d_i, can take magnitude 1 (e.g., for a jump from 0 to -1) or 2 (for a jump from -1 to $+1$ or from $+1$ to -1), which

[2] Consequently, a single drawing is an equivalent of 1MCS.

is its largest possible value and simultaneously the largest possible single-agent supply or demand for the stocks in a single round. We assume the agent's supply or demand is realized at once, in the same round, and the given agent buys or sells some stocks[3].

Evidently, after the purchase or the sale, the agent becomes an advocate of the decision it made and may encourage its neighbors to do the same. Actually, it is also in its best interest, as the one who purchased wants the price to increase, and this is realized when more agents buy, and vice versa in the case of a sale. In this way the mechanism of mutual imitation works. The cycle of the agents' activity and communication is repeated.

To sum up, the model corresponds to the real-world, where investors buy or sell stocks imitating the past behavior of their colleagues, and the more balanced the market situation is, the more willingly they do it.

This reinterpretation of the spin variable prevents the model market from the paradox mentioned in Sec. 5.1, where the price does not change, although all the agents buy (or sell) stocks. Additionally, this approach provides a desired correlation between the return volatility and the trading volume in the model. Another essential consequence of this approach is that the threshold slows down both the fast increase and decrease of the price, providing a specific price damping, which cannot itself induce a change in the trend, nor even any oscillations.

5.2.1 The noise in the model

In our simulations we predominantly used the noise from the Weierstrass–Mandelbrot (WM) probability distribution (cf. Kutner (1999a) and refs. therein):

$$p(x) = \left(1 - \frac{1}{K}\right) \sum_{j=0}^{\infty} \frac{1}{K^j} \cdot \frac{1}{2}\delta(|x| - b_0 b^j), \ \ K, b > 1, \ \ b_0 > 0. \tag{5.4}$$

The reasons for using this distribution are as follows:

(i) Emotions on the market are sometimes very strong; therefore the power-law tails of WM distribution may be more useful to describe these emotions than a simple Gaussian one.

[3]In a real-life situation, a purchase of a stock is always associated with the sale of another stock of the same asset by another investor. In our model, we simplify that view, neglecting this feature (i.e., we assume an unlimited market liquidity or consider only a part of the whole market), similarly as in many other econophysical financial-market models – cf. Chapter 4.

(ii) WM distribution enables the control of its variance and exponent. The variance may take varied finite values and also be infinite; the distribution may take both Lévy and non-Lévy forms asymptotically.

WM distribution is a discrete one, having spikes at $x = \pm b_0 b^j,\ j = 0, 1, 2, \ldots$ Its variance is given by

$$\sigma^2 = b_0^2 \frac{1 - 1/K}{1 - b^2/K} \text{ for } b^2/K < 1; \tag{5.5}$$

otherwise, it is infinite.

One can easily prove that for $|x| \gg 1/\ln b$ Eq. (5.4) takes the approximated form

$$p(x) = \frac{1 - 1/K}{\ln K} \frac{b_0^\beta}{|x|^{1+\beta}}, \tag{5.6}$$

where $\beta = \frac{\ln K}{\ln b}$, and the variance of the above given approximate PDF is finite only for $\beta > 2$; otherwise, it is infinite.

Although Eq. (5.6) represents a power-law distribution, only the part of it nonexceeding the maximal range $[-(4J+\lambda); 4J+\lambda]$ is essential for the dynamics of the system. Inside this range, spin value s_i can vary between $-1, 0$, and $+1$. Above its right border, s_i can only take value $+1$, while below its left border it can only take value -1. The reason for such behavior is the threshold character of the social impact function (5.1).

5.2.2 Some additional formulas and assumptions

After Barro (1972), in ABMs we usually consider the logarithmic return, r, proportional to the properly defined excess demand ED (cf. Eqs. (4.13), (4.14), and (4.34) from Chapter 4 together with the discussion therein). In our model this feature can be written as

$$r_\tau(t) = \ln P(t) - \ln P(t - \tau) = \frac{1}{\Lambda} ED_\tau(t), \tag{5.7}$$

where P is price,

$$ED_\tau(t) = \sum_{i=1}^{N} [s_i(t) - s_i(t - \tau)] = N[M(t) - M(t - \tau)], \tag{5.8}$$

and τ is delayed time in units of rounds (for $\tau = 1$ we have $r_1 \equiv r$), while the coefficient Λ can be interpreted as the market depth. It is worth noting that the excess demand $ED_\tau(t) \neq 0$ only if the state of at least one agent changes. Naturally, the changes of states may cancel each other, leaving the excess demand unchanged. Thus, the price P changes if and only if the magnetization M changes, i.e., when there is an imbalance between the overall demand and supply on the market, as in a real-life situation.

To cope with an occasional trapping of the system into the fully ferromagnetic state, when all agents have the same value of their spin variables, either $+1$ or -1 (this can be interpreted as a fully illiquid market; cf. similar problem for the Lux–Marchesi model, at the end of Sec. 4.4.1 from the previous chapter, and Fig. 5.1 below for some illustration of this behavior), we introduced an exogenous factor to act as a market maker. It abruptly restores the system to a paramagnetic state with randomly oriented spins. After that, we continue the computations until the next such reset[4].

5.3 The model results and comparing them with empirical data

5.3.1 Basic results

After Sieczka and Hołyst (2008) we set the lattice size $N = 32 \times 32 = 1024$ agents and at the beginning of the simulation we drew a random configuration of the spin values. During the simulation, the magnetization frequently tended to fluctuate around a fixed value (cf. inset plot in Fig. 5.2). We found this value can change during the simulation and the magnetization jumps from one equilibrium value to another (degeneration of the equilibrium state) with the set of possible values clearly specified for the simulation of a particular lattice size $n \times n$. Such behavior is a result of the tendency of the system to fall into steady states, defined by the symmetry of the lattice, in this case two-dimensional square symmetry. Such steady states are quite stable, so the system can persist in each of them for a long time (see Fig. 5.3).

For an increasing value of the threshold parameter λ, a phase transition between walk- and noise-like behavior of the magnetization evolution occurs (cf. Fig. 5.4).

[4]The convergence to a fully ordered state is not a question of preponderance of one of the fractions (-1, 0, or $+1$). When there is an even drawing for each of these values, there are still many restarts. See (Pinto et al., 2014) for some other attempt to avoid such ordering behavior in the Ising model.

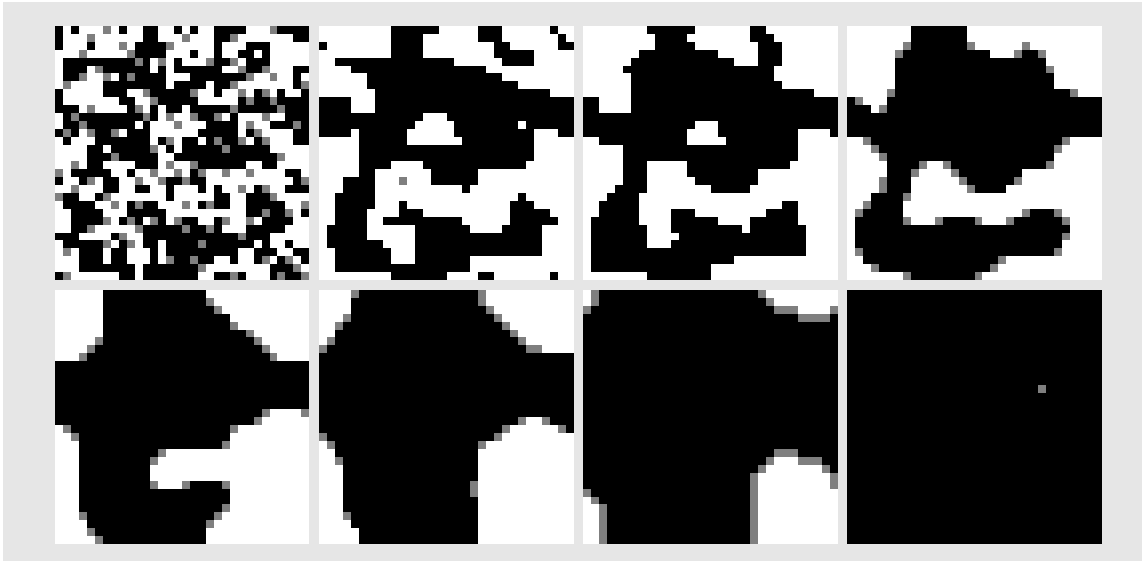

Figure 5.1: Illustration of the fast convergence of the system to a fully paramagnetic state. From left to right, up to down: frames 1, 6, 11, 21, 201, 351, 501, 784 of the example simulation (for parameters $J = 1, \lambda = 2, K = 5, b = 2, b_0 = 0.2$). The spin values are: $+1$ (white), 0 (gray), and -1 (black). It can be seen that the neutral position (spin value 0, gray) is taken by a relatively small number of traders.

Return statistics from the model simulation reveal power-law tails for varied values of time lag τ, as in Fig. 5.5. Because the noise, ϵ_i, is cut off by values $-(4J + \lambda)$ on the left and $4J + \lambda$ on the right (cf. Eqs. (5.1) and (5.2)), for parameters as in Fig. 5.5 only components with $j = 0, 1, 2, 3, 4$ are significant values in the sum in Eq. (5.4). For most values of τ, the distributions reveal explicit power-law tails with the exponent close to $1 + \beta$, where $\beta = \ln K/\ln b = \ln 5/\ln 2 = 2.322$. In other words, they reproduce the statistics from Eq. (5.6), even despite the aforementioned cut-off of the noise and the fact that the range $[-(4J+\lambda), 4J+\lambda]$ is narrower than its equivalent in the SH model. Only for large values of τ does the power-law character of the statistics break down, nevertheless, fat tails still occur, as opposed to the SH model, where the authors obtained convergence to Gaussian tails for the largest values of τ.

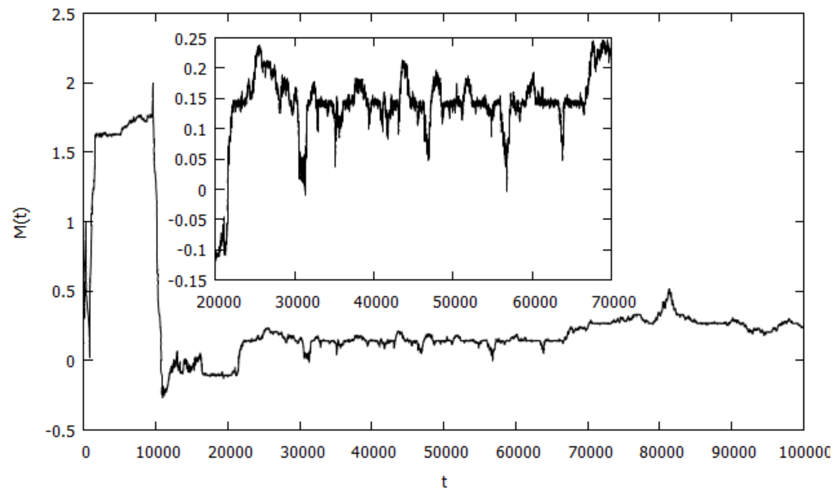

Figure 5.2: A sample magnetization run for the model simulation for parameters $J = 1, \lambda = 1, K = 5, b = 2, b_0 = 0.2$ with the abrupt transitions from the ferromagnetic to paramagnetic state visible at the beginning (approximately the first 10 000 time steps). Inset: the zoom for t between 20 000 and 70 000.

The autocorrelation function[5] of absolute returns in the model are slowly decaying, and they frequently resemble an exponentially truncated power-law function (cf. Fig. 5.6), which is consistent with empirical findings (Arneodo et al., 1998; Raberto et al., 2002; Lillo et al., 2004; Cont, 2007).

5.3.2 Empirical-data comparison

As was indicated in Sec. 5.1, the model is capable of reproducing a wide range of empirical results. Firstly, return variograms with volatility clustering resemble the real-market ones (Fig. 5.7). Secondly, cumulative absolute-return distributions for stock markets of essentially different capitalizations, with varying slope of tails are reproduced quite well by our model (by varying few driven parameters; Fig. 5.8). As presented in Fig. 5.8d, the model distributions show some nontrivial structures for larger values of time lag τ. Fortunately, they agree quite well with the empirical data within their attainable range. A distribution comparison for a

[5] Due to the fact that time in our model is discrete (measured in units of rounds), the autocorrelation function is defined as $C(x(t), x(t+\Delta t)) = \frac{\langle (x(t)-\mu)(x(t+\Delta t)-\mu)\rangle}{\langle (x(t)-\mu)^2\rangle}$; $\mu = \langle x(t)\rangle$. Thus, the length of time intervals between subsequent returns is not taken into consideration, herein.

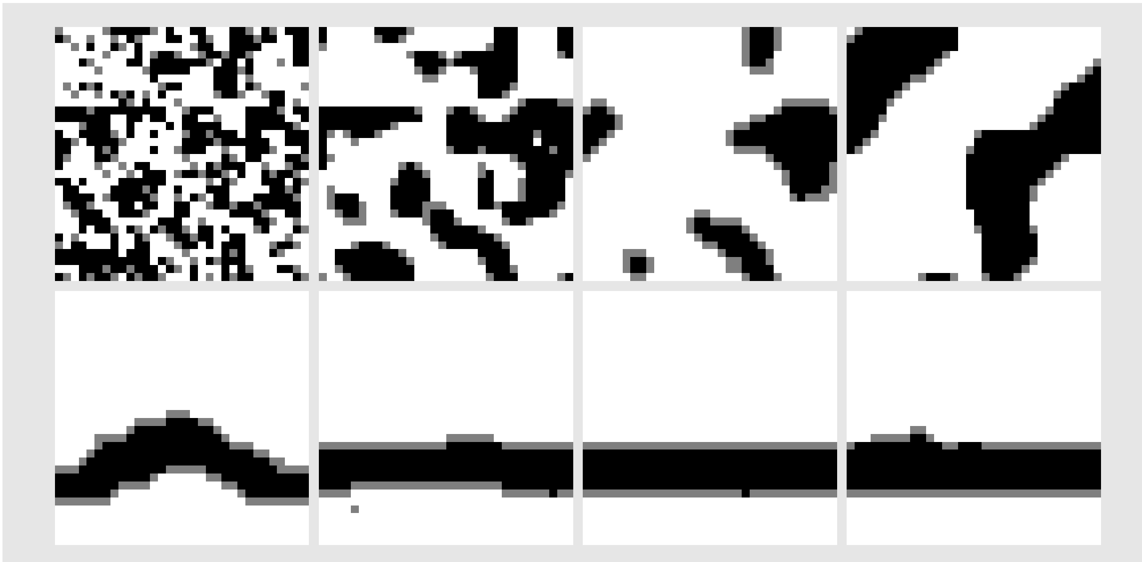

Figure 5.3: Illustration of the system falling to a steady state. From left to right, up to down: frames 2954, 2959, 2989, 3259, 3849, 4549, 6049, and 9999 of the example simulation (for the parameters $J = 1, \lambda = 2, K = 5, b = 2, b_0 = 0.2$). The white color indicates spin value $+1$, gray for 0, and black for -1.

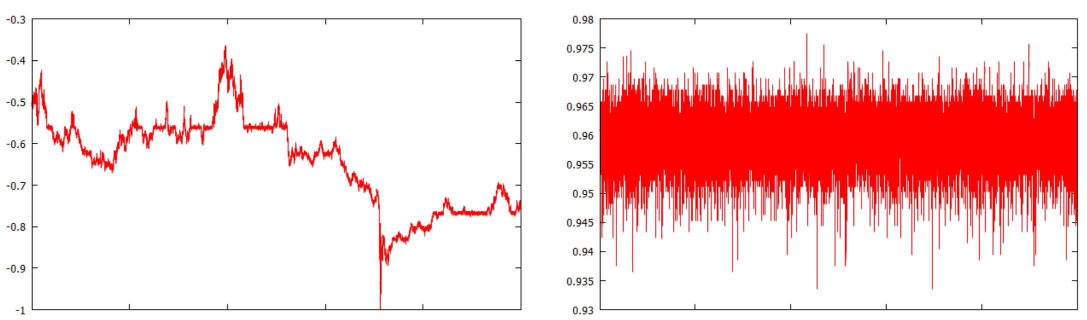

Figure 5.4: Phase transition of the magnetization evolution characteristics for parameters $J = 1, K = 5, b = 2, b_0 = 0.2$, and $\lambda = 2.2$ (on the left) and 2.3 (on the right).

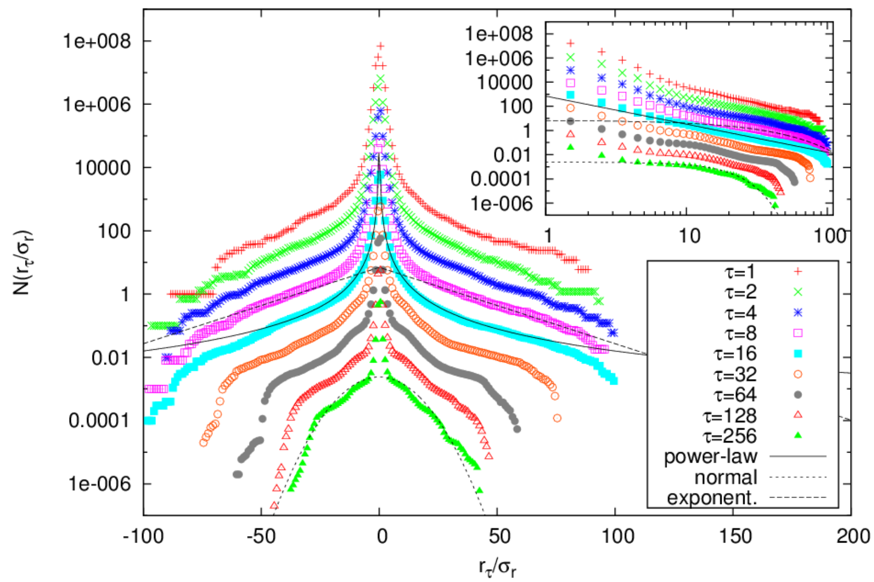

Figure 5.5: Numerical statistics of returns $r_\tau(t)$ for $J = 1, \lambda = 1, K = 5, b = 2, b_0 = 0.2$, and different τ (see the legend). For viewing purposes, the returns are rescaled by their standard deviations and the histograms are shifted upward with successive powers of 10. Power-law (solid line), Gaussian (denoted as normal; dotted line) and exponential (dashed line) functions fitted to selected histograms are shown. An exponent for the power law is $3.322 = 1 + \beta$ (see the text for more details).

full range of the return values (i.e., positive and negative) is shown in Fig. 5.9. It is evident that our model reproduces both the absolute and usual market-return statistics.

Also, the behavior of the returns' autocorrelation function is reproduced well within the frame of our approach. In case of raw (not absolute) returns, the empirical autocorrelation function is usually fast decaying with only initial values significant. For short time resolutions (i.e., for so-called tick data) for time lag $\tau = 1$ it is expected to be negative due to the antipersistent character of the financial time series on short time scales, related to the bid-ask bounce effect (Arneodo et al., 1998; Cont, 2001; Gubiec and Kutner, 2010; Preis, 2011). Substantially, this behavior characterizes the autocorrelation function from our model shown in Fig. 5.10.

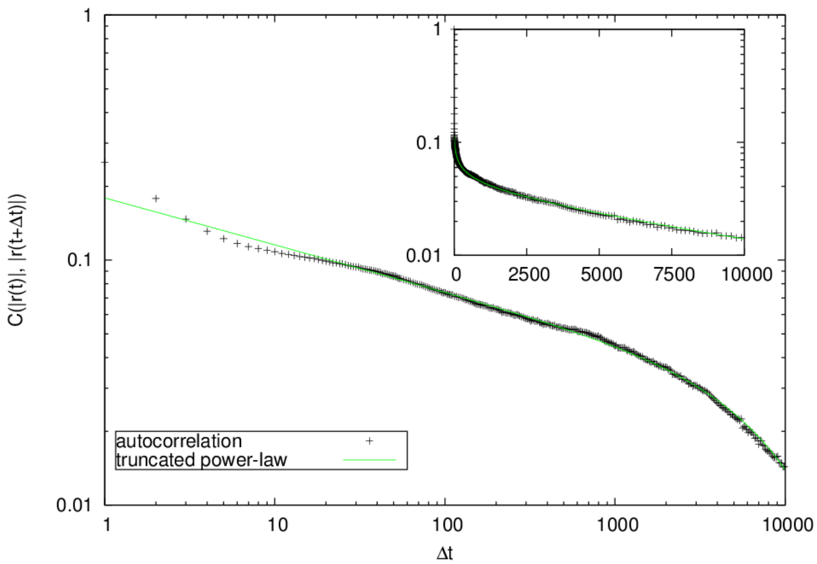

Figure 5.6: The autocorrelation function of absolute returns for $J = 1, \lambda = 1, K = 5, b = 2, b_0 = 0.21$ on log-log and (inset) lin-log scale (black pluses). The green solid line is the fit of an exponentially truncated power-law function.

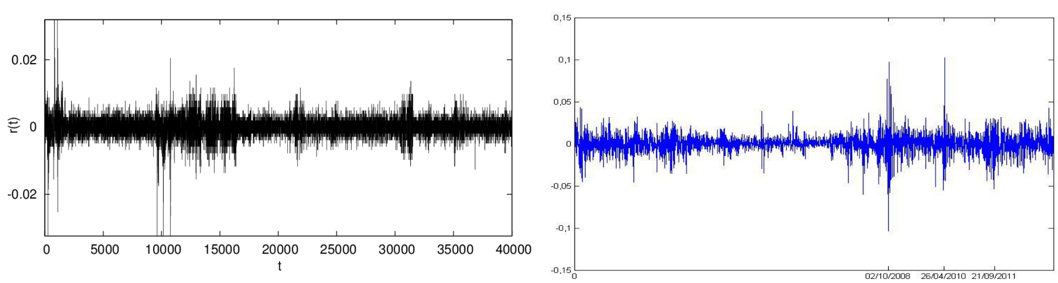

Figure 5.7: Returns in time for (on the left) an example run of the cunning-agents model simulation for the parameters $J = 1, \lambda = 1, K = 5, b = 2, b_0 = 0.2$, and (on the right) PSI20 settlement prices from 24 January 2000 to 24 May 2013 (taken from Pascoal and Monteiro (2014)).

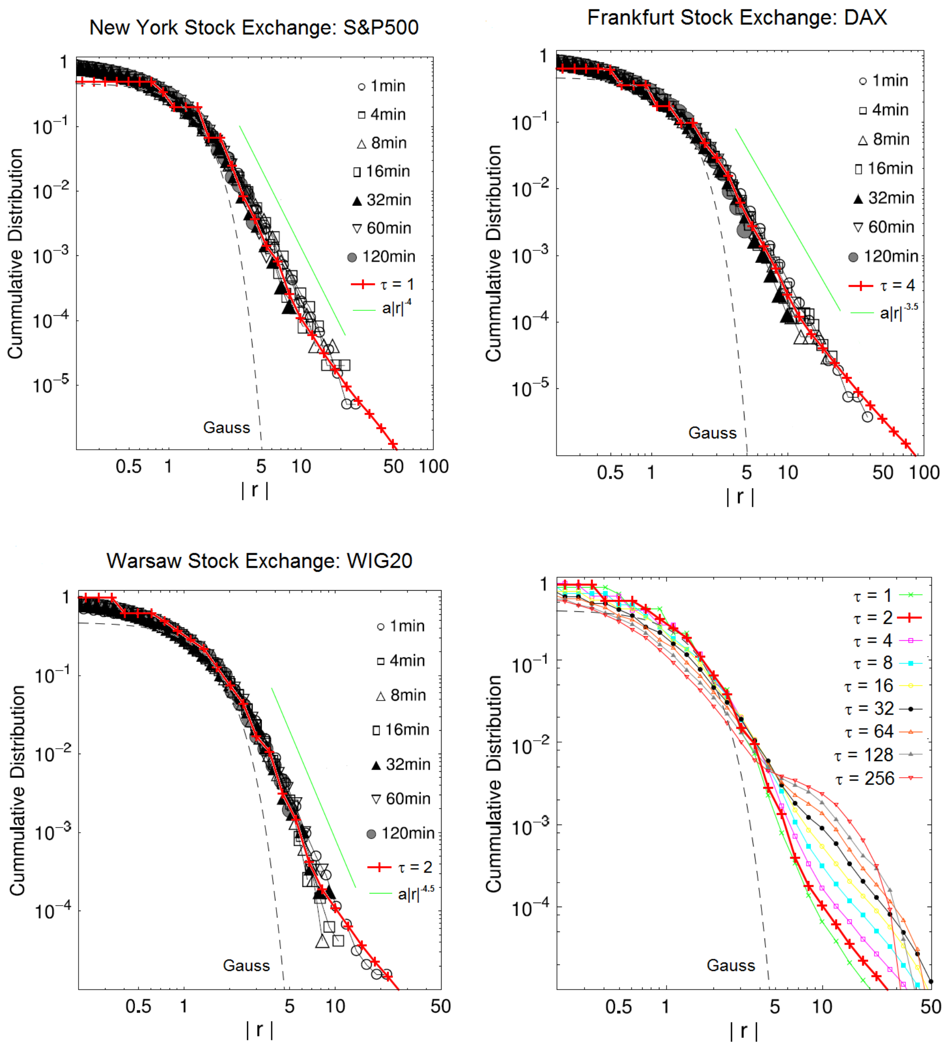

Figure 5.8: (a)-(c) A comparison between cumulative distributions of absolute returns for the model and real-market data (matching by the trial and error method with a fit accuracy of about 10%). The theoretical results are shown by markers connected by segments (in red), while the empirical results are denoted by markers connected by dashed curves (in black). (a) For the American S&P500 index, $J = 1$, $\lambda = 2.2$, $K = 6.5$, $b = 2$, $b_0 = 0.2$, $\tau = 1$. (b) For the German DAX, $J = 1$, $\lambda = 2.1$, $K = 6.5$, $b = 2$, $b_0 = 0.2$, $\tau = 4$. (c) For the Polish WIG20, $J = 1$, $\lambda = 2.2$, $K = 5$, $b = 2$, $b_0 = 0.25$, $\tau = 2$. The green straight lines in plots (a)-(c) denote a power law driven by exponents equal to 4.0, 3.5, and 4.5, respectively. (d) For instance, the model statistics for different values of τ (see the legend for details) but for other parameters the same as in (c). Dashed single curves show Gaussian predictions. Empirical data reprinted from Drożdż et al. (2007).

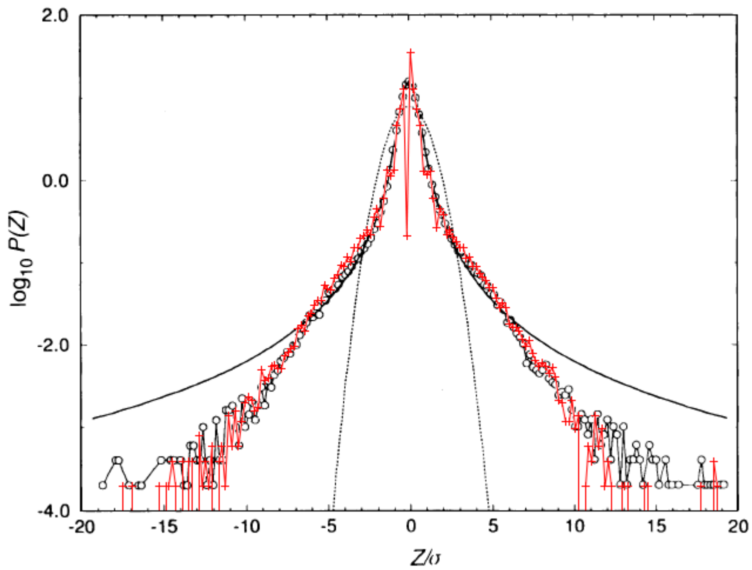

Figure 5.9: The distribution of one-minute returns for the S&P500 index from January 1984 to December 1989 (black empty circles and solid segments) and the model returns r_1 for 500 000 time steps, $J = 1, \lambda = 1, K = 5, b = 2, b_0 = 0.2$ (red pluses and solid segments). Additionally, the solid line shows symmetrical Lévy stable distribution, while the dotted line shows Gaussian distribution. It can be seen that the empirical distribution is reproduced by our model in the entire range of returns (matching was done by the trial and error method, with the fit accuracy of about 10%). Empirical data reprinted from Mantegna et al. (1995).

Comparison of the case of the absolute-return autocorrelation function is shown in Fig. 5.11. Evidently, the shapes of the numerical and the empirical curve are very similar; also the slopes of the tails of both autocorrelation functions agree quite well with each other. Although similar results regarding the usual as well as the absolute-return autocorrelation function were obtained previously by other authors, I have not found the actual reproduction of either of them by an agent-based model, which would comprise the initial significant negative value of the first one as well as the power law until the 10^3 time step for the second one, as in our model (cf. Arthur et al. (1996); Farmer (1999); Iori (1999); Lux and Marchesi (1999); Bouchaud (2000); Giardina et al. (2001); Izumi and Ueda (2001); Jefferies et al. (2001); LeBaron (2001); Raberto et al. (2001); Tay and Linn (2001); Farmer and Joshi (2002); Hommes (2002); Tesfatsion (2002); Giardina and Bouchaud (2003); Alfarano et al. (2005); Kozłowska et al. (2006);

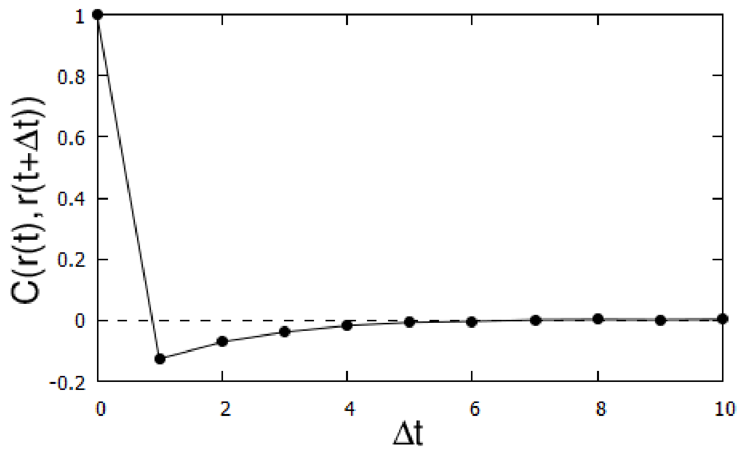

Figure 5.10: The autocorrelation function of returns in the cunning-agents model for the parameters $J = 1, \lambda = 1, K = 5, b = 2, b_0 = 0.2$. The dashed horizontal line stands for ordinate zero.

LeBaron (2006); Mizuno et al. (2006); Preis et al. (2006); Cont (2007); Zhou and Sornette (2007); Chiarella et al. (2009); Tedeschi et al. (2009); Feng et al. (2012); Thurner et al. (2012)).

5.3.3 Interevent-times description[6]

As shown previously in this section, the main stylized facts from financial markets are already reproduced well by the cunning-agents model. Additionally, Fig. 5.12 shows a comparison of our model predictions with the formula (3.12) for a distribution of the interevent times from Chapter 3. It is clear that the numerical predictions of the cunning-agents model agree with the superstatistics model as well as with the empirical data.

To the best of our knowledge, this comparison, presented initially in Denys et al. (2014), was the first successful attempt at reproducing the universality discovered in Ludescher and Bunde (2014) by a microscopic, agent-based model and, together with the successful reproduction of the empirical return autocorrelation (cf. Sec. 5.3.2), it constitutes the next significant contribution to the field of knowledge presented in this thesis. This description was made a year before the description by Gontis et al. (2016) (cf. Sec. 4.8 in the previous chapter) and it appears to

[6]Based on (Denys et al., 2014).

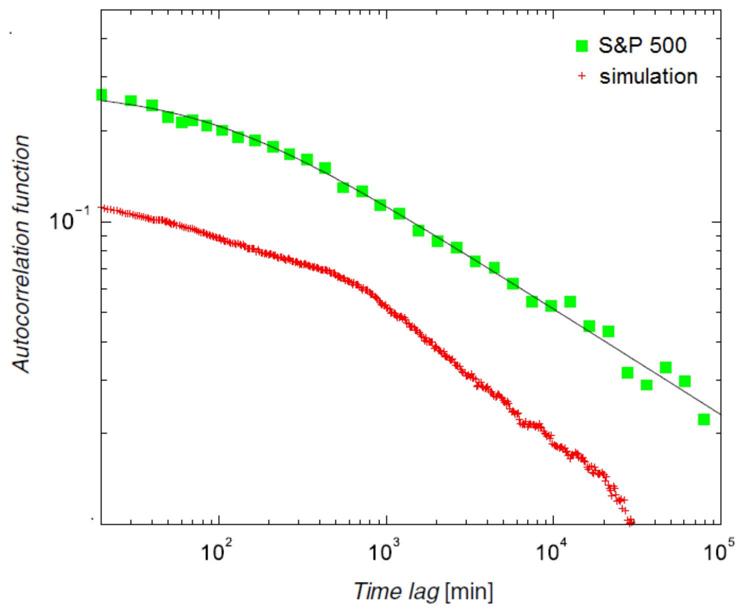

Figure 5.11: The autocorrelation function of absolute returns for S&P500 (green points) and of $|r_1(t)|$ from our model (red points) for $J = 1$, $\lambda = 2.1$, $K = 6.5$, $b = 2.1$, and $b_0 = 0.2$. Given that the trading day on the NYSE lasts 6.5 h, a single time step, herein (i.e., $\tau = 1$) calibrates as approximately 1 min. Empirical data, including the dashed curve drawn with a rough estimation, reprinted from Stanley et al. (2010).

also be more coherent, for we used the mean *quantile* interevent time R_Q instead of the ordinary momentum mean to distinguish between different thresholds of the interevent-time statistics. Moreover, we did not rescale the statistics, taking into account the possibility of infinite moments.

Furthermore, this is a confirmation of the correctness of both models, i.e., the numerical agent-based model of financial markets presented in this chapter and the analytical superstatistics model of the interevent times presented in Chapter 3, or at least evidence of their mutual agreement.

5.4 Summary

The present chapter concerned an agent-based model of financial markets based on the Potts model known from statistical physics. The model is in fact a reinter-

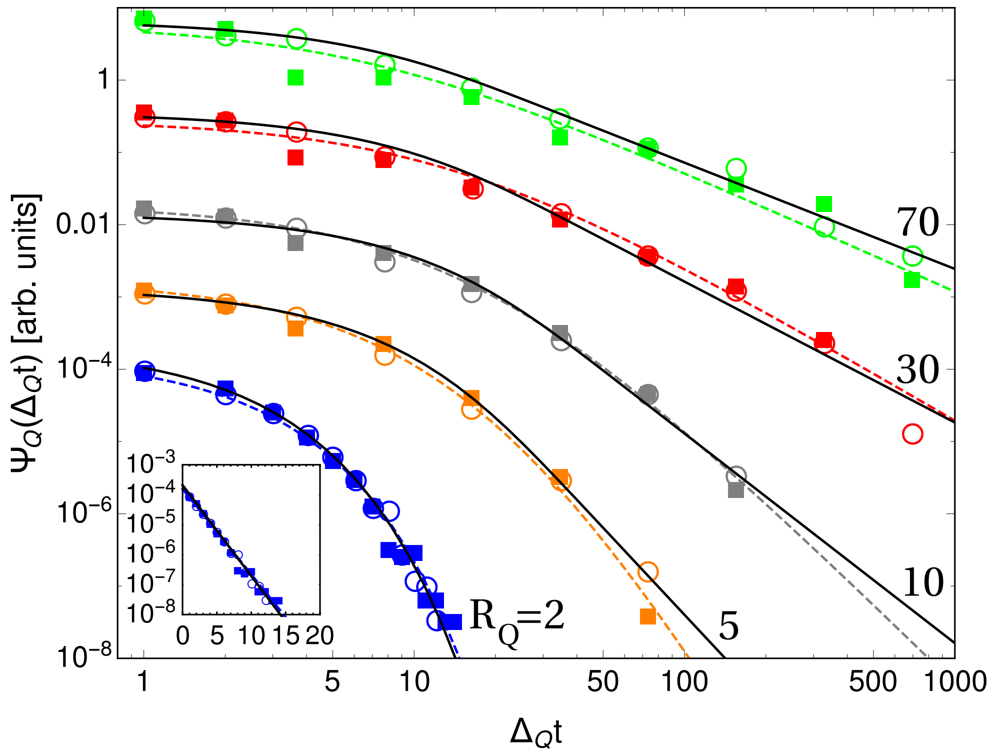

Figure 5.12: Superstatistics, $\psi_Q(\Delta_Q t)$, vs. interevent time, $\Delta_Q t$, in log-log scale for example simulation data from our model (full colored squares) for $R_Q = 2, 5, 10, 30, 70$ and $\tau = 10^3$ [rounds] (estimated equivalent of a single trading day), and for the parameters $J = 1, \lambda = 2, K = 5, b = 2, b_0 = 0.2$ (hence, the Pareto exponent $\beta = \ln K / \ln b > 2$, cf. Eqs. (5.4) and (5.6)). The inset is plot of $\psi_Q(\Delta_Q t)$ vs. $\Delta_Q t$ in semilogarithmic scale for $R_Q = 2$. The black solid curves are predictions of our formula (3.12) with the parameters of the fits the same as in Fig. 3.5 (see Tab. 3.6), while the dashed curves are q-exponential fits. Additionally, empty colored circles are the empirical data taken from Ludescher et al. (2011). Matching was done by the trial and error method with the fit accuracy of about 10%. Compare with Fig. 3.5.

pretation of the Sieczka–Hołyst threshold model of financial markets, namely, we changed the meaning of the spin variable, leaving the mathematics unchanged. The spin value in the reinterpretation stands for the current market state of an agent, i.e., whether it took short or long position, or is neutral at the moment; it does not stand for its action (buying, selling, or staying inactive) as in the original version. As in the SH model, this value is then passed to the neighboring agents as advice or an example that a given agent gives them. In turn, the action in this approach is defined by a *change* of the spin. Thereby, we secure the model against the paradox from the original version of the SH model, where everybody buys stocks but the corresponding price does not change.

The approach outlined above, i.e., the spin variable being rather the agent's state than its action, is novel in terms of the models of market game and constitutes an original contribution to econophysical market modeling[7]. Indeed, I separated opinion and decision, which hitherto have been generally dealt with together. In this way, I connected the concepts of opinion and decision modeling within the model of cunning agents. Such an approach resembles the one from some sociophysical models, where the spin frequently stands for an agent's opinion, which is rather a state (cf. Grabowski and Kosiński (2006)). One may conclude that in this approach the decision is more difficult, since it requires change of the state, and thus the model is more "passive" than the prevailing Ising-like market models.

Furthermore, to take into account the possibility of some intense emotions in the model, we used the Weierstrass–Mandelbrot noise in the agents' activity, instead of the canonical Gaussian one. We verified that, despite the presence of a threshold in the model, the results obtained for Gaussian and Weierstrass–Mandelbrot noises are quite different. For instance, for the WM distribution we obtained a power-law return and absolute-return statistics for (almost) the whole time-resolution range, while for the normal distribution the corresponding range is significantly shorter. We presume that this difference is caused by a dissimilarity between both noises within the maximal range of their impact (for results presented in Sec. 5.3 it was $[-(4J+\lambda), 4J+\lambda]$, where λ is the maximal threshold value, and J is the strength of the nearest neighbors' interaction). The discreteness of the WM distribution may also affect the results somehow.

We simulated the dynamics of the agents' activities and for the output data obtained from these simulations we calculated some basic quantities, such as the statistics (histograms) or autocorrelation function of both returns and absolute

[7]However, this approach appears to be natural and understandable, since the state of a spin indicates the state of an investor and the action is associated with the change of this state.

returns (Sec. 5.3.1). We compared the results of the simulations with the corresponding empirical data and obtained a good agreement (Sec. 5.3.2). Particularly, we reproduced the shape of both the usual and absolute-value autocorrelation function, as well as the interevent-time distribution considered in this thesis. **To the best of our knowledge, the interevent-time statistics comparison was the first successful one for an agent-based model (cf. Sec. 5.3.3), which constitutes another original contribution to the field of knowledge presented herein.**

What is more, the comparison of the model predictions with the interevent-time empirical data (cf. Fig. 5.12) was made without rescaling the relevant statistics. Ipso facto we allowed a mean value of the interevent time to be infinite, which distinguishes our approach from that used later in Gontis et al. (2016) (based on Yamasaki et al. (2005)) and makes it better suited to market reality. The reconstruction of varied essential stylized facts, notably autocorrelation-function shapes and interevent-time statistics, confirms the legitimacy of our approach. On top of this, I have not found in the literature the actual reproduction of either the usual or absolute-value autocorrelation function by an agent-based model, comprising the initial significant negative value of the first one (for short time scales) as well as power-law decay until the 10^3 time step for the second one, which our model provides (cf. Arthur et al. (1996); Farmer (1999); Iori (1999); Lux and Marchesi (1999); Bouchaud (2000); Giardina et al. (2001); Izumi and Ueda (2001); Jefferies et al. (2001); LeBaron (2001); Raberto et al. (2001); Tay and Linn (2001); Farmer and Joshi (2002); Hommes (2002); Tesfatsion (2002); Giardina and Bouchaud (2003); Alfarano et al. (2005); Kozłowska et al. (2006); LeBaron (2006); Mizuno et al. (2006); Preis et al. (2006); Cont (2007); Zhou and Sornette (2007); Chiarella et al. (2009); Tedeschi et al. (2009); Feng et al. (2012); Thurner et al. (2012)).

Extension of our approach to some more realistic topologies of the social network, as well as some other noises assumed, appears to be possible. It would be particularly interesting to incorporate order-book mechanics into the model and to observe its influence on the results. Evidently, the model in the present version is perfectly symmetrical, therefore breaking this symmetry would be desirable. Finally, some other modifications and generalizations of the basic version, adjusting it to the market reality, would be appreciated as well.

Furthermore, the role of abrupt transitions in the model mechanics shall be thoroughly considered. Some reasonable possibility to dispose of them without losing the obtained market similarities would be welcome. Another interesting question is how the presence of the threshold in the model, in particular the limit for the noise influence on an agent's state established by it, affects the shape of statistics and autocorrelation of returns, e.g., their exponential cut-off at their very end. Regarding the Weierstrass–Mandelbrot noise, the interesting question would be explaining the observed β-dependence of the results, i.a., the shape of return statistics, and their behavior as the time scale of the returns, τ, increases.

* * *

To conclude, the presentation of the numerical cunning-agents model made in this chapter, after the essential analytical model of superstatistics presented in Chapter 3, accomplishes the overall description of the interevent-time statistics in the present thesis. In the next chapter (6) a number of final comments are provided in order to summarize the whole picture presented up to this point.

6

Final remarks

After a detailed description of the problem of interevent times on financial markets and in geophysical data, the last chapter of this thesis provides a comparison of the goal given in the introduction with the actual results presented in the body of the thesis (Sec. 6.1). At the end, some possible further research directions related to the problems concerned herein are sketched (Sec. 6.2).

6.1 Conclusions

The goal of the thesis, presented in Chapter 1, was to provide a consistent description of the problem of the times between excessive events, that is, extending a positive threshold. This description had to be achieved under two complementary approaches: (i) analytical and (ii) numerical. The goal was achieved in the manner described below.

- First, I derived an analytical, closed form for the distribution of interevent times. My approach is founded on the extreme value theory (EVT) and the continuous-time random walk (CTRW) valley model (Scher and Montroll, 1975; Pfister and Scher, 1978; Weiss, 2005; Schulz and Barkai, 2015), reinterpreted and extended for this purpose to the case of stretched-exponential relaxation time.

- I used the bivariate Weibull distribution to consider the dependence of subsequent interevent times.

- I demonstrated that both the shape exponent α_Q and the relaxation time τ_Q present in the final formula depend only on the threshold Q, which leads to the collapse observed in the empirical data. Therefore, the obtained formula may be treated as "universal".

- I verified that the formula describes data collapse observed for a wide range of financial data for different assets (stocks, indices, currencies, or goods) and time scales (minutes, hours, days, and months).
- In this way I showed that considering extreme events together with some dependence between them is sufficient for describing the observed threshold phenomena.

This description, presented previously in Denys et al. (2016a,b), is more appropriate than the one based on the Tsallis q-exponential function given recently in Ludescher et al. (2011), though the single-variable return distribution in the form of a q exponential is also taken into consideration here. Our approach is reasonable, as we have a complete derivation of the final formula. **The derivation, without assuming any specific form of an underlying distribution of the returns[1], together with a thorough empirical-data comparison and some propositions for application in economics and geophysics, constitutes an original contribution to the field of knowledge presented in the first part of the thesis.** Moreover, the description of geophysical data mentioned above gives our work an interdisciplinary character.

- Secondly, using a three-state two-dimensional Potts model known from statistical physics, reinterpreted as agent-based numerical model of financial markets.
- Spins in the model, s_i, are investors ("spinsons") imitating their nearest neighbors on a model social network by taking similar market positions, that is, long position for $s_i = +1$, a short one for $s_i = -1$, and a neutral state for $s_i = 0$.
- The actual investment decision is in this case the change of the spin variable, $d_i(t) = s_i(t) - s_i(t-1)$, which is intuitively understandable.

To the best of our knowledge, such interpretation of a spin variable, as a market state, rather than the activity of an investor, is different to the interpretation assumed beforehand in financial-market ABMs (cf. Kwapień and Drożdż

[1] However, we used the Weibull distribution as a reference case in our further considerations, for its bivariate forms known – cf. Sec. 3.2.2 from Chapter 3.

(2012); Sornette (2014); Tsallis (2016)) and it constitutes the key concept of the model. **This novelty, together with the successful empirical comparisons with the usual and absolute-value return autocorrelation[2] and with interevent-time distributions (cf. Sec. 5.3.3 in the previous chapter), is another original contribution to the field of knowledge, presented in the second part of the thesis.**

Additionally, the comparison of the numerical results with the interevent-time statistics confirms the consistency of both our models, i.e., the analytical one and the numerical one (cf. Denys et al. (2014)). However, as first and higher moments of interevent times, $\Delta_Q t$, may not exist, our approach, i.e., taking the quantile mean R_Q instead of the usual first moment as a control variable, appears to be more relevant than the competitive, subsequent description by Gontis et al. (2016).

In this way, both aims of the thesis were fulfilled, extending the knowledge and understanding of stochastic processes, agent-based modeling, financial markets and geophysical systems.

Incidentally, both models – the analytical and the numerical ones – are based on the concept of a threshold value. However, in the case of the analytical model (Chapter 3) the threshold is used as a crucial control variable, distinguishing significant events from unimportant ones and specifying the shape of the resultant superstatistics of interevent times. As regards the numerical case (Chapter 5), we use a threshold at a microscopic level of agent interaction, to determine whether its neighbors' impact together with its own beliefs are sufficient to convince it to buy or sell some stocks. Therefore, both thresholds, though having different senses, are used to determine the relevance of some information that we possess, which is typical for socio- and econophysical considerations.

[2]I have not found the actual reproduction of either the usual or absolute-value autocorrelation function by an ABM with the initial significant negative value of the usual one (for short time scales) as well as power-law decay until 10^3 time step for the absolute one, which are provided by our model; cf. Arthur et al. (1996); Farmer (1999); Iori (1999); Lux and Marchesi (1999); Bouchaud (2000); Giardina et al. (2001); Izumi and Ueda (2001); Jefferies et al. (2001); LeBaron (2001); Raberto et al. (2001); Tay and Linn (2001); Farmer and Joshi (2002); Hommes (2002); Tesfatsion (2002); Giardina and Bouchaud (2003); Alfarano et al. (2005); Kozłowska et al. (2006); LeBaron (2006); Mizuno et al. (2006); Preis et al. (2006); Cont (2007); Zhou and Sornette (2007); Chiarella et al. (2009); Tedeschi et al. (2009); Feng et al. (2012); Thurner et al. (2012).

6.2 Future work

The work summarized above opens the way for some further studies that could extend the present view. As for the analytical model, the observed superscaling of the scaling exponent indicates that there may exist some additional level of classifying relaxation processes in systems; however the deeper physical meaning of this is still not well understood.

Moreover, we actually obtained a complete formalism for calculating the superstatistics (cf. Eq. (3.9)), where using (i) different forms of conditional distribution $\psi_Q(\Delta_Q t|\varepsilon)$ and (ii) the conjecture linking the relaxation time $\tau_Q(\varepsilon)$ and the mean discrete interevent time R_Q, required for a specific purpose, we may obtain varied forms of the final formula, not only the one presented in this thesis, Eq. (3.12).

Substantially, our work distinguishes a whole class of potential interevent-time models, however, there is no evidence that the model presented here is a unique one. In other words, our work narrows the scope of possible solutions of the problem instead of indicating the single one.

It is evident that we managed to show here not only the single-variable description, but also the **two-variable** one, by using the bivariate Weibull distribution. The next step would be to consider some trivariate distributions, in order to determine the narrowest PDF class that we could apply. Generally speaking, one may consider the correlations between some distant interevent times; possibly in this way we could justify the key conjecture linking the relaxation time and the return distribution used in our model, Eq. (3.11).

The possible application of our analytical approach for simulating a value-at-risk time series (see Sec. 3.5.1 from Chapter 3) is only a preproposal; the actual use of our model in insurance or investment practice requires works confirming its usefulness in some specific cases. The profit analysis that was outlined here (Sec. 3.5.2) should be extended to data of much better accuracy than the ones that we actually used, e.g., to accurately determine the parameters of fits and examine an equivalent of our superscaling (the scaling of scaling exponent) hypothesis for the case of profits.

Additionally, the geophysical data fit (Sec. 3.5.3) could be better than the one obtained; it may require reconsidering the assumptions that we made and possibly propose a modification of the current version. Actually, since the essential concept of the model does not indicate any specific kind of the represented data, the possible applications to biological and physical noises, and some other kinds of noises, appears to be feasible, but the geophysical-application case shows that each of them would probably require a whole separate study to adjust the specific assumptions of the model to each particular case.

Regarding the numerical model, in addition to the remarks already made (see Sec. 5.4 from Chapter 5), one could attempt to reconstruct the multifractal spectra of a real-life return time series, as the model has not been analyzed in this matter yet (cf. Grech and Czarnecki (2009)). As for analytical work, it would be a challenge to solve the three-state Potts model used in our approach and obtain the result in a closed form. Consequently, it might enable us to calculate the value of the superscaling exponent ζ present in our analytical model (cf. Secs. 3.3.1 and 3.4 from Chapter 3). In this way, we would obtain this (macroscopic) quantity using the microscopic assumptions of our ABM. In the long run, the goal for the cunning-agents model would be the Holy Grail of market modeling, namely the prediction of market behavior, or at least a partial one.

To sum up, the thesis contains a number of innovative results combining both fundamental considerations and their applications, thus it may be useful for some large establishments operating on markets that deal, on a daily basis, with the key problem of excessive losses (e.g., insurance companies or financial institutions). Although the description of the problem of interevent times given in this thesis may be extended and improved, a general direction was actually indicated. The further studies required might be able to answer, at least partially, a fundamental question of the *ab initio* laws of market economy.

Appendix

Derivation of interevent-time distribution[1]

To derive the distribution $\psi_Q^{\pm}(\Delta_Q t)$ we use the second part of Eq. (3.9), i.e.,

$$\psi_Q^{\pm}(\Delta_Q t) = -\frac{\int_Q^{\infty} \psi_Q^{\pm}(\Delta_Q t|\varepsilon)\,\mathrm{d}\left(\int_\varepsilon^{\infty} D(\varepsilon')\,\mathrm{d}\varepsilon'\right)}{\int_Q^{\infty} D(\varepsilon)\,\mathrm{d}\varepsilon},$$

and based on Eq. (3.11) we rewrite it as follows,

$$\psi_Q^{\pm}(\Delta_Q t) = \frac{1}{z(Q)}\frac{1}{\tau_Q(0)} I_Q^{\pm}, \tag{1}$$

where auxiliary variable

$$z = z(\varepsilon) \stackrel{\text{def.}}{=} \int_\varepsilon^{\infty} D(\varepsilon')\,\mathrm{d}\varepsilon', \tag{2}$$

and integral

$$\begin{aligned} I_Q^{\pm} &\stackrel{\text{def.}}{=} \int_0^{\tau R_Q^{-1}} z^{\pm 1/\alpha_Q^{\pm}} \exp\left(-z^{\pm 1/\alpha_Q^{\pm}} \frac{\Delta_Q t}{\tau_Q^{\pm}(0)}\right) \mathrm{d}z \\ &= \frac{1}{1 \pm 1/\alpha_Q^{\pm}} \int_0^{\tau R_Q^{-1}} \exp\left(-z^{\pm 1/\alpha_Q^{\pm}} \frac{\Delta_Q t}{\tau_Q^{\pm}(0)}\right) \mathrm{d}\left(z^{1 \pm 1/\alpha_Q^{\pm}}\right). \end{aligned} \tag{3}$$

[1]Based on Appendix in Denys et al. (2016b).

From the identity

$$\mathrm{d}\left(z^{1\pm 1/\alpha_Q^\pm}\right) = \mathrm{d}\left(z^{\pm 1/\alpha_Q^\pm}\frac{\Delta_Q t}{\tau_Q^\pm(0)}\right)^{\alpha_Q^\pm(1/\alpha_Q^\pm \pm 1)} \times \left(\frac{\tau_Q^\pm(0)}{\Delta_Q t}\right)^{1\pm 1/\alpha_Q^\pm} \tag{4}$$

and Eqs. (1) and (2) we obtain the final formula,

$$\psi_Q^\pm(\Delta_Q t) = \frac{1}{\tau_Q^\pm(Q)}\frac{\alpha_Q^\pm}{\left(\Delta_Q t/\tau_Q^\pm(Q)\right)^{1\pm\alpha_Q^\pm}} \times \Gamma^\pm\left(1 \pm \alpha_Q^\pm, \frac{\Delta_Q t}{\tau_Q^\pm(Q)}\right), \tag{5}$$

where

$$\Gamma^+\left(1+\alpha_Q^+, \frac{\Delta_Q t}{\tau_Q^+(Q)}\right) = \int_0^{\frac{\Delta_Q t}{\tau_Q^+(Q)}} u^{\alpha_Q^+}\exp(-u)\,\mathrm{d}u \tag{6}$$

is the lower incomplete gamma function with

$$u \stackrel{\text{def.}}{=} z^{1/\alpha_Q^+}\frac{\Delta_Q t}{\tau_Q^+(0)} \tag{7}$$

taken and, complementary,

$$\Gamma^-\left(1-\alpha_Q^-, \frac{\Delta_Q t}{\tau_Q^-(Q)}\right) = \int_{\frac{\Delta_Q t}{\tau_Q^-(Q)}}^{\infty} u^{-\alpha_Q^-}\exp(-u)\,\mathrm{d}u \tag{8}$$

is the upper incomplete gamma function with

$$u \stackrel{\text{def.}}{=} z^{-1/\alpha_Q^-}\frac{\Delta_Q t}{\tau_Q^-(0)} \tag{9}$$

taken. Hence, we use the relation

$$d\left(u^{1\pm\alpha_Q^\pm}\right) = (1\pm\alpha_Q^\pm)u^{\pm\alpha_Q^\pm}\,\mathrm{d}u \tag{10}$$

in both "+" and "−" cases.

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