Notes on Infra-Red effects in QED

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Abstract

In this note, we discuss the infrared problem in quantum electrodynamics and its resolution via Faddeev–Kulish dressings. We begin by connecting the classical definition of radiation, as presented in standard electrodynamics textbooks, with quantum field-theoretic expressions. We then analyze the infrared divergences that arise in Feynman diagrams and demonstrate that their exponentiation leads to a trivial S-operator when acting on the standard Fock space built from states with a finite number of external photons. Next, we construct Faddeev–Kulish dressings and show how they eliminate infrared divergences. Finally, we briefly describe how the soft theorems predict a measurable phenomenon of the memory effect.

Introduction

In undergraduate physics courses, it is usually convenient to distinguish between an electron (thought of as a ball with electric charge e and mass m_0 and electric field which it produces. However, introduction of the concept of a charged particle stripped from its electric field leads to two very severe problems. Only if we realize that charged particles separated from their electro-magnetic field do not exists in real world, we can make sense of QFT results!

First issue is the famous problem of infinite energy of an electron with its own electric field [1]. Since distance from electron to itself, r, is zero, interaction energy is infinite, $U = \lim_{r\to 0} \frac{e^2}{4\pi}\frac{1}{r} = \infty$. Since "$E = mc^2$", it is tempting to conclude that electrons have infinite mass $m_0 + U/c^2$ - so they cannot move. But this is clearly nonsense - mass of an electron is 0.5 GeV. We need to *renormalize* the mass - i.e. say that the parameter $m_0 = 0.5\ \text{GeV}/c^2 - U/c^2$ - that is, the *physical* electron with mass 0.5 GeV *is* the ball with electric charge e and mass parameter m_0 together with its own Coulomb field.

In these notes, we will focus on the second problem. When a charged particle "wiggles" or accelerates, it sources an electro-magnetic field disturbance $\delta F_{\mu\nu}$ which decays slowly at infinity, as $\delta F_{\mu\nu} \sim \mathcal{O}(r^{-1})$. Such electromagnetic modes are called "radiation."

Consider a scattering event involving charged particles (with renormalized mass - assume that we have already taken care of the Coulombic self-interaction of particles). In every non-trivial scattering event, the charged particles accelerate (so that final velocities are different than initial velocities), and hence, they produce radiation. Therefore, a non-trivial scattering event of charged particles with no radiation both at the beginning and in the end of an experiment is not possible! Quantum-mechanically, given an initial state of n charged particles (without photons) $|p_1, \dots, p_n\rangle$ and final state of n' charged particles with different momenta (and without photons) $|p'_1,\ldots,p'_{n'}\rangle$, the following S-matrix element vanishes:

$$\langle p'_1,\ldots,p'_{n'}|\,\hat{S}\,|p_1,\ldots,p_n\rangle = 0. \tag{0.1}$$

This suggests that the most convenient basis for QED S-matrix is *not* comprised of single-particle states, but rather by *dressed states* describing charged particles together with the acceleration radiation (known also as *bremsstrahlung*).

In the first section we will show that the standard free QFT modes of electromagnetic field coincide with the definition of radiation that can be found in textbooks on classical electrodynamics [1].

In the second section we shall focus on the structure divergences of QED Feynman diagrams in the limit as energy of a single external photon goes to zero (we say that the external photon is *soft*). It turns out that the behavior of amplitudes in this *soft limit* contains a lot of interesting information about the theory. Next, in the second section, we will show how the soft (infra-red) divergences exponentiate for diagrams corresponding to the above-discussed $p_1,\ldots,p_n \to p'_1,\ldots,p'_{n'}$ scattering, leading to $\langle p'_1,\ldots,p'_{n'}|\,\hat{S}\,|p_1,\ldots,sp_n\rangle \sim \exp(-\infty) = 0$. In the third section, we will discuss the construction of asymptotic states that contain acceleration radiation, and show that the corresponding S-matrix is non-trivial. A detailed, self-contained analysis of IR divergences in QED can be also found in the classic textbook [2].

At the end of this note, we discuss the simplest example of so-called "memory effects" [3–5].

0.1 Basic conventions and gauge-fixing

The main subject of study in this note is a theory of U(1) gauge field A_μ coupled to matter fields Φ, with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\text{matter}}(\Phi, A). \tag{0.2}$$

For computational simplicity, throughout these notes, Feynman-diagrammatic calculations will be performed, depending on the situation, for either scalar QED, corresponding to

$$\mathcal{L}_{\text{matter}} = -(D_\mu\phi)^\dagger(D^\mu\phi) + V(\phi^\dagger\phi), \tag{0.3}$$

with arbitrary potential function V dependent on U(1)-invariant variable $\phi^\dagger\phi$, or spinor QED, corresponding to:

$$\mathcal{L}_{\text{matter}} = \bar{\psi}(i\not{D} - m)\psi. \tag{0.4}$$

In the end of each calculation, we will generalize our results to an arbitrary matter coupled to the U(1) gauge field.

We will work in the Lorentz gauge, $\partial_\mu A^\mu = 0$, supplemented by temporal gauge condition, $A_0 = 0$. We can always do this, since, given $\tilde{A}_\mu$, we take $A_\mu = \tilde{A}_\mu + \partial_\mu\lambda$ with λ satisfying

$$\partial_\mu\partial^\mu\lambda = -\partial^\mu\tilde{A}_\mu. \tag{0.5}$$

The Lorentz condition fixes U(1) transformations only up to functions satisfying the wave equation, $\partial_\mu \partial^\mu \lambda' = 0$. Then, equations of motion for A_μ are equivalent to a non-homogenous Klein-Gordon equation

$$\Box A_\mu = J_\mu, \tag{0.6}$$

with current $J^\mu = \frac{\partial \mathcal{L}_{\text{matter}}}{\partial A_\mu}$.

In the Lorentz gauge, we can also fix the residual gauge freedom by imposing the temporal gauge condition:

$$A_0 = 0, \tag{0.7}$$

since, given $\tilde{A}_0 \neq 0$, we can choose λ' such that $\partial_0 \lambda' = -\tilde{A}_0$ and $\partial^2 \lambda' = 0$.[1]

1 What is radiation? - radiative phase space of electromagnetism

In undergraduate textbooks, one usually defines electomagnetic radiation as the part of the Poynting vector $\vec{S} = \vec{E} \times \vec{B}$ that decays as "$\frac{1}{r}$" at infinity, $r \to \infty$ (r is the standard radial distance from the origin of a chosen coordinate system). But this definition is not elegant. Since $\vec{S}$ is not a Lorentz-invariant object, it is not directly measurable.

For instance, consider a Coulomb field of a point charge e. In the rest frame of the charge $\vec{E} = \frac{e}{4\pi} \frac{\vec{r}}{r^3}$, $\vec{B} = 0$, and hence $\vec{E} \times \vec{B} = 0$. However, if we make a Lorentz boost, both electric and magnetic fields in the new frame are nonzero $\vec{E}' \neq 0 \neq \vec{B}'$, and $\vec{E}' \times \vec{B}'$.

We clearly need a more geometric way to properly describe electromagnetic radiation. To prepare ourselves for a proper definition, let us first briefly review causal structure of Minkowski spacetime.

1.1 Aspects of geometry of Minkowski spacetime

Let (t, r, θ, φ) be the standard spherical coordinates in Minkowski space. The Minkowski metric is:

$$\eta = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathring{q}, \tag{1.1}$$

where $\mathring{q}$ is the metric on the unit round sphere, $\mathring{q} = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$. Introduce advanced and retarded time coordinates, u and v, as:

$$v = t + r, \quad u = t - r. \tag{1.2}$$

In these coordinates, the Minkowski metric reads

$$\eta = -\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r + r^2 \mathring{q} \tag{1.3}$$

$$= -\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + r^2 \mathring{q} \tag{1.4}$$

$$= -\mathrm{d}u\mathrm{d}v + \frac{1}{4}(u - v)^2 \mathring{q}. \tag{1.5}$$

[1]With $A_0 = 0$, Lorentz condition reduces to the Coulomb condition, however, now the residual gauge transformations are time-independent (in Coulomb gauge there are more residual gauge transformations since they are generically time-dependent).

The range of u, v-coordinates is $-\infty < u \leq v < \infty$.

We will also frequently use stereographic coordinates on S^2:

$$z = e^{i\varphi} \cot(\theta/2), \quad \bar{z} = e^{-i\varphi} \cot(\theta/2). \tag{1.6}$$

In these coordinates

$$\mathring{q} = \frac{4 \mathrm{d}z \mathrm{d}\bar{z}}{(1 + z\bar{z})^2} \equiv 2 q_{z\bar{z}} \mathrm{d}z \mathrm{d}\bar{z}, \tag{1.7}$$

where $q_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$. The inverse metric has two nonzero components, $q^{z\bar{z}} = q^{\bar{z}z} = \frac{1}{q_{z\bar{z}}} = \frac{1}{2}(1 + z\bar{z})^2$. Volume form on S^2 is

$$\mathrm{vol}_{S^2} = \sin(\theta) \mathrm{d}\theta \wedge \mathrm{d}\varphi = i q_{z\bar{z}} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{1.8}$$

1.1.1 Penrose diagram for Minkowski space

Structure of the spacetime at "$r \to \infty$" can be understood by introducing yet another set of coordinates:

$$U = \arctan(u), \quad V = \arctan(v), \tag{1.9}$$

with $-\frac{\pi}{2} < U \leq V \leq +\frac{\pi}{2}$. Then

$$\eta = -\frac{1}{\cos^2 U \cos^2 V} \left(\mathrm{d}U \mathrm{d}V + \frac{1}{4} \sin^2(U - V) \mathring{q} \right) \equiv \Omega^2 \hat{\eta}, \tag{1.10}$$

where $\Omega = (\cos U \cos V)^{-1}$ and metric $\hat{\eta}$ is regular at $U, V \to \pm\pi/2$:

$$\hat{\eta} = -\mathrm{d}U \mathrm{d}V + \frac{1}{4} \sin^2(U - V) \mathring{q}. \tag{1.11}$$

It is well-known that Weyl-rescaling a metric $g_{\mu\nu} \mapsto \Omega^2(x) g_{\mu\nu}$ preserves null (lightlike) geodesics. In our simple case, lightlike trajectories are given by $u =$ const. or $v =$ const. for Minkowski metric, and $U =$ const. or $V =$ const. in the rescaled metric $\hat{\eta}$. Hence, $\hat{\eta}$ yields the same causal structure as η. Since coordinates U and V have finite range, suppressing angular coordinates θ, φ, we can draw a finite-size diagram of the entire spacetime:

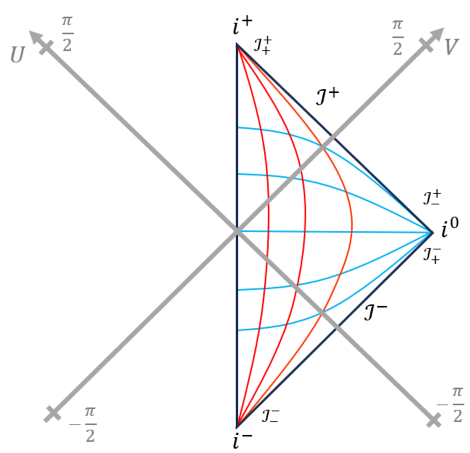


Figure 1: Penrose diagram of Minkowski spacetime. Red lines represent lines of constant r, and blue lines represent constant time slices $t =$ const. Every point on the diagram represents a 2-sphere (of conformally-rescaled radius $\frac{1}{4} \sin^2(U - V)$).

Point $i^+ = (U = \pi/2, V = \pi/2)$ is called "future infinity", and it is where all timelike curves end. In particular, we achieve it in the limit $t \to \infty$ with fixed r. Similarly, "past infinity" $i^- = (U = -\pi/2, V = -\pi/2)$ corresponds to the limit $t \to -\infty$ with fixed r. Spacetime boundary $\mathcal{J}^+$ called "future null infinity" is $\mathcal{J}^+ = \{-\pi/2 < U < \pi/2, V = \pi/2\}$. It is reached in the limit $r \to \infty$ with $u = t - r =$ const. fixed. Past null infinity $\mathcal{J}^- = \{U = -\pi/2, -\pi/2 < V < \pi/2\}$, is reached in the limit $r \to \infty$ with $v = t + r =$ const. Line $U = V$ is the "middle" of space, i.e., $r = 0$. We also introduced notation $\mathcal{J}^+_+$ and $\mathcal{J}^+_-$ for for future and past boundary of $\mathcal{J}^+$ (both $\mathcal{J}^+_\pm$ are spheres). Similarly, $\mathcal{J}^-_\pm$ are future and past boundary of $\mathcal{J}^-$. Even though on the above diagram it seems that spheres $\mathcal{J}^+_-$, i^0, $\mathcal{J}^-_+$ coincide, it is important to distinguish between them in order to impose antipodal boundary conditions [5,6]. Conformal diagram nicely illustrates causal structure of spacetime, but it does not preserve distances between points - region near i^0 is very large (it is infinite) even though on the diagram it looks small.

It is convenient to unfold the above diagram to represent antipodal points on the spheres. Regions $\mathcal{J}^\pm$ are of particular interest to us, since in a free theory, massless particles (and hence, radiation) follow lines of unit slope and cross points on the spheres at $\mathcal{J}^-$ ($\mathcal{J}^+$) at retarded (advanced) times v (u). Massive particles never reach $\mathcal{J}^\pm$. Timelike geodesics start at i^- and end on i^+. This is illustrated on the following diagram:

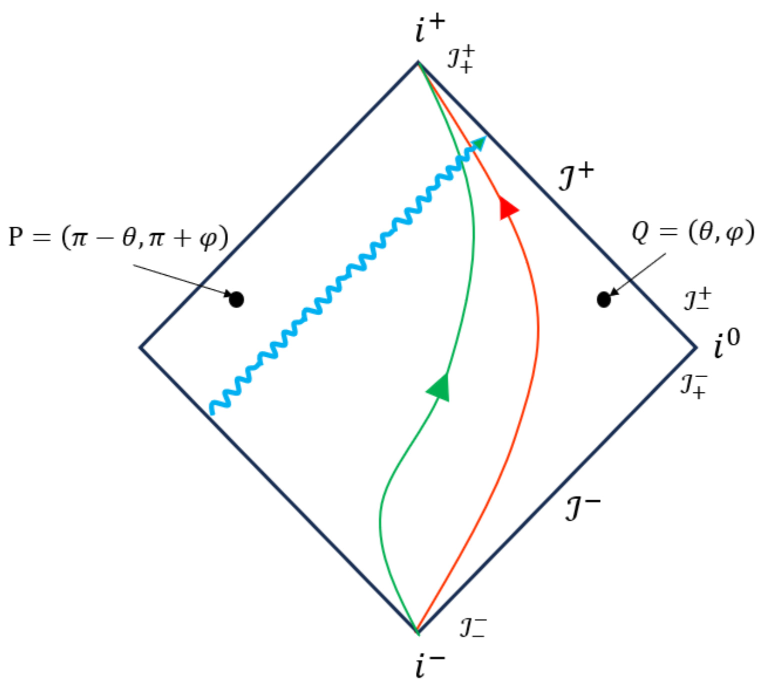


Figure 2: Penrose diagram of Minkowski space, where two points on the diagram represent a single sphere S^2. For example, points P and Q are antipodal points on a 2-sphere. Red and green lines illustrate timelike trajectories. Wiggly blue line represents a lightlike trajectory.

1.2 Conventions for 4-momenta and polarizations

Let us now spell out our conventions for free field expansions and introduce useful parametrizations of 4-momenta and polarization vectors. Free U(1) gauge field can be written as:

$$A_\mu(x) = \int \widetilde{dq}\left(a_\mu(\vec{q})e^{iq\cdot x} + a^\dagger_\mu(\vec{q})e^{-iq\cdot x}\right), \tag{1.12}$$

where $a_\mu(\vec{q}) = \sum_{\alpha=\pm} a_\alpha(\vec{q})\varepsilon^\alpha_\mu(\vec{q})$, and $\varepsilon^\alpha_\mu(\vec{q})$ are circular polarization vectors, and $\widetilde{dq} = \frac{d^3\vec{q}}{2|\vec{q}|(2\pi)^3}$ is a Lorentz-invariant measure of integration over photon momenta. In quantum theory, $a^\dagger_\alpha(\vec{q})$ is the complex conjugate of the annihilation operator $a_\alpha(\vec{q})$ satisfying canonical commutation relations, $[a_\alpha(\vec{q}), a^\dagger_\beta(\vec{q}')] = (2\pi)^3 2|\vec{q}|\delta^3(\vec{q}-\vec{q}')$. In classical theory $a^\dagger_\alpha(\vec{q})$ is simply complex conjugate of the Fourier mode $a_\alpha(\vec{q})$. We will not introduce a separate notation for classical fields and quantum field operators. It should be clear from the context to which of these two objects we are referring to in a given paragraph.

One can parametrize null momenta q^μ as:

$$\begin{aligned} q^\mu &= (q^0, q^1, q^2, q^3) \\ &= \omega(1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ &= \frac{\omega}{1+w\bar{w}}(1+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-w\bar{w}) \equiv \frac{\sqrt{2}\omega}{1+w\bar{w}}\hat{q}^\mu, \end{aligned}$$

where q^0, q^1, q^2, q^3 are the standard components in cartesian coordinate system (t, x^1, x^2, x^3). We also defined:

$$\hat{q}^\mu(w,\bar{w}) = \frac{1}{\sqrt{2}}(1+w\bar{w}, w+\bar{w} - i(w-\bar{w}), 1-w\bar{w}). \tag{1.13}$$

Given a photon with 4-velocity q^μ, ω is the frequency of the photon measured by an observer moving with 4-velocity $u^\mu = (\partial_t)^\mu = (\partial_u)^\mu$, namely $\omega = q_\mu u^\mu$. Given $\hat{q}(w,\bar{w})$, circular polarization vectors are defined as:

$$\begin{aligned} \varepsilon^\mu_+(w,\bar{w}) &= \partial_w \hat{q}^\mu - f(q)q^\mu = \frac{1}{\sqrt{2}}(\bar{w}, 1, -i, -\bar{w}) - f(q)q^\mu, \\ \varepsilon^\mu_-(w,\bar{w}) &= \partial_{\bar{w}} \hat{q}^\mu - g(q)q^\mu = \frac{1}{\sqrt{2}}(w, 1, +i, -w) - g(q)q^\mu, \end{aligned} \tag{1.14}$$

with functions $f(q) = f(\omega, w, \bar{w})$, $g(q) = g(\omega, w, \bar{w})$ chosen so that polarization vectors have zero u-components, $\varepsilon^u_+ = 0 = \varepsilon^u_-$ (since we want to work in the temporal gauge). They obey the standard orthogonality relations:

$$\eta_{\mu\nu}(\varepsilon^\mu_\alpha)^*\varepsilon^\nu_\beta = \delta_{\alpha\beta}, \quad \varepsilon^\mu_\alpha(w,\bar{w})\hat{q}_\mu(w,\bar{w}) = 0. \tag{1.15}$$

We will use the following volume form on $\mathcal{J}^+$:

$$\mathrm{d}u \wedge \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi = du \wedge iq_{z\bar{z}}\mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{1.16}$$

1.3 Radiative phase space

Let us now define radiation as part of electromagnetic field that carries energy to $\mathcal{J}^+$.

1.3.1 Energy flux

Energy flux of electromagnetic field through a codimension 1 spacelike surface Σ, as seen by observer with 4-velocity ξ^μ, is

$$\mathcal{F}_{\xi,\Sigma}[A] = \int_\Sigma d^3x\sqrt{h}\, n^\mu T_{\mu\nu}\xi^\nu, \tag{1.17}$$

where h is the determinant of the metric induced on Σ, n^μ is the unit normal to Σ, and $T_{\mu\nu}$ is the electromagnetic stress tensor:

$$T_{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}(x)}\int\sqrt{-g}\left(-\tfrac{1}{4}F_{\rho\sigma}F_{\kappa\lambda}g^{\rho\sigma}g^{\kappa\lambda}\right) = F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}. \tag{1.18}$$

We can take the limit of Σ approaching $\mathcal{J}^+$, with an observer moving near $\mathcal{J}^+$ with 4-velocity $\xi^\mu\partial_\mu = \partial_t = \frac{\partial u}{\partial t}\partial_u + \frac{\partial r}{\partial t}\partial_r = \partial_u$. Then $n^\mu = (\partial_u)^\mu$, and flux through $\mathcal{J}^+$ is:

$$\mathcal{F}_{\mathcal{J}^+} = \lim_{r\to\infty}\int_{-\infty}^{\infty} \mathrm{d}u(\partial_u)^\mu \int_{S^2} r^2\mathrm{vol}_{S^2}\, T_{\mu\nu}(\partial_u)^\nu, \tag{1.19}$$

where vol_{S^2} is the volume form on the unit round sphere. The uu-component of the stress tensor is:

$$T_{uu} = F_{u\rho}F_{u\sigma}\eta^{\rho\sigma} = \frac{2}{r^2}q^{z\bar{z}}F_{uz}F_{u\bar{z}} \tag{1.20}$$

since $\eta_{uu} = 0$, $\eta^{rz} = \eta^{r\bar{z}} = 0$ and $\eta^{z\bar{z}} = \frac{1}{r^2}q^{z\bar{z}}$. The factor of $r^{-2}q^{z\bar{z}}$ cancels with $r^2 q_{z\bar{z}}$ from the volume form vol_{S^2}, and we obtain:

$$\mathcal{F}_{\mathcal{J}^+} = 2\int_{-\infty}^{\infty} du \int_{\mathbb{C}} d^2z \lim_{r\to\infty}(F_{uz}F_{u\bar{z}}). \tag{1.21}$$

1.3.2 Asymptotic expansions

Requiring $\mathcal{F}_{\mathcal{J}^+}$ to be finite suggests the following fall-off conditions at $\mathcal{J}^+$:

$$F_{uA}(u,r,z,\bar{z}) = F^{(0)}_{uA}(u,r,z,\bar{z}) + \frac{1}{r}F^{(1)}_{uA}(u,r,z,\bar{z}) + \mathcal{O}(r^{-2}), \tag{1.22}$$

with radiative degrees of freedom encoded in $F^{(0)}_{uA}(u,r,z,\bar{z})$, with index $A = z,\bar{z}$ running over stereographic coordinates.

In retarded coordinates $u,r,z,\bar{z}$, temporal gauge condition reads:

$$0 = A_t = \frac{\partial u}{\partial t}A_u + \frac{\partial r}{\partial t}A_r = A_u. \tag{1.23}$$

Therefore:

$$F_{uA}(u,r,z,\bar{z}) = \partial_u A_A(u,r,z,\bar{z}), \tag{1.24}$$

which implies:

$$A_A(u,r,z,\bar{z}) = A^{(0)}_A(u,z,\bar{z}) + \frac{1}{r}A^{(1)}_A(u,z,\bar{z}) + \mathcal{O}(r^{-2}). \tag{1.25}$$

Radial component $A_r(u, r, z\bar{z})$ is determined by the Lorentz gauge condition, which (generically, in curved spacetime) reads $\nabla^\mu A_\mu = 0$.

Fall-off condition for $A_r(u, r, z, \bar{z})$ can be determined from finiteness of the electric charge. Total electric charge of sources of radiation that reached $\mathcal{J}^+$ is defined as:

$$Q = \int_{\mathcal{J}^+_+} \star F = \int_{\mathcal{J}^+_+} F_{ur} \star \mathrm{d}u \wedge \mathrm{d}r = \int_{S^2} F_{ur}\, r^2 \sin(\theta) \mathrm{d}\theta \wedge \mathrm{d}\varphi. \tag{1.26}$$

It is finite provided that $F_{ur} = \mathcal{O}(r^{-2})$. Since $F_{ur} = \partial_u A_r - \partial_r A_u = \partial_u A_r + \mathcal{O}(r^{-2})$, we have:

$$A_r = \frac{1}{r^2} A_r^{(2)}(u, z, \bar{z}) + \mathcal{O}(r^{-3}). \tag{1.27}$$

Coefficients $A_r^{(k \geq 2)}$ are fully determined by $\nabla_\mu A^\mu = 0$. Radiative modes are fully described by two functions on $\mathcal{J}^+$, $A_z^{(0)}(u, z, \bar{z})$, $A_{\bar{z}}^{(0)}(u, z, \bar{z})$.

1.3.3 Radiative modes from free-field expansion

Let us now show that fall-off conditions (1.25), (1.27) are fully consistent with the standard free field expansion in plane waves:

$$A_\mu(x) = \int \widetilde{dk} \left(a_\mu(\vec{k}) e^{ik \cdot x} + a_\mu^\dagger(\vec{k}) e^{-ik \cdot x} \right), \tag{1.28}$$

Writing $\vec{k}$ in spherical coordinates, $\vec{k} = \omega(\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$, where θ is the angle between $\vec{k}$ and $\vec{x} = r\hat{x}$, we obtain:

$$A_\mu(x) = \int_0^\infty \frac{\omega d\omega}{2(2\pi)^3} \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi \left(a_\mu(\vec{k}) e^{-i\omega u - i\omega r(1-\cos(\theta))} + \text{h.c.} \right). \tag{1.29}$$

At $r \to \infty$ we can use stationary phase approximation. Then, the integral over θ, φ localizes at $\theta = 0$ - we can put $\theta = 0$ everywhere except in the rapidly changing exponents. At $\theta = 0$ vector $\hat{k}$ coincides with the direction of $\vec{x}$. Let us denote $\vec{x} = r\hat{x}$. Then, we can write

$$A_\mu(x) \overset{r\to\infty}{\sim} \int_0^\infty \frac{\omega d\omega}{8\pi^2} \left(a_\mu(\omega\hat{x}) \int_{-1}^1 d(\cos\theta) e^{-i\omega u - i\omega r(1-\cos(\theta))} + \text{h.c.} \right). \tag{1.30}$$

In the stationary phase approximation $\int_{-1}^1 d(\cos\theta) e^{-i\omega r(1-\cos(\theta))} \approx \frac{1}{i\omega r}$, and thus

$$A_\mu(x) \overset{r\to\infty}{\sim} \frac{1}{8\pi^2 i} \frac{1}{r} \int_0^\infty d\omega \left(a_\mu(\omega\hat{x}) e^{-i\omega u} - a_\mu^\dagger(\omega\hat{x}) e^{i\omega u} \right). \tag{1.31}$$

Because circular polarizations $\varepsilon_\mu^\alpha(\hat{x})$ do not have u, r-components, $\varepsilon_r^\alpha(\hat{x}) = 0 = \varepsilon_u^\alpha(\hat{x})$, result (1.31) implies that A_r and A_u decay at $r \to \infty$ faster than $1/r$, which is consistent with conditions $A_u = 0$ and $A_r \sim \mathcal{O}(r^{-1})$. $A_z, A_{\bar{z}}$ are of order $\mathcal{O}(1)$, since $\partial x^\mu / \partial z \sim \mathcal{O}(r)$ for Cartesian coordinates $x^\mu = (t, x^1, x^2, x^3)$.

More specifically, for $\vec{k}=\omega\hat{x}$ polarization vectors (1.14) are given by:

$$\varepsilon_+^\mu=\frac{1}{r\sqrt{q_{z\bar{z}}}}(\partial_z)^\mu,\quad \varepsilon_-^\mu=\frac{1}{r\sqrt{q_{z\bar{z}}}}(\partial_{\bar{z}})^\mu,\quad \varepsilon_{+\mu}=r\sqrt{q_{z\bar{z}}}(\mathrm{d}\bar{z})_\mu,\quad \varepsilon_{-\mu}=r\sqrt{q_{z\bar{z}}}(\mathrm{d}z)_\mu, \tag{1.32}$$

where $q_{z\bar{z}}=\frac{2}{(1+z\bar{z})^2}$ and therefore:

$$a_\mu(\omega\hat{x})=r\sqrt{q_{z\bar{z}}}\left(a_-(\omega\hat{x})(\mathrm{d}\bar{z})_\mu+a_+(\omega\hat{x})(\mathrm{d}z)_\mu\right), \tag{1.33}$$

$$A_z(u,r,z,\bar{z})=\frac{\sqrt{q_{z\bar{z}}}}{8\pi^2 i}\int_0^\infty \mathrm{d}\omega\left(a_+(\omega\hat{x})e^{-i\omega u}-a_-^\dagger(\omega\hat{x})e^{i\omega u}\right)=A_z^{(0)}(u,z,\bar{z}). \tag{1.34}$$

The bottom line is that radiative modes are fully described by the standard free field coefficients $a_\mu(\omega\hat{x}(z,\bar{z}))$ and $a_\mu^\dagger(\omega\hat{x}(z,\bar{z}))$, which are Fourier modes of $A_z^{(0)}(u,z,\bar{z})$ and $A_{\bar{z}}^{(0)}(u,\bar{z},z)$. Upon quantization $a_\pm$, $a_\pm^\dagger$ become creation and annihilation operators. Then, the above results tell us that asymptotic photon states describe radiation - nothing surprising!

1.4 Towards Feynman-diagrammatic description

In QFT, we calculate the profile of electromagnetic flied, like $F_{uz}^{(0)}$, as expectation value of the corresponding field operator, $\hat{F}_{uz}^{(0)}$. For example, for scattering event with initial state $|\text{in}\rangle$ and final state $|\text{out}\rangle$, the corresponding field profile at $\mathcal{J}^+$ (i.e., at the end of the scattering experiment) is [7]:

$$F_{uz}^{(0)}=\partial_u A_z^{(0)}=\langle\text{out}|\,\hat{F}_{uz}^{(0)}\hat{S}\,|\text{in}\rangle=-\frac{1}{8\pi^2}\int_0^\infty d\omega\,\omega\,e^{-i\omega u}\,\langle\text{out}|\,a_+(\omega\hat{x})\hat{S}\,|\text{in}\rangle\,. \tag{1.35}$$

$F_{uz}^{(0)}$ is simply given by a scattering amplitude with an extra external photon! At the tree level, amplitude $\langle\text{out}|\,a_+(\omega\hat{x})\hat{S}\,|\text{in}\rangle$ is a meromorphic function of ω, so we can Laurent-expand it at $\omega=0$. Since the integral (1.35) should be convergent at $\omega\to 0$, the Laurent expansion must start at order ω^{-1}:

$$\langle\text{out}|\,a_\pm(\omega\hat{x})\hat{S}\,|\text{in}\rangle=\omega^{-1}c_\pm^{(-1)}(\hat{x})+c_\pm^{(0)}(\hat{x})+\omega c_\pm^{(1)}(\hat{x})+\cdots\,. \tag{1.36}$$

One can check using Feynman-diagrammatic calculations (and we shall do this in the next section) that this is indeed the case.

We can rewrite the equation (1.36) as follows:

$$\langle\text{out}|\,a_\pm(\omega\hat{x})\hat{S}\,|\text{in}\rangle=\left(\omega^{-1}S_\pm^{(0)}(\hat{x})+S_\pm^{(1)}(\hat{x})+\omega S_\pm^{(2)}(\hat{x})+\cdots\right)\langle\text{out}|\,\hat{S}\,|\text{in}\rangle\,, \tag{1.37}$$

where $S_\pm^{(k)}$ are differential operators acting on the original amplitude $\langle\text{out}|\,\hat{S}\,|\text{in}\rangle$. It turns out that the behavior of the amplitude at low energies $\omega\to 0$ is very interesting. We refer to a low-energy photon as "soft" and will henceforth call equation (1.37) the "soft expansion." In QED, the first two coefficients in expansion (1.37) are *universal* that is, they maintain the same form regardless of the matter fields to which the electromagnetic field is coupled. Moreover, we will demonstrate that "soft factors" $S_\pm^{(0)}$, $S_\pm^{(1)}$ are related to conservation laws of the theory.

The expression (1.35) is also critical from an experimental point of view. The waveform $F^{(0)}_{uz}$ is measurable (we will discuss this in more detail in section 5). If we can quickly calculate Laurent coefficients $c^{(k)}$ for small integers k, we know about low-energy radiation produced in a given scattering process. Notably, Feynman-diagrammatic calculations are often much more efficient than standard methods for solving Maxwell's equations with specific sources.

Importantly, the above analysis can be extended by replacing photons with spin-2 particles - i.e., gravitons. In this case, we will soon be able to precisely measure waveforms of gravitational waves produced in scatterings of some stars or black holes in new gravitational wave detectors like LISA [8–10]. This presents an exciting opportunity to test Einstein's theory of general relativity in a novel regime!

1.5 Remark: Newman-Penrose coefficients

In this section we will briefly describe a more elegant, geometric definition of radiative electromagnetic modes.

Let us first take a null tetrad, that is, a frame field $(l, n, m, \bar{m})$ such that the only nonzero contractions w.r.t. a given metric $g_{\mu\nu}$ are:

$$g_{\mu\nu} l^\mu n^\nu = -1, \quad g_{\mu\nu} m^\mu \bar{m}^\nu = 1, \tag{1.38}$$

in particular, $l_\mu l^\mu = 0 = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu$.

We say that $(l, n, m, \bar{m})$ is adapted to a given null geodesic congruence if we can write l as $l = \partial/\partial u$, where u is an affine parameter of the null geodesics. Then any other null tetrad adapted to this congruence has the form:

$$(l, \hat{n}, \hat{m}, \hat{\bar{m}}), \tag{1.39}$$

where:

$$\hat{n} = n + f(x)\bar{m} + \bar{f}(x)m + \frac{1}{2}|f(x)|^2 l, \tag{1.40}$$

$$\hat{m} = e^{i\chi(x)}(m + f(x)l), \tag{1.41}$$

with functions $\chi(x) \in \mathbb{R}$, $f(x) \in \mathbb{C}$. To verify this we need to simply check that scalar products of $l, \hat{n}, \hat{m}, \hat{\bar{m}}$ recover (1.38).

An example of a null frame field is $(\partial_u, \partial_v, \sqrt{q^{z\bar{z}}}\partial_z, \sqrt{q^{z\bar{z}}}\partial_{\bar{z}})$, where $v = u + 2r$. It is adapted to a null geodesic congruence forming $\mathcal{I}^+$. Given $(l, n, m, \bar{m})$, we can define *Newman-Penrose coefficients*:

$$\Phi_0 = F_{\mu\nu} l^\mu \bar{m}^\nu = \frac{1}{r}\Phi_0^0 + \frac{1}{r^2}\Phi_0^1 + \cdots, \tag{1.42}$$

$$\Phi_1 = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + m^\mu \bar{m}^\nu) = \frac{1}{r^2}\Phi_1^0 + \frac{1}{r^3}\Phi_1^1 + \cdots, \tag{1.43}$$

$$\Phi_2 = F_{\mu\nu} m^\mu n^\nu = \frac{1}{r^3}\Phi_2^0 + \frac{1}{r^4}\Phi_2^1 + \cdots. \tag{1.44}$$

This is a more geometric way of writing fall-off conditions (1.27), (1.25). One can show that Φ_0 does not depend on the choice of the null tetrad adapted to a given null geodesic congruence, that is, it does not change under transformations $n \mapsto \hat{n}$, $m \mapsto \hat{m}$, with $\hat{n}$, $\hat{m}$ given by equations (1.40) and (1.41). If $\Phi_0 = 0$, then Φ_1 is invariant under $(n, m) \mapsto (\hat{n}, \hat{m})$, and if both Φ_0 and Φ_1 are zero, $\Phi_0 = \Phi_1 = 0$, then Φ_2 is invariant under $(n, m) \mapsto (\hat{n}, \hat{m})$.

Similar conditions imposed on gravitational field are usually referred to as "peeling property" of the Weyl Tensor [11, 12].

2 Soft Theorems

2.1 Soft photon theorem

Consider an $(N + 1)$-point amplitude in U(1) gauge theory coupled to certain matter fields, $A_{N+1}(1, 2, \ldots, N; s)$, where $(N + 1)$-st particle (called "s") is a photon with frequency ω, 4-momentum $q = \omega\hat{q}$, and polarization $\varepsilon_\alpha(\hat{q})$. The *soft photon theorem* [13, 14] states that in the limit of vanishing energy of the photon, $\omega \to 0$, the amplitude A_{N+1} behaves as:

$$A_{N+1} \overset{\omega\to 0}{\sim} \left(\omega^{-1} S_\alpha^{(0)}(\hat{q}) + S_\alpha^{(1)}(\hat{q}) + \mathcal{O}(\omega)\right) A_N(1, 2, \ldots, N), \tag{2.1}$$

with universal Laurent coefficients:

$$S_\alpha^{(0)}(\hat{q}) = \sum_{a=1}^{N} Q_a \eta_a \frac{(p_a \cdot \varepsilon_\alpha)}{p_a \cdot \hat{q}}, \tag{2.2}$$

$$S_\alpha^{(1)}(\hat{q}) = \frac{1}{2} \sum_{a=1}^{N} Q_a \eta_a \frac{1}{p_a \cdot \hat{q}} \hat{q}_\mu \varepsilon_{\alpha\nu} J_a^{\mu\nu}, \tag{2.3}$$

where $\eta_a = +1$ if the particle a is outgoing and $\eta_a = -1$ if the particle a is incoming. $J_a^{\mu\nu} = L_a^{\mu\nu} + S_a^{\mu\nu}$ is the angular momentum operator corresponding to the a'th particle. Orbital angular momentum operator in momentum space is:

$$L_{\mu\nu}^{(a)} = p_{a\mu} \frac{\partial}{\partial p_a^\nu} - p_{a\nu} \frac{\partial}{\partial p_a^\mu}. \tag{2.4}$$

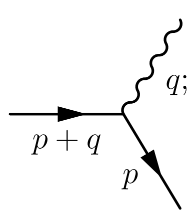
For a beautiful interpretation of soft theorems as Ward identities, see [5].

2.2 Argument for the leading soft photon theorem

Consider scalar electrodynamics:

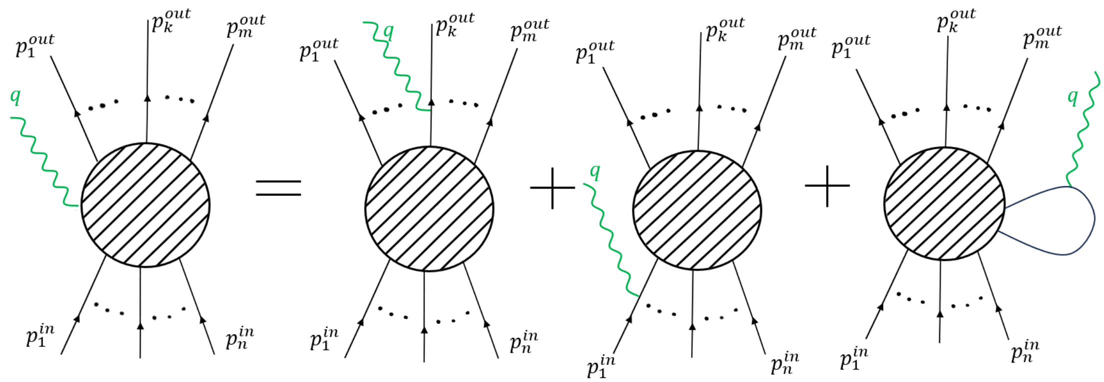
$$\mathcal{L}_{SQED} = -(D_\mu \Phi^\dagger)(D^\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.5}$$

where $D_\mu \Phi = \partial_\mu \Phi - ieA_\mu \Phi$. From this Lagrangian we can read-off the Feynman rule for 3-point vertex:



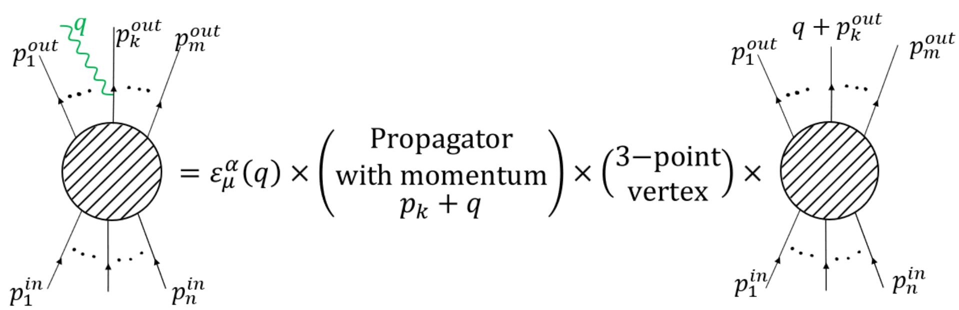
$$= e(2p^\mu + q^\mu) \tag{2.6}$$

Consider a scattering process with n incoming charged scalar particles, m outgoing charged scalars and an outgoing photon with 4-momentum momentum $q^\mu = \omega \hat{q}^\mu$, with vanishing frequency $\omega \to 0$. Let us denote the corresponding scattering amplitude as $A_{n+m+1} = A_{n+m+1}(\omega\hat{q}; p_1^{in}, \ldots, p_n^{in}, p_1^{out}, \cdots, p_m^{out})$. The amplitude may also contain external photon legs, but, since they are not important for our end result, we will not write them explicitly. A_{n+m+1} can be written as a sum of terms, when the soft photon is attached to an external line and to an internal line:



(2.7)

Let's focus on the first diagram, on the right hand side. It is almost the same as the diagram without the amplitude without the soft photon (with the k'th external momentum p_k^{out} replaced by $p_k^{out} + q$):



(2.8)

Scalar propagator carrying momentum $p_k^{out} + q$, with p_k^{out} and q on-shell can be written as:

$$\frac{-i}{(p_k^{out} + q)^2 + m^2} = \frac{-i}{2p_k^{out} \cdot q} \sim \mathcal{O}(\omega^{-1}) \tag{2.9}$$

If the k'th outgoing particle has charge Q_a^{out}, then the additional vertex factor is simply $iQ_a^{out}(2p^\mu + q^\mu)$. Since $q^\mu \varepsilon_\mu^\alpha(q) = 0$, the right-hand side of the above equation can be written as:

$$\sum_{a=1}^{m} Q_a^{out} \frac{p_a^{out} \cdot \varepsilon^\alpha(q)}{p_a^{out} \cdot q} A_{n+m}(p_1^{in}, \cdots, p_a^{out} + q, \cdots p_m^{out}). \tag{2.10}$$

By crossing symmetry, an ingoing scalar particle with momentum p and charge Q_a is equivalent to an outgoing scalar particle with momentum $-p$ and charge $-Q_a$. Therefore, the second diagram on the right-hand side of (2.7) contributes:

$$-\sum_{a=1}^{n} Q_a^{in} \frac{p_a^{in} \cdot \varepsilon^\alpha(q)}{p_a^{in} \cdot q} A_{n+m}(p_1^{in}, \cdots, p_a^{in} - q, \cdots p_m^{out}). \tag{2.11}$$

Both (2.10) and (2.11) diverge as ω^{-1} at $\omega \to 0$.

Note that generically internal momenta in a tree-level are not on shell, so

$$\frac{-i}{(p_{\text{internal}} + q)^2 + m^2} \xrightarrow{\omega \to 0} \frac{-i}{p_{\text{internal}}^2 + m^2} \sim \mathcal{O}(1). \tag{2.12}$$

Therefore, the third diagram on the right-hand side of (2.7) is regular in the limit $\omega \to 0$. It takes the form:

$$\varepsilon_\mu^\alpha(\hat{q}) \cdot N^\mu(\omega, \hat{q}), \tag{2.13}$$

with

$$N_\mu(\omega, \hat{q}) = N_\mu^{(0)}(\hat{q}) + \omega N_\mu^{(1)}(\hat{q}) + \omega^2 N_\mu^{(2)}(\hat{q}) + \cdots . \tag{2.14}$$

Summarizing, the full amplitude $A_{n+m+1}(\omega\hat{q}; p_1^{in}, \ldots, p_m^{out})$ can be written as:

$$A_{n+m+1}(\omega\hat{q}; p_1^{in}, \ldots, p_m^{out}) = \sum_{a \in \text{in} \cup \text{out}} Q_a \eta_a \frac{p_a \cdot \varepsilon^\alpha}{p_a \cdot q} A_{n+m}(p_1^{\text{in}}, \ldots, p_a + \eta_a q, \cdots, p_m^{out}) + \varepsilon_\mu^\alpha N^\mu, \tag{2.15}$$

with index a running over both initial and final particles. We also defined $\eta_a = +1$ for outgoing particle, and $\eta_a = -1$ for incoming particles. At the leading order in ω we obtain:

$$A_{n+m+1}(\omega\hat{q}; p_1^{in}, \ldots, p_m^{out}) = \frac{1}{\omega} \sum_{a \in \text{in} \cup \text{out}} Q_a \eta_a \frac{p_a \cdot \varepsilon^\alpha(\hat{q})}{p_a \cdot \hat{q}} A_{n+m}(p_1^{in}, \ldots, p_m^{out}) + \mathcal{O}(\omega^0), \tag{2.16}$$

in agreement with (2.2).

2.3 Subleading soft theorem from Ward identity and charge conservation

In this section we will determine the form of the subleading soft factor, $S_\alpha^{(1)}$. For notational simplicity, for now-on we shall restrict to scattering particles involving only outgoing particles (we will put $n = 0$ and denote $p_a^{out} = p_a$). With crossing symmetry, our results can be easily generalized to include incoming particles too.

Recall that, because of U(1)-symmetry, amplitude $A_{m+1} = \varepsilon_\mu^\alpha A_{m+1}^\mu$ must satisfy Ward identity, i.e. it must vanish if we replace $\varepsilon_\alpha^\mu(\hat{q})$ with q^μ:

$$0 = q_\mu A_{m+1}^\mu = \sum_a Q_a A_m(p_1, \ldots, p_a + q, \ldots, p_m) + q_\mu N^\mu. \tag{2.17}$$

where in the last equality we used (2.15) with ε_α^μ replaced by q^μ. At the leading order in ω, i.e. $\mathcal{O}(\omega^0)$, we obtain:

$$0 = \hat{q}_\mu S_\alpha^{(0)\mu}(\hat{q}) A_m = \sum_a Q_a A_m(p_1, \ldots, p_m). \tag{2.18}$$

Including incoming particles gives us:

$$\sum_{a\in\text{in}\cup\text{out}} Q_a\eta_a = 0 \;\Leftrightarrow\; \sum_{a\in\text{in}} Q_a^{in} = \sum_{a\in\text{out}} Q_a^{out}. \tag{2.19}$$

This is nothing but charge conservation law. We learn that soft theorems together with Ward identities imply conservation laws extends to non-Abelian gauge theories and gravity. Notably, in the case of gravity, leading soft theorem implies 4-momentum conservation, and subleading soft theorem implies conservation of angular momentum [15].

Let us now return to (2.15):

$$A_{m+1} = \sum_{a=1}^{m} Q_a \frac{p_a\cdot\varepsilon^\alpha}{p_a\cdot q} A_m(p_1,\ldots,p_a+q,\ldots,p_m) + \varepsilon^\mu_\alpha N_\mu(\omega,\hat{q},p_1,\ldots,p_m). \tag{2.20}$$

Extracting the $\mathcal{O}(\omega^0)$-contribution arising from the above term, and then replacing ε^μ_α with q^μ gives us:

$$\sum_{a=1}^{m} Q_a\hat{q}^\mu \frac{\partial}{\partial p_a^\mu} A_m(p_1,\ldots,p_a,\ldots,p_m) + \hat{q}^\mu N_\mu^{(0)}(\omega,\hat{q},p_1,\ldots,p_m) = 0. \tag{2.21}$$

This equation must hold for an arbitrary $\hat{q}^\mu$, and therefore:

$$N_\mu^{(0)}(\omega,\hat{q},p_1,\ldots,p_m) = -\sum_{a=1}^{m} Q_a \frac{\partial}{\partial p_a^\mu} A_m(p_1,\ldots,p_a,\ldots,p_m). \tag{2.22}$$

Substituting this equation back into (2.20), we obtain:

$$\begin{aligned} A_{m+1} &= \mathcal{O}(\omega^{-1}) + \sum_{a=1}^{m} Q_a \left(\frac{p_{a\mu}\varepsilon^\mu_\alpha}{p_a\cdot\hat{q}} \hat{q}^\nu \frac{\partial}{\partial p_a^\nu} - \varepsilon^\nu_\alpha \frac{\partial}{\partial p_a^\nu} \right) A_m + \mathcal{O}(\omega) \\ &= \mathcal{O}(\omega^{-1}) + \sum_{a=1}^{m} \frac{Q_a}{p_a\cdot\hat{q}} \hat{q}^\mu\varepsilon^\nu_\alpha \left(p_{a\mu}\frac{\partial}{\partial p_a^\nu} - p_{a\nu}\frac{\partial}{\partial p_a^\mu} \right) A_m + \mathcal{O}(\omega), \end{aligned} \tag{2.23}$$

recovering the subleading soft factor (2.2).

2.4 Remark: Subleading soft factor receive loop corrections

Since loop corrections do not violate charge conservation and Ward identities, the leading soft factor cannot receive corrections from loop diagrams. Factor

$$\sum_a Q_a\eta_a \frac{p_a\cdot\varepsilon_\alpha}{p_a\cdot q}, \tag{2.24}$$

is the only Lorentz-invariant expression of order $\mathcal{O}(\omega^{-1})$ that can be constructed from p_a, q, ε_α and Q_a which vanishes when ε_α is replaced with q and charge conservation is assumed. Hence, it must be protected from loop corrections.

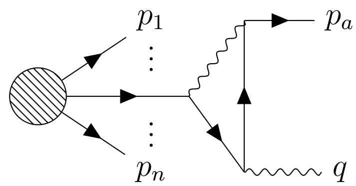
However, in [16] authors argued that the subleading soft photon theorem is modified at 1-loop order:

$$S_\alpha^{(1)\text{1-loop}} = \sum_a \frac{Q_a \eta_a}{p_a \cdot q_s} \varepsilon^\mu_\alpha q^\nu \left(p_{a\mu} \frac{\partial}{\partial p_a^\nu} - p_{a\nu} \frac{\partial}{\partial p_a^\mu} \right) (K^{\text{reg.}} A_m), \tag{2.25}$$

with

$$K^{\text{reg}} = \frac{i}{2} \sum_{a \neq b} q_a q_b \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 - i\epsilon} \frac{(2p_a - l)\cdot(2p_b + l)}{(2p_a \cdot l - l^2 + i\epsilon)(2p_b \cdot l + l^2 - i\epsilon)}. \tag{2.26}$$

This expression arises from loop corrections to the 3-point vertex to which the soft photon line is attached:



(2.27)

Moreover, it turns out that the analysis of loop corrections to soft theorems is even more complicated. In the seminal work [17], Sahoo and Sen have shown that the expansion of QED (or in gravity) amplitudes in small frequency of a photon (or gravity) contains logarithmic terms.

3 Is QED S-matrix trivial?

Accelerating charged particles are known to emit radiation. Somewhat unexpectedly, this simple statement has a surprising consequences for QED S-matrix. It implies that any non-trivial scattering event of charged particles with no radiation both at the beginning and in the end of an experiment is not possible! In QFT this means that, given an initial state of n charged particles (without photons) $|p_1, \ldots, p_n\rangle$ and final state of n' charged particles with different momenta (and without photons) $|p'_1, \ldots, p'_{n'}\rangle$, the following S-matrix element vanishes:

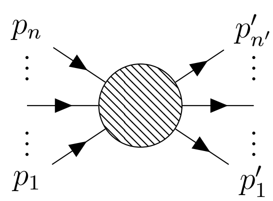
$$\langle p'_1, \ldots, p'_{n'} | \hat{S} | p_1, \ldots, p_n \rangle = 0. \tag{3.1}$$

This result was firstly derived in [18], but we shall follow here the presentation of [19].

Consider QED (coupled to some matter fields) with IR cutoff λ, i.e. assume that 3-momenta $\vec{q}$ have norm $|\vec{q}| \geq \lambda$. Let us call photons with momentum $\vec{q}$ s.t. $\Lambda \geq |\vec{q}| \geq \lambda$ soft.

Denote the n-point amplitude in the theory with cutoff λ as $\mathcal{A}_n^{(\Lambda)}$:

$$A_n^{(\Lambda)} =$$

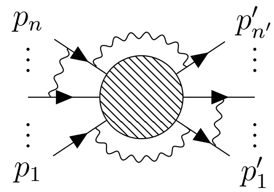


(3.2)

Similarly, $A_n^{(\Lambda)}$ is an amplitude in the theory with cutoff Λ (meaning that all lines - both external and internal) carry 3-momentum greater than Λ.

For small Λ, λ (compared to some interesting energy scale, like a mass of a matter particle) comes from $A_n^{(\Lambda)}$ with soft particles attached to external legs of $A_n^{(\Lambda)}$ since soft theorems indicate that they are of order λ^{-1}:

$$A_n^{(\lambda)} \approx \sum_{\text{soft exchanges}}$$



$$, \tag{3.3}$$

where the sum runs over the photon lines carrying soft momenta q, $\lambda < |\vec{q}| < \Lambda$. The blob denotes the amplitude $A_n^{(\Lambda)}$. In (3.3), photon lines attached to the external legs carry soft momenta $\Lambda \geq |\vec{q}| \geq \lambda$. A diagram with N soft photon loops comes with a symmetry factor:

$$2^N N!, \tag{3.4}$$

where 2^N corresponds to the changes of the orientation of each line and $N!$ corresponds to permutations of the soft lines.

Using soft theorems, we can rewrite (3.3) as:

$$A_n^{(\lambda)} \approx \sum_{N=0}^{\infty} \frac{1}{N!} 2^{-N} \left(\sum_{\alpha=\pm} \int_{\lambda \leq |\vec{q}| \leq \Lambda} \frac{d^3 q}{(2\pi)^3} S_\alpha^{(0)}(q) \frac{-i}{q^2 - i\epsilon} S_\alpha^{(0)}(-q) \right)^N A_n^{(\Lambda)}, \tag{3.5}$$

where the leading soft factor is:

$$S_\alpha^{(0)}(q) = \sum_{a \in \text{in} \cup \text{out}} Q_a \eta_a \frac{p_a \cdot \varepsilon_\alpha(q)}{p_a \cdot q - i\epsilon\eta_a}. \tag{3.6}$$

Since $A_n^{(\Lambda)}$ is independent of N, the sum over N gives an exponential factor:

$$\begin{aligned} A_n^{(\lambda)} &\approx \sum_{N=0}^{\infty} \frac{1}{N!} \left[\frac{-i}{2} \sum_{a,b} \int_{\mathbb{R}} \frac{dq^0}{2\pi} \int_{\Lambda \geq |\vec{q}| \geq \lambda} \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^2 Q_a Q_b \eta_a \eta_b (p_a^\mu \Pi_{\mu\nu}(q) p_b^\nu)}{(p_a \cdot q - i\epsilon\eta_a)(-p_b \cdot q - i\epsilon\eta_b)(q^2 - i\epsilon)} \right]^N A_n^{(\Lambda)} \\ &= \exp\left[\frac{1}{2} \sum_{a,b} Q_a Q_b \eta_a \eta_b I_{ab} \right] A_n^{(\Lambda)}, \end{aligned} \tag{3.7}$$

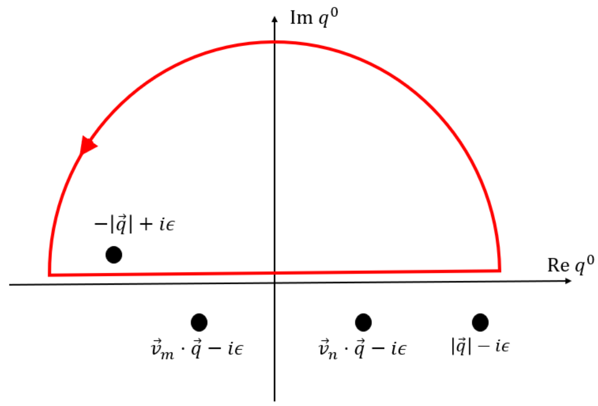
where we defined:

$$\begin{aligned} I_{ab} &= \int \frac{dq^0}{2\pi} \int_{\Lambda \geq |\vec{q}| \geq \lambda} \frac{d^3\vec{q}}{(2\pi)^3} \frac{i(p_a \cdot p_b)}{(p_a \cdot q - i\epsilon\eta_a)(p_b \cdot q + i\epsilon\eta_b)(q^2 - i\epsilon)} \\ &= \int \frac{dq^0}{2\pi} \int_{\Lambda \geq |\vec{q}| \geq \lambda} \frac{d^3\vec{q}}{(2\pi)^3} \frac{i p_a \cdot p_b}{(-(q^0)^2 + \vec{q}^2 - i\epsilon)(-q^0 p_a^0 + \vec{q} \cdot \vec{p}_a - i\epsilon\eta_a)(-q^0 p_b^0 + \vec{q} \cdot \vec{p}_b + i\epsilon\eta_b)}. \end{aligned} \tag{3.8}$$

In the above expression the integrand has poles at:

$$q^0 = \pm|\vec{q}| \mp i\epsilon, \quad q^0 = \frac{\vec{p}_a \cdot \vec{q}}{p_a^0} - i\epsilon\eta_a, \quad q^0 = \frac{\vec{p}_b \cdot \vec{q}}{p_b^0} + i\epsilon\eta_b. \tag{3.9}$$

For $\eta_a = -1$, $\eta_b = +1$ we can close the integration contour over q^0 in the upper-half complex plane, picking up only the contribution from the residue $q^0 = -|\vec{q}| + i\epsilon$:



This gives us:

$$\begin{aligned} I_{kl} &= i \int_{\Lambda \geq |\vec{q}| \geq \lambda} \frac{d^3\vec{q}}{(2\pi)^3} \frac{i(p_a \cdot p_b)}{2|\vec{q}|(|\vec{q}|p_a^0 + \vec{q}\cdot\vec{p}_a)(|\vec{q}|p_b^0 + \vec{q}\cdot\vec{p}_b)} \\ &= -\frac{1}{2(2\pi)^3} \frac{p_a \cdot p_b}{p_a^0\, p_b^0} \int_\lambda^\Lambda \frac{d\omega}{\omega} \int_{S^2} d^2\hat{q} \frac{1}{(1+\vec{v}_a\cdot\hat{q})(1+\vec{v}_b\cdot\hat{q})} \\ &= \frac{1}{2(2\pi)^3} \log\left(\frac{\Lambda}{\lambda}\right) J(\vec{v}_a, \vec{v}_b), \end{aligned} \tag{3.10}$$

where $d^2\hat{q}$ is the volume form on the unit round sphere S^2 and we defined $\vec{v}_a \equiv \vec{p}_a/p_a^0$, $\omega \equiv |\vec{q}|$. In the last line, we defined:

$$J(\vec{v}_a, \vec{v}_b) = -\frac{p_a \cdot p_b}{p_a^0\, p_b^0} \int_{S^2} d^2\hat{q} \frac{1}{(1+\vec{v}_a\cdot\hat{q})(1+\vec{v}_b\cdot\hat{q})} = (1-\vec{v}_a\cdot\vec{v}_b) \int_{S^2} d^2\hat{q} \frac{1}{(1+\vec{v}_a\cdot\hat{q})(1+\vec{v}_b\cdot\hat{q})} \tag{3.11}$$

If we assume that the external particles have non-zero mass, the integral over S^2 is finite since $|\vec{v}_a| < 1$ implies $(1 + \vec{v}_a \cdot \hat{q}) > 0$. For massless external particles, we may have collinear divergences [20].

For $\eta_a = 1 = \eta_b$ the integrand of (3.8) has two residues in the upper half of the complex q^0 plane (UHP). By closing the integration contour in the UHP, we pick contributions from poles $-|\vec{q}| + i\epsilon$ and $q^0 = \vec{v}_b \cdot \vec{q} + i\epsilon$. Contribution from $-|\vec{q}| + i\epsilon$ is the same as in the previously-discussed case (with $\eta_a = -1 = -\eta_b$).

Residue at $q^0 = \vec{v}_b \cdot \vec{q} + i\epsilon$ gives us:

$$\begin{aligned} &\frac{p_a \cdot p_b}{(2\pi)^3} \int_{\Lambda \geq |\vec{q}| \geq \lambda} \frac{d^3\vec{q}}{p_a^0 p_b^0} \frac{1}{\vec{q}^2(-\vec{v}_b\cdot\vec{q} + \vec{v}_a\cdot\vec{q} - i\epsilon)(1-(\vec{v}_b\cdot\hat{q})^2 - i\epsilon)} = \\ &\quad = -(1-\vec{v}_a\cdot\vec{v}_b) \log\left(\frac{\Lambda}{\lambda}\right) \int_{S^2} d^2\hat{q} \frac{1}{(1-(\hat{q}\cdot\vec{v}_b)^2)(\hat{q}\cdot(\vec{v}_a-\vec{v}_b) - i\epsilon)}. \end{aligned} \tag{3.12}$$

The factor $(1-(\hat{q}\cdot\vec{v}_b)^2)^{-1}$ is regular for massive external particles and gives only collinear singularities for massless external particles. The principal part of $(\hat{q}\cdot(\vec{v}_a-\vec{v}_b)-i\epsilon)^{-1}$ is odd, and hence, its integral vanishes. Therefore, the integral:

$$\int_{S^2} d^2\hat{q}\frac{1}{(1-(\hat{q}\cdot\vec{v}_b)^2)(\hat{q}\cdot(\vec{v}_a-\vec{v}_b)-i\epsilon)} = i\pi\int_{S^2} d^2\hat{q}\frac{\delta(\hat{q}\cdot(\vec{v}_a-\vec{v}_b))}{(1-(\hat{q}\cdot\vec{v}_b)^2)} \tag{3.13}$$

is purely imaginary.

Analysis of the two other cases, $\eta_k=-1=\eta_k$ and $\eta_k=1=-\eta_k$, is analogous. We conclude that the real part of I_{mn} is given by:

$$\operatorname{Re} I_{ab} = \frac{1}{2(2\pi)^3}\log\left(\frac{\Lambda}{\lambda}\right)J(\vec{v}_a,\vec{v}_b), \tag{3.14}$$

$$J(\vec{v}_a,\vec{v}_b) = (1-\vec{v}_a\cdot\vec{v}_b)\int_{S^2} d^2\hat{q}\frac{1}{(1+\vec{v}_a\cdot\hat{q})(1+\vec{v}_b\cdot\hat{q})}, \tag{3.15}$$

independently of the signs of η_a,η_b. In the rest frame of particle b, we have $\vec{v}_b=0$ and $\vec{v}_a=(0,0,\beta_{ab})$ is the relative velocity between particles a and b. Then, we can rewrite (3.15) as

$$J(\vec{v}_a,\vec{v}_b) = 2\pi\int_{-1}^{1} d(\cos\theta)(1+\beta_{ab}\cos\theta)^{-1} = \frac{2\pi}{\beta_{ab}}\log\left(\frac{1+\beta_{ab}}{1-\beta_{ab}}\right). \tag{3.16}$$

In an arbitrary frame, the relative velocity is given by:

$$\beta_{ab} = \sqrt{1-\frac{m_a^2m_b^2}{(p_a\cdot p_b)^2}}. \tag{3.17}$$

Equipped with the above formulae, we can rewrite (3.3) as:

$$|A_n^{(\lambda)}| = \exp\left[\frac{1}{2}\sum_{a,b}Q_aQ_b\eta_a\eta_b\frac{1}{2(2\pi)^3}\frac{2\pi}{\beta_{ab}}\log\left(\frac{1+\beta_{ab}}{1-\beta_{ab}}\right)\right]|A_n^{(\Lambda)}| = \left(\frac{\lambda}{\Lambda}\right)^B|A_n^{(\Lambda)}|, \tag{3.18}$$

where

$$B = -\frac{1}{(4\pi)^2}\sum_{a,b}Q_aQ_b\eta_a\eta_b\frac{1}{\beta_{ab}}\log\left(\frac{1+\beta_{ab}}{1-\beta_{ab}}\right). \tag{3.19}$$

It turns out that for any non-trivial scattering event we have $B>0$. For example, consider scattering of a test particle on an external potential (that is, $1\to 1$ scattering in a non-trivial background). Then $a,b=0,1$ with 0 (1) corresponding to initial (final) particle. Asume that $Q_0=Q_1=e$. Relative velocities are:

$$\beta_{00}=\beta_{11}=0,\quad \beta_{01}=\beta_{10}>0, \tag{3.20}$$

so that:

$$\begin{aligned} B &= -\frac{e^2}{(4\pi)^2}\cdot 2\eta_0\eta_1\log\left(\frac{1+\beta_{01}}{1-\beta_{01}}\right) - \frac{e^2}{(4\pi)^2}\cdot 2\lim_{\beta_{00}\to 0}\frac{1}{\beta_{00}}\log\left(\frac{1+\beta_{00}}{1-\beta_{00}}\right) = \\ &= \frac{2e^2}{(4\pi)^2}\left(\frac{1}{\beta_{01}}\log\left(\frac{1+\beta_{01}}{1-\beta_{01}}\right)-2\right). \end{aligned} \tag{3.21}$$

Function $\frac{1}{\beta_{01}}\log\left(\frac{1+\beta_{01}}{1-\beta_{01}}\right)$ is greater than 2 for all $\beta_{01} \in (0,1)$, and thus, $B \geq 0$.

Eq. (3.18) implies that in the limit $\lambda \to 0$ we have $|A_n^\lambda| \to 0$. For arbitrarily small Λ our approximation of soft exchanges with the leading soft factors becomes more and more accurate. Hence, we see that all non-trivial scattering processes have zero probability if we restrict external particles to have finite momenta $|\vec{q}| \geq \Lambda$. This is just a QFT way of saying that, since particles in a non-trivial scattering accelerate, they emit radiation which contains photons of arbitrarily small frequency - non-trivial scattering of charged particles without bremsstrahlung is impossible. We need to find a way to encode radiation in asymptotic states of the theory.

4 Faddeev-Kulish construction of asymptotic states

Note that in the previous section we used the standard QFT framework, and hence, we made an implicit assumption that the theory is free in the far past and future. However, this assumption is true only if the scattered particles do not interact if they are far-separated. But his assumption does not hold if the particles mediating interactions are massless (since in this case the interaction potential $V(r) \sim \frac{1}{r}$ lacks the fast-decaying exponential factor e^{-mr}). In particular, it is not valid in electrodynamics.

In this section we will show that by including the long-ranged electromagnetic interactions in asymptotic states resolves the problem of vanishing S-matrix from the previous section and forces us to include bremsstrahlung. The formalism presented in this section was originally developed in [21].

Consider interaction Hamiltonian of spinor QED:

$$V(t) = -\int_{t=\text{const.}} d^3\vec{x}\, A_\mu j^\mu = -e\int d^3\vec{x} A_\mu(t,\vec{x}) :\bar{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x}): . \tag{4.1}$$

We want to analyze $V(t)$ at $t \to \pm\infty$, where we suspect that V is small (although non-vanishing!). Hence, we approximate A_μ and ψ with free-theory expressions, namely:

$$A_\mu(x) = \int \widetilde{dk}\left(a_\mu(\vec{k})e^{ik\cdot x} + a_\mu^\dagger(\vec{k})e^{-ik\cdot x}\right), \tag{4.2}$$

$$\psi(x) = \sum_{s=\pm}\int \widetilde{dp}\left(b_s(\vec{p})u_s(\vec{p})e^{ip\cdot x} + d_s^\dagger(\vec{p})v_s(\vec{p})e^{-ip\cdot x}\right), \tag{4.3}$$

where $a_\mu(\vec{k}) = \sum_{\alpha=\pm}\varepsilon_\mu^\alpha(\hat{k})a_\alpha(\vec{k})$ and:

$$[a_\alpha(\vec{k}), a_\beta^\dagger(\vec{k}')] = \delta_{\alpha\beta}\tilde{\delta}(\vec{k}-\vec{k}'), \tag{4.4}$$

$$[b_s(\vec{p}), b_{s'}^\dagger(\vec{p}')] = \delta_{ss'}\tilde{\delta}(\vec{p}-\vec{p}') = [d_s(\vec{p}), d_{s'}^\dagger(\vec{p}')], \tag{4.5}$$

$$(\not{p}+m)u(\vec{p}) = 0 = (-\not{p}+m)v(\vec{p}), \tag{4.6}$$

where $\tilde{\delta}(\vec{k}-\vec{k}') = 2E_k(2\pi)^3\delta^3(\vec{k}-\vec{k}')$.

We expect that asymptotic dynamics of the our system is governed by the Hamiltonian:

$$H_{as} = H_0 + V(t), \tag{4.7}$$

where H_0 is the free theory Hamiltonian. The corresponding evolution operator, $U(t)$, satisfies:

$$i\frac{d}{dt}U(t) = (H_0 + V(t))U(t). \tag{4.8}$$

Substituting ansatz $U(t) = e^{-iH_0 t}Z(t)$, we obtain:

$$i\frac{d}{dt}Z(t) = V(t)Z(t). \tag{4.9}$$

Lemma:
At $t \to \pm\infty$ potential $V(t)$ takes the form:

$$V(t) \overset{t\to\pm\infty}{\sim}= V_{as}(t) = -e\int \widetilde{dk}\int \widetilde{dp}\frac{p^\mu}{E_p}\hat{N}(\vec{p})\left[a_\mu(\vec{k})e^{i(p\cdot k)t/E_p} + a^\dagger_\mu(\vec{k})e^{-i(p\cdot k)t/E_p}\right], \tag{4.10}$$

where the particle number operator $\hat{N}(\vec{p})$ is:

$$\hat{N}(\vec{p}) = \sum_{s=\pm}\left[b^\dagger_s(\vec{p})b_s(\vec{p}) - d^\dagger_s(\vec{p})d_s(\vec{p})\right], \tag{4.11}$$

and $E_p = \sqrt{\vec{p}^2 + m^2}$.

Proof:
Let's write V in terms of mode expansions:

$$\begin{aligned}V(t) = &- e\int d^3x \sum_{s,s'}\int \widetilde{dk}\widetilde{dp}\widetilde{dq}\Big[b^\dagger_s(\vec{p})b_{s'}(\vec{q})\bar{u}_s(p)\gamma^\mu u_{s'}(q)e^{i(-p+q)\cdot x}\\ &+ b^\dagger_s(\vec{p})d^\dagger_{s'}(\vec{q})\bar{u}_s(p)\gamma^\mu v_{s'}(q)e^{i(-p-q)\cdot x} - d^\dagger_{s'}(\vec{q})d_s(\vec{p})\bar{v}_s(p)\gamma^\mu v_{s'}(q)e^{i(p-q)\cdot x}\\ &- b_{s'}(\vec{q})b_s(\vec{p})\bar{v}_s(p)\gamma^\mu u_{s'}(q)e^{i(p+q)\cdot x}\Big]\left(a_\mu(\vec{k})e^{ik\cdot x} + a^\dagger_\mu(\vec{k})e^{-ik\cdot x}\right)\end{aligned} \tag{4.12}$$

The integral over $\vec{x}$ gives momentum-conserving delta functions, $(2\pi)^3\delta^{(3)}(\vec{p}\pm\vec{q}\pm\vec{k})$. Thus, we can easily integrate over q. After multiplying the two brackets, we get eight terms, each with time dependence of the type:

$$\exp\left[i\left(\pm q^0 \pm p^0 \pm |\vec{k}|\right)t\right],$$

where $q^0 = \sqrt{(\vec{p}+\vec{k})^2 + m^2}$. At early and late times ($t \to \pm\infty$) we can use the saddle-point approximation. Terms which depend on the sum $q^0 + p^0$ vanish, since, for every $\vec{k}$ and $\vec{p}$ kinematically $q^0 + p^0 \pm |\vec{k}| \neq 0$. For the other terms, we have a saddle at $|\vec{k}| \to 0$. This means that only soft photons are responsible for long-range interactions!

Take the terms proportional to $\exp\big(i(p^0 - q^0 + k^0)t\big)$. After integration over $\vec{x}$ and $\vec{q}$ we get:

$$-\frac{e}{2}\sum_{s,s'=\pm}\int \widetilde{dp}\,\widetilde{dk}\,\frac{\bar{u}_s(p)\gamma^\mu u_{s'}(p+k)}{\sqrt{(\vec{k}+\vec{p})^2+m^2}}b^\dagger_s(\vec{p})b_{s'}(\vec{p}+\vec{k})a^\dagger_\mu(\vec{k})e^{i(p^0-q^0+k^0)\cdot t}. \tag{4.13}$$

In the expansion in small $|\vec{k}|$ we need to keep track of the variables that depend only on the direction (and not the length) of $\vec{k}$, e.g. polarization $\varepsilon^\mu_\alpha(\vec{k})$. Using $q^0 \approx p^0 + (\vec{p}\cdot\vec{k})/p^0$, $ik^0t = ik^0p^0/p^0$ and the special case of Gordon identities, $\bar{u}_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu\delta_{ss'}$ we arrive at

$$-e\sum_{s,s'=\pm}\int\widetilde{dk}\int\widetilde{dp}\,\frac{p^\mu}{p^0}\,b^\dagger_s(\vec{p})b_s(\vec{p})a^\dagger_\mu(\vec{k})e^{-i(k\cdot p)\cdot t/p^0}. \tag{4.14}$$

In the same fashion we can manipulate the term proportional to $b^\dagger(p)b(q)a(k)$. The only difference is that the sign of $\vec{k}$ changes (everywhere except in $a_\mu(\vec{k})$):

$$-e\sum_{s,s'=\pm}\int\widetilde{dk}\int\widetilde{dp}\,\frac{p^\mu}{p^0}\,b^\dagger_s(\vec{p})b_s(\vec{p})a_\mu(-\vec{k})e^{-i(k\cdot p)\cdot t/p^0}. \tag{4.15}$$

One can similarly calculate expressions proportional to $d^\dagger_s(p)d_s(p)$. This completes the proof. ♣

Note that V_{as} can be written as:

$$V_{as}(t) = \int\widetilde{dk}\;J^\mu_{as}(\vec{k},t)\left[a^\dagger_\mu(-\vec{k}) + a_\mu(\vec{k})\right], \tag{4.16}$$

where

$$J^\mu_{as}(\vec{k},t) = -e\int\widetilde{dp}\,\frac{p^\mu}{p^0}e^{i\frac{p\cdot k}{p^0}t}\sum_{s=\pm}[b^\dagger_s(\vec{p})b_s(\vec{p}) - d^\dagger_s(\vec{p})d_s(\vec{p})]. \tag{4.17}$$

J^μ_{as} is a classical current of asymptotic particles moving with constant velocities $\vec{p}/E_{\vec{p}}$.

4.1 Asymptotic States

In this section we will describe how free theory states (which are asymptotic states of an interacting theory if the interactions are short-ranged) are modified in the presence of asymptotic potential $V_{as}(t)$. See Appendix A for a short review of the standard construction of asymptotic states and definition of S-matrix, when there is no long-range potential.

The previous analysis tells us that QED is not free at $t \to \pm\infty$. Asymptotic states evolve under Hamiltonian:

$$H = H_0 + V_{as}(t). \tag{4.18}$$

The corresponding evolution operator, $U(t,t_0)$, is given by:

$$i\frac{d}{dt}U(t,t_0) = (H_0 + V_{as}(t))U(t,t_0), \quad U(t_0,t_0) = \mathbb{1}. \tag{4.19}$$

By writing $U(t,t_0) = e^{-iH_0(t-t_0)}Z(t,t_0)$, we can reduce the above equation to:

$$i\frac{d}{dt}Z(t,t_0) = e^{iH_0\Delta t}Ve^{-iH_0\Delta t}Z(t,t_0), \quad Z(t_0,t_0) = \mathbb{1}, \tag{4.20}$$

where $\Delta t = t - t_0$ and $V(t) = -e\int d^3x\bar{\psi}(t,\vec{x})\not{A}(t,\vec{x})\psi(t,\vec{x})$. $\psi(t,\vec{x}) = e^{iH_0t}\psi(\vec{x})e^{-iH_0t}$, $A_\mu(t,\vec{x}) = e^{iH_0t}A_\mu(\vec{x})e^{-iH_0t}$ are free fields, with mode expansions (4.2).

According to the above lemma, $V(t)$ approaches $V_{as}(t)$ at $t \to \pm\infty$. Let $Z_{as}(t,t_0)$ satisfy $i\frac{\mathrm{d}}{\mathrm{d}t}Z_{as}(t,t_0) = V_{as}(\Delta t)Z_{as}(t,t_0)$. Then, we can define "in" and "out" states $|\Psi^{\pm}_{\alpha}\rangle$ as:

$$e^{-iH(t-t_0)}\int \mathrm{d}\alpha\, f(\alpha)\left|\psi^{\pm}_{\alpha}\right\rangle \overset{t\to\pm\infty}{\sim} e^{-iH_0(t-t_0)}Z_{as}(t,t_0)\int \mathrm{d}\alpha\, f(\alpha)\left|\Phi_{\alpha}\right\rangle . \tag{4.21}$$

Equivalently, we can write

$$\left|\Psi^{\pm}_{\alpha}\right\rangle = \Omega(\pm\infty)\left|\Phi_{\alpha}\right\rangle, \quad \text{with} \quad \Omega(t,t_0) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}Z_{as}(t,t_0). \tag{4.22}$$

remembering that the equality is in the distributional sense - it is true only when smeared with $f(\alpha)$.

With this modification of asymptotic states, S-matrix elements are given by:

$$\begin{aligned} S_{\beta\alpha} &= \langle\Psi^{+}_{\beta}|\Psi^{-}_{\alpha}\rangle \\ &= \lim_{t\to+\infty, t'\to-\infty} \langle\Phi_{\beta}|\, Z^{\dagger}_{as}(t,t_0)e^{iH_0(t-t_0)}e^{-iH(t-t_0)}e^{iH_0(t_0-t')}Z_{as}(t',t_0)\left|\Phi_{\alpha}\right\rangle \\ &= \left\langle\Phi^{Dr+}_{\beta}\right|\hat{S}\left|\Phi^{Dr,-}_{\alpha}\right\rangle, \end{aligned} \tag{4.23}$$

where $\hat{S}$ is the standard $\hat{S}$-operator, defined perturbatively via the Dyson series:

$$\hat{S} = \lim_{t\to+\infty, t'\to-\infty} e^{iH_0(t-t')}e^{-iH(t-t')}e^{-iH_0t'} = \mathsf{T}\exp\left(\int_{-\infty}^{+\infty} dt\, V(t)\right), \tag{4.24}$$

and the "dressed states" $\left|\Phi^{Dr,\pm}_{\alpha}\right\rangle$ are:

$$\left|\Phi^{Dr,\pm}_{\alpha}\right\rangle = \lim_{t\to\pm\infty} Z_{as}(t,t_0)\left|\Phi_{\alpha}\right\rangle \equiv Z^{\pm}_{as}\left|\Phi_{\alpha}\right\rangle. \tag{4.25}$$

4.2 Calculation of the dressing operators

Recall that operator $Z_{as}(t,t_0)$ satisfies:

$$i\frac{\mathrm{d}}{\mathrm{d}t}Z_{as}(t,t_0) = V_{as}(t-t_0)Z_{as}(t,t_0), \tag{4.26}$$

with

$$V_{as}(t,t_0) = -e\int \widetilde{dk}\int \widetilde{dp}\frac{p^{\mu}}{E_p}\hat{N}(\vec{p})\left(a_{\mu}(\vec{k})e^{i(p\cdot k)(t-t_0)/E_p} + \text{h.c.}\right). \tag{4.27}$$

Since $V_{as}(t,t_0)$ is linear in photon creation and annihilation operators, and the particle number operators $\hat{N}(\vec{p})$ commute with each other, $[N(\vec{p}), N(\vec{p}')] = 0$, we have:

$$[V_{as}(t), [V_{as}(t'), V_{as}(t'')]] = 0. \tag{4.28}$$

This allows us to find an explicit solution to the equation (4.26) using the Magnus expansion [22], namely:

$$Z_{as}(t,t_0) = \mathsf{T}\exp\left(-i\int_{t_0}^{t} dt' V_{as}(t')\right) = \exp\left(\sum_{k=1}^{\infty} A_k(t)\right), \tag{4.29}$$

where

$$A_1(t) = \int_{t_0}^{t} dt' \, V_{as}(t'), \tag{4.30}$$

$$A_2(t) = -\frac{i}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, [V_{as}(t_1), V_{as}(t_2)], \tag{4.31}$$

$$A_3(t) = -\frac{1}{4} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \, [V_{as}(t_1), [V_{as}(t_2), V_{as}(t_3)]], \tag{4.32}$$

with obvious generalization to $A_{4,5,...}$. Because of (4.28), the sum $\sum_k A_k$ truncates at $k = 2$, and we can compactly express $Z_{as}(t)$ as

$$Z_{as}(t) = \exp\left[-R(t) + R(t_0) + i\Phi(t, t_0)\right], \tag{4.33}$$

where

$$\begin{aligned} R(t) &= -ie \int^{t} ds \int \widetilde{dk} \, \widetilde{dp} \, \frac{p^\mu}{E_p} N(\vec{p}) \left(a_\mu(\vec{k}) e^{i(p\cdot k)s/E_p} + \text{h.c.} \right) \\ &= -e \int \widetilde{dk} \int \widetilde{dp} \, \frac{p^\mu}{p \cdot k} N(\vec{p}) \left(a_\mu(\vec{k}) e^{i(p\cdot k)t/E_p} - a^\dagger_\mu(\vec{k}) e^{-i(p\cdot k)t/E_p} \right). \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} \Phi(t, t_0) &= \frac{i}{2} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \, [V_{as}(s_1), V_{as}(s_2)] \\ &= \frac{ie^2}{2} \int \widetilde{dk} \, \widetilde{dp_1} \, \widetilde{dp_2} \, \hat{N}(\vec{p}_1) \hat{N}(\vec{p}_2) \frac{\sum_\alpha (p_1 \cdot \epsilon_\alpha)(p_2 \cdot \epsilon^*_\alpha)}{(p_1 \cdot k)(p_2 \cdot k)} \times \\ &\quad \times \left(\left(e^{i(p_1\cdot k)t/E_1} - e^{i(p_1\cdot k)t_0/E_1} \right) \left(e^{i(p_2\cdot k)t/E_2} - e^{i(p_2\cdot k)t_0/E_2} \right) - \text{h.c.} \right). \end{aligned} \tag{4.35}$$

Since $\Phi(t, t_0)$ commutes will all other operators, it contributes only an overall phase factor to scattering amplitudes. It is not interesting and we shall suppress it in the following analyses - we define $Z_{as}(t) = \exp[-R(t) + R(t_0)]$.

As noted in [23], one should regularize the above expressions by substituting:

$$p \cdot k \to p \cdot k - i\epsilon \, \text{sgn}(t) \tag{4.36}$$

where $\text{sgn}(t) = t/|t|$. Then, in the limit $t \to \pm\infty$, $R(t)$ vanishes, and:

$$\begin{aligned} \Phi(t, t_0) &= \frac{i}{2} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \, [V_{as}(s_1), V_{as}(s_2)] \\ &= \frac{ie^2}{2} \int \widetilde{dk} \, \widetilde{dp_1} \, \widetilde{dp_2} \, \hat{N}(\vec{p}_1) \hat{N}(\vec{p}_2) \frac{\sum_\alpha (p_1 \cdot \epsilon_\alpha)(p_2 \cdot \epsilon^*_\alpha)}{(p_1 \cdot k)(p_2 \cdot k)} \left(e^{it_0 k\cdot(p_1/E_1 + p_2/E_2)} - \text{h.c.} \right). \end{aligned} \tag{4.37}$$

Choosing $t_0 = 0$ we obtain: $\Phi(\pm\infty, 0) = 0$ and:

$$R(t_0 = 0) = e \int \widetilde{dk} \int \widetilde{dp} \, \frac{p^\mu}{p \cdot k} N(\vec{p}) \left(a^\dagger_\mu(\vec{k}) - a_\mu(\vec{k}) \right) \equiv R. \tag{4.38}$$

Then, the dressing operator of "in" and "out" states takes a simple form:

$$Z_{as}(t \to \pm\infty, t_0 = 0) = e^R, \tag{4.39}$$

so that

$$\left|\Phi_\alpha^{Dr,\pm}\right\rangle = \lim_{t\to\pm\infty} Z_{as}(t, t_0 = 0) \left|\Phi_\alpha\right\rangle = e^R \left|\Phi_\alpha\right\rangle. \tag{4.40}$$

Note that operator R is anti-hermitian, and hence, exponent e^R is unitary.

4.3 Cancellation of IR divergences

Recall that in Section 3 we argued that IR divergences caused by soft momenta running in loops between external legs of Feynman diagrams exponentiate, resulting in a trivial S-matrix of undressed charged particle states (it is equal to $\mathbb{1}$ on the Fock space of the undressed states). In this section we shall argue that Faddeev-Kulish dressings cancel these divergences. This implies that S-matrix of dressed states is not trivial. See [24] for a slightly different argument for cancellation of IR divergences.

Let us consider a scattering amplitude with n dressed charged particles, A_n. By crossing symmetry, we can without loss of generality restrict to scattering configurations without final particles. Thus, we write:

$$A_n = \langle 0| \hat{S} e^R \left|\text{in}\right\rangle, \tag{4.41}$$

where R is the dressing operator, and $\hat{S}$ is the S-operator (4.24). Let us mark an energy scale Λ such that all particles in the state $\left|\text{in}\right\rangle$ have momenta $\vec{q}$ greater than Λ, $|\vec{q}| \geq \Lambda$.

When acting on initial state of n particles with electric charges Q_a and a finite number of photons (with some polarizations α_{ph} and momenta $\vec{k}_{\text{ph}}$):

$$\left|\text{in}\right\rangle = \left(\prod_{a=1}^{n} b_a^\dagger(\vec{p}_a)\right)\left(\prod_{\text{photons}} a_{\alpha_{\text{ph}}}^\dagger(\vec{k}_{\text{ph}})\right)\left|0\right\rangle, \tag{4.42}$$

we can express R as:

$$R = e\sum_{a=1}^{n} Q_a \int \widetilde{dk} \frac{p_a^\mu}{(p_a\cdot k)} \hat{N}(\vec{p}_a)\left(a_\mu^\dagger(\vec{k}) - a_\mu(\vec{k})\right) \equiv \int \widetilde{dk}\, f^\mu(\vec{k})\left(a_\mu^\dagger(\vec{k}) - a_\mu(\vec{k})\right), \tag{4.43}$$

where:

$$f^\mu(\vec{k}) = e\sum_{a=1}^{n} Q_a \frac{p_a^\mu}{(p_a\cdot k)}. \tag{4.44}$$

Using the Baker-Campbell-Hausdorf formula, we can express e^R as:

$$e^R = e^{\int \widetilde{dk}\, f^\mu(\vec{k})\left(a_\mu^\dagger(\vec{k}) - a_\mu(\vec{k})\right)} = e^{\int \widetilde{dk} f^\mu(\vec{k}) a_\mu^\dagger(\vec{k})} e^{-\int \widetilde{dk} f^\mu(\vec{k}) a_\mu(\vec{k})} e^{-\frac{1}{2}\int \widetilde{dk} f(\vec{k})\cdot\Pi(\vec{k})\cdot f(\vec{k})}, \tag{4.45}$$

where:

$$f(\vec{k})\cdot\Pi(\vec{k})\cdot f(\vec{k}) = f_\mu(\vec{k}) f_\nu(\vec{k}) \Pi_{\mu\nu}(\vec{k}) = e^2 \sum_{a,b=1}^{n} Q_a Q_b \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}, \tag{4.46}$$

and projection operator:

$$\Pi^{\mu\nu}(\vec{k}) = \sum_{\alpha=\pm} \varepsilon^{\mu}_{\alpha}(\vec{k})\varepsilon^{\nu}_{\alpha}(\vec{k}). \tag{4.47}$$

This allows us to express A_n as:

$$A_n = e^{-\frac{e^2}{2}\sum_{a,b=1}^{n} Q_a Q_b \int \widetilde{dk} \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}} \langle 0|\, \hat{S}\, e^{\int \widetilde{dk} f^{\mu}(\vec{k}) a^{\dagger}_{\mu}(\vec{k})}\, e^{-\int \widetilde{dk} f^{\mu}(\vec{k}) a_{\mu}(\vec{k})} \,|\text{in}\rangle\,. \tag{4.48}$$

Note that the integral $\int \widetilde{dk}\, \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}$ is divergent at $\vec{k} \to 0$, and therefore, we introduce an IR cutoff $\lambda < \Lambda$, restricting the integration region to $|\vec{k}| \geq \lambda$ and modifying the S-operator $\hat{S}$ to $\hat{S}^{(\lambda)}$, such that it acts only on states with momenta $|\vec{k}| \geq \lambda$, and in the corresponding loop integrals one sums only over momenta $|\vec{q}| \geq \lambda$. We aim to take the limit $\lambda \to 0$ at the end of this calculation.

Next, commutation relation of operators $e^{-\int \widetilde{dk} f^{\mu}(\vec{k}) a_{\mu}(\vec{k})}$ and $a^{\dagger}_{\alpha}(\vec{q})$,

$$e^{-\int \widetilde{dk} f^{\mu}(\vec{k}) a_{\mu}(\vec{k})} a^{\dagger}_{\alpha}(\vec{q}) = \left(a^{\dagger}_{\alpha}(\vec{q}) - \varepsilon^{\mu}_{\alpha}(\vec{q}) f_{\mu}(\vec{q})\right) e^{-\int \widetilde{dk} f^{\mu}(\vec{k}) a_{\mu}(\vec{k})} \tag{4.49}$$

implies that

$$\begin{aligned} A_n =& e^{-\frac{e^2}{2}\sum_{a,b=1}^{n} Q_a Q_b \int_{|\vec{k}|\geq\lambda} \widetilde{dk} \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}} \langle 0|\, \hat{S}^{(\lambda)}\, e^{\int_{|\vec{k}|\geq\lambda} \widetilde{dk}\, f^{\mu}(\vec{k}) a^{\dagger}_{\mu}(\vec{k})} \times \\ & \times \prod_{a=1}^{n} b^{\dagger}_{a}(\vec{p}_a) \prod_{\text{photons}} \left(a^{\dagger}_{\alpha_{\text{ph}}}(\vec{k}_{\text{ph}}) - \varepsilon^{\mu}_{\alpha_{\text{ph}}}(\vec{k}_{\text{ph}}) f_{\mu}(\vec{k}_{\text{ph}})\right) |0\rangle\,. \end{aligned} \tag{4.50}$$

We obtain a sum of amplitudes with different numbers of external photons. Part of the dressing operator e^{R}, $e^{-\int_{\vec{k}\geq\lambda} \widetilde{dk} f^{\mu}(\vec{k}) a_{\mu}(\vec{k})}$, gives a finite contribution to A_n, and does not modify the state of the initial charged particles.

Let us now consider the operator:

$$\exp\left(\int_{|\vec{k}|\geq\lambda} \widetilde{dk}\, f^{\nu}(\vec{k}) a^{\dagger}_{\nu}(\vec{k})\right) = \sum_{N=1}^{\infty} \frac{1}{N!} \left(\int_{|\vec{k}|\geq\lambda} \widetilde{dk}\, f^{\nu}(\vec{k}) a^{\dagger}_{\nu}(\vec{k})\right)^{N} \tag{4.51}$$

The N'th term in the above expansion is equivalent to attaching N new external photons, with smeared polarizations $\sum_{\alpha=\pm}\int_{|\vec{k}|\geq\lambda} \widetilde{dk}\, f_{\mu}(\vec{k})\varepsilon^{\mu}_{\alpha}(\vec{k})$. Hence, at sufficiently small momenta $\vec{k}$, we can replace $a^{\dagger}_{\alpha}(\vec{k})$ with the leading soft factor,

$$S_{\alpha}(\vec{k}) = e\sum_{a=1}^{n} Q_a \frac{p_a \cdot \varepsilon_{\alpha}(\vec{k})}{p_a \cdot k}. \tag{4.52}$$

We can split the integral over $\vec{k}$ as:

$$\int_{|\vec{k}|\geq\lambda} \widetilde{dk}\, f^{\nu}(\vec{k}) a^{\dagger}_{\nu}(\vec{k}) = \left(\int_{\Lambda\geq\vec{k}\geq\lambda} + \int_{|\vec{k}|\geq\Lambda}\right) \widetilde{dk}\, f^{\nu}(\vec{k}) a^{\dagger}_{\nu}(\vec{k}) \tag{4.53}$$

For sufficiently small Λ, with a high accuracy we can approximate the integral over $|\vec{k}| \in (\lambda, \Lambda)$ by replacing $a^\dagger_\mu(\vec{k})$ with the leading soft factor (4.52). We approximate:

$$\exp\left(\int_{|\vec{k}|\geq\lambda} \widetilde{dk}\, f^\nu(\vec{k}) a^\dagger_\nu(\vec{k})\right) \simeq \exp\left(e\sum_{\alpha=\pm}\sum_{a=1}^{n} Q_a \int_{\Lambda\geq|\vec{k}|\geq\lambda} \widetilde{dk}\, \frac{(p_a\cdot\varepsilon_\alpha(\vec{k}))(f(\vec{k})\cdot\varepsilon_\alpha(\vec{k}))}{k\cdot p_a}\right) \times$$
$$\times \exp\left(\int_{|\vec{k}|\geq\Lambda} \widetilde{dk}\, f^\nu(\vec{k}) a^\dagger_\nu(\vec{k})\right) \tag{4.54}$$

Thus, A_n can be estimated as:

$$A_n = e^{-\frac{e^2}{2}\sum_{a,b=1}^n Q_a Q_b \int_{|\vec{k}|\geq\lambda} \widetilde{dk} \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}} e^{e\sum_{a=1}^n Q_a \int_{\Lambda\geq|\vec{k}|\geq\lambda} \widetilde{dk}\, \frac{p_a\cdot\Pi(\vec{k})\cdot f(\vec{k})}{k\cdot p_a}} A_{n,\Lambda,\lambda}, \tag{4.55}$$

where:

$$A_{n,\Lambda,\lambda} = \langle 0|\, \hat{S}^{(\lambda)}\, e^{\int_{|\vec{k}|\geq\Lambda} \widetilde{dk}\, f^\nu(\vec{k}) a^\dagger_\nu(\vec{k})} \prod_{a=1}^{n} b^\dagger_a(\vec{p}_a) \prod_{\text{photons}} \left(a^\dagger_{\alpha_{\rm ph}}(\vec{k}_{\rm ph}) - \varepsilon^\mu_{\alpha_{\rm ph}}(\vec{k}_{\rm ph}) f_\mu(\vec{k}_{\rm ph})\right)|0\rangle\,. \tag{4.56}$$

The argument from Section 3 implies that, for sufficiently small Λ and λ, we can approximate amplitude $A_{n,\Lambda,\lambda}$ as:

$$A_{n,\Lambda,\lambda} = \exp\left(-\frac{e^2}{2}\sum_{a,b} Q_a Q_b \int_{\Lambda\geq|\vec{k}|\geq\lambda} \widetilde{dk}\, \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}\right) A_{n,\Lambda,\Lambda} \tag{4.57}$$

with finite

$$A_{n,\Lambda,\Lambda} = \langle 0|\, \hat{S}^{(\Lambda)}\, e^{\int_{|\vec{k}|\geq\Lambda} \widetilde{dk}\, f^\mu(\vec{k}) a^\dagger_\mu(\vec{k})} \prod_{a=1}^{n} b^\dagger_a(\vec{p}_a) \prod_{\text{photons}} \left(a^\dagger_{\alpha_{\rm ph}}(\vec{k}_{\rm ph}) - \varepsilon^\mu_{\alpha_{\rm ph}}(\vec{k}_{\rm ph}) f_\mu(\vec{k}_{\rm ph})\right)|0\rangle\,, \tag{4.58}$$

where in $\hat{S}^{(\Lambda)}$ we sum only over loop momenta $\vec{l}$ greater than Λ, $|\vec{l}| \geq \Lambda$. Combining the above expression with (4.55), we obtain:

$$\begin{aligned} A_n &= e^{-\frac{e^2}{2}\sum_{a,b=1}^n Q_a Q_b \int_{|\vec{k}|\geq\lambda} \widetilde{dk} \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}} e^{+\frac{e^2}{2}\sum_{a,b=1}^n Q_a Q_b \int_{\Lambda\geq|\vec{k}|\geq\lambda} \widetilde{dk} \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}} A_{n,\Lambda,\Lambda} \\ &= \exp\left(-\frac{e^2}{2}\sum_{a,b=1}^{n} Q_a Q_b \int_{|\vec{k}|\geq\Lambda} \widetilde{dk} \frac{p_a\cdot\Pi(\vec{k})\cdot p_b}{(p_a\cdot k)(p_b\cdot k)}\right) A_{n,\Lambda,\Lambda}. \end{aligned} \tag{4.59}$$

This is a finite result independent of λ. In the limit $\lambda \to 0$ we obtain a finite, nonzero result, fixing the IR problem from the previous section. Faddeev-Kulish dressings cancell infra-red divergences.

5 Memory Effect

In this section, we will provide a basic explanation of how to measure the waveform $F^{(0)}_{uz}$. Our discussion will primarily follow the arguments presented in [4], which were also covered in a remarkably illustrative presentation of aspects of IR physics by Miller [6] (a highly recommended read).

Consider a test charge Q immersed in a viscous fluid, which position is determined by the Newton's second law:

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \vec{F}_{\text{ext}} - b\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, \tag{5.1}$$

where b is a viscosity coefficient, $-b\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ is the drag force caused by the fluid, and $\vec{F}_{\text{ext}}$ is an external force acting on a charge. Then, the difference between initial and final velocity of the charged particle is:

$$m\Delta\vec{v} = \int_{-\infty}^{\infty} dt\, \vec{F}_{\text{ext}} - b\Delta\vec{x}. \tag{5.2}$$

If the particle starts at rest, and $\vec{F}_{\text{ext}}$ is turned on only for a finite period of time, then, because of the viscosity of the fluid, the particle will be at rest also at $t \to \infty$. Hence, $\Delta\vec{v} = 0$ and:

$$\Delta\vec{x} = \frac{1}{b}\int dt\, \vec{F}_{\text{ext}}. \tag{5.3}$$

Now, consider a large number of such probe particles with charge Q, immersed in the fluid, arranged in a large sphere and moving initially near $\mathcal{J}^-$, and then $\mathcal{J}^+$. Assume that they surround a group of moving charges emitting radiation. If the radiation sources move slowly, the probe particles at $\mathcal{J}^\pm$ feel only electric, and not magnetic field. Force acting on the probe particles near $\mathcal{J}^+$ is $\vec{F}_{\text{ext}} = QF^{(0)}_{uA}\frac{\partial}{\partial z^A}$, where z^A are coordinates on the sphere at $\mathcal{J}^+$. Similarly, near $\mathcal{J}^-$ we have $\vec{F}_{\text{ext}} = QF^{(0)}_{vA}\frac{\partial}{\partial z^A}$. Recall that according to the fall-off conditions (1.25) (1.27), force in the radial direction is negligible far away from the radiation sources. Therefore, relative change of positions of test particles on the celestial sphere is:

$$\Delta x_A = \frac{Q}{b}\left(\int_{-\infty}^{\infty} du\, F_{uA} + \int_{-\infty}^{\infty} dv\, F_{vA}\right). \tag{5.4}$$

This shift is fully determined by soft theorems! Note that:

$$\begin{aligned}
\int_{-\infty}^{\infty} du\, \langle\text{out}|\, F_{uz}\hat{S}\,|\text{in}\rangle &= \frac{1}{8\pi}\int_{-\infty}^{\infty} du \int_0^{\infty} d\omega\, \omega e^{i\omega u}\, \langle\text{out}|\, a_+(\omega\hat{x})\hat{S}\,|\text{in}\rangle \\
&= \frac{1}{8\pi}\int_0^{\infty} d\omega\, \delta(\omega)\omega\, \langle\text{out}|\, a_+(\omega\hat{x})\hat{S}\,|\text{in}\rangle \\
&= \frac{1}{8\pi}\lim_{\omega\to 0}\omega\, \langle\text{out}|\, a_+(\omega\hat{x})\hat{S}\,|\text{in}\rangle \\
&= \sum_{a\in\text{in}\cup\text{out}} \eta_a \frac{Q_a}{8\pi}\frac{\varepsilon_+\cdot p_a}{\hat{q}(z,\bar{z})\cdot p_a}\, \langle\text{out}|\,\hat{S}\,|\text{in}\rangle\,,
\end{aligned} \tag{5.5}$$

where in the last step of the derivation we used the soft theorem for positive-helicity soft photon.

On the other hand, the temporal gauge condition implies $F^{(0)}_{uz} = \partial_u A^{(0)}_z$, and thus, $\int_{-\infty}^{\infty} du\, F^{(0)}_{uz} = A^{(0)}_z|_{\mathcal{J}^+_+} - A^{(0)}_z|_{\mathcal{J}^+_-}$. Similarly, $\int_{-\infty}^{\infty} dv\, F^{(0)}_{vz} = A^{(0)}_z|_{\mathcal{J}^-_+} - A^{(0)}_z|_{\mathcal{J}^-_-}$. Using antipodal matching conditions , $A_z^{(0)}(z,\bar{z})|_{\mathcal{J}^+_-} = A_z^{(0)}(z,\bar{z})|_{\mathcal{J}^-_+}$ [5, 6], we obtain:

$$\Delta A_z^{(0)} = A_z^{(0)}|_{\mathcal{J}^+_+} - A_z^{(0)}|_{\mathcal{J}^-_-} = \int_{-\infty}^{\infty} du\, F_{uz} + \int_{-\infty}^{\infty} dv\, F_{vz} = \sum_{a\in \text{in}\cup\text{out}} \frac{Q_a \eta_a}{4\pi} \frac{\varepsilon_+ \cdot p_a}{\hat{q}(z,\bar{z})\cdot p_a}. \tag{5.6}$$

Note that $\Delta A_z^{(0)}$ is total derivative:

$$\Delta A_A^{(0)}(z,\bar{z}) = \frac{\partial}{\partial z^A} \sum_a \eta_a \frac{Q_a}{4\pi} \log(\hat{q}(z,\bar{z})\cdot p_a). \tag{5.7}$$

But this cannot imply that $\Delta A_A^{(0)}$ is a gauge-dependent quantity. It is directly related to the measurable shift Δx^A. Remember that in the temporal gauge residual gauge transformations are time-independent. Differences between field configurations at different times, like $\Delta A_z^{(0)} = A_z^{(0)}|_{\mathcal{J}^+_+} - A_z^{(0)}|_{\mathcal{J}^-_-}$ are measurable, physical quantities.

Let us also emphasize that $\Delta A_z^{(0)}$ depends only on initial and final velocities of the radiating particles. It is independent of the details of the process that caused acceleration of the particles (that caused radiation). One can say that $\Delta A_A^{(0)}$ (and thus also Δx^A) "remembers" initial velocities of charged particles. Hence, the phenomenon of the shift Δx^A is commonly referred to as *memory effect* [3–5].

5.1 Subleading memory effect

One can also set up experiment that measures other modes of the electromagnetic radiation, for example:

$$\int_{-\infty}^{\infty} du\, u F_{uz}^{(0)}(u,z,\bar{z}). \tag{5.8}$$

Its expectation value at the end of a scattering experiment is fully determined by the subleading soft factor $S_+^{(1)}(z,\bar{z})$:

$$\begin{aligned}
\int_{-\infty}^{\infty} du\, u F_{uz} &= \frac{1}{8\pi} \int_{-\infty}^{\infty} du\, u \int_0^{\infty} d\omega\, \omega e^{i\omega u} \langle \text{out}|\, a_+(\omega \hat{x}(z,\bar{z}))\hat{S}\, |\text{in}\rangle \\
&= \frac{-i}{8\pi} \int_{-\infty}^{\infty} du \int_0^{\infty} d\omega\, (\partial_\omega e^{i\omega u})\omega \langle \text{out}|\, a_+(\omega \hat{x}(z,\bar{z}))\hat{S}\, |\text{in}\rangle \\
&= \frac{-i}{8\pi} \int_{-\infty}^{\infty} du \int_0^{\infty} d\omega\, e^{i\omega u} \partial_\omega(\omega \langle \text{out}|\, a_+(\omega \hat{x}(z,\bar{z}))\hat{S}\, |\text{in}\rangle) \\
&= \frac{i}{8\pi} S_+^{(1)} \langle \text{out}|\, \hat{S}\, |\text{in}\rangle\, .
\end{aligned} \tag{5.9}$$

A physical phenomenon directly dependent on the quantity $\int_{-\infty}^{\infty} du\, u F_{uz}^{(0)}$ is referred to as "subleading memory effect" [25].

Summary

In this note we set out to clarify the infrared structure of quantum electrodynamics. Beginning from classical considerations, we showed that the standard free QFT modes of electromagnetic field coincide with the "undergraduate definition of radiation" and the proper definition via Newman-Penrose coefficients. Then, considering generic waveform profile, we showed that Feynman diagrams involving particles with zero energy are expected to be divergent.

In the second section, with Feynman-diagrammatic analysis we confirmed the intuition from the first section. We computed the leading and subleading contributions to tree-level amplitudes in the limit of vanishing frequency of an external photon ("in the soft-photon limit") and showed that they are controlled by universal expressions ("soft factors"). Because of the direct relation between the leading soft factor and charge conservation, the leading soft factor is believed to be protected from loop corrections (the leading soft photon theorem is true at any loop order). We also discussed how the subleading soft photon factor is modified at 1-loop.

In the third section we showed that the soft theorems imply IR divergences of amplitudes due to soft photons running in loops. These divergences exponentiate, leading to a vanishing amplitude between any initial and final states with finite numbers of photons.

This failure of the standard definition of S-matrix can be traced directly to the presence of long-range interactions, which persist in QED even when the particles are widely separated in space and time. A resolution of this problem is provided by the Faddeev-Kulish construction of asymptotic states. We derived the form of the dressing operators and showed how they cancel IR divergences to all orders in perturbation theory. The resulting dressed S-matrix is finite and non-trivial.

We finished our analysis in section 4, by showing that the soft factors appearing in amplitudes correspond to measurable quantities, such as the electromagnetic memory, predicting the net displacement of test charges induced by radiation.

We conclude by noting that the ideas developed here are not specific to QED. The infrared structure of gravity and non-Abelian gauge theories exhibits many analogous features, including soft theorems, infinite-dimensional symmetry algebras, and memory effects [5]. The study of these effects has led to the development of celestial holography, a program that aims to recast scattering amplitudes as correlation functions of a conformal field theory living on the celestial sphere [26, 27], hinting at a relation between infrared physics and quantum gravity.

To deepen the understanding of topics described in this note, we encourage the reader to explore several related directions. These include the study of asymptotic symmetries in gauge and gravitational theories [5, 28, 29], the role of soft modes in black hole physics [30, 31], and the S-matrix bootstrap [32–34]. On the more practical side, infrared effects are intimately related to post-Minkowskian expansions, eikonal approximations, and waveform predictions for gravitational wave detectors [35, 36].

A Fundamentals of scattering theory

Key question for scattering theorist: "Given initial state $|\text{in}\rangle$, what is the probability of measuring state $|\text{out}\rangle$.

Assume that "in" and "out" states contain well-separated, non-interacting particles, relatively well-localized in momentum space"

$$|\text{in}\rangle = |\{p_1, s_1, a_1\}, \ldots, \{p_n, s_n, a_n\}\rangle \, , \tag{A.1}$$

where p_a, s_a are 4-momentum and helicity of the a'th particle, respectively. a_a collectively denotes other degrees of freedom of the particle.

Poincaré transformations of "in" and "out" states:

$$U(\Lambda, b)\,|\{p_a, s_a, a_a\}\rangle = e^{ib_\mu \sum_a p_a^\mu} \sum_{s_1' \cdots s_n'} \left(\prod_{a=1}^{n} D_{s_a s_a'}[W(\Lambda p_a)] \right) |\{\Lambda p_a, s_a', a_a\}\rangle \, , \tag{A.2}$$

where $D_{s_a s_a'}$ is the spin-s_a representation of the little group transformation of p_a corresponding to Λ. For massless particles we have $D_{ss'} = \delta_{ss'} e^{is\theta(\Lambda p)}$ with momentum-dependent phase $\theta(\Lambda p) \in \mathbb{R}$, and for massive partickes $D_{ss'}$ is a $(2j+1)$-dimensional unitary irrep. of SO(3).

We choose relativistic normatization of states:

$$_{\text{in}}\langle p, s, a | p', s', a' \rangle_{in} = \delta_{ss'} \delta_{aa'} \tilde{\delta}(p, p'), \tag{A.3}$$

where $\tilde{\delta}(p, p') = (2\pi)^{d-1} 2E_p \delta^{d-1}(\vec{p} - \vec{p}')$ and $E_p = \sqrt{\vec{p}^2 + m^2}$.

Let vectors $|\Psi_\alpha^+\rangle$ ($|\Psi_\alpha^-\rangle$) form an orthonormal basis of "out" ("in") states,

$$\mathbb{1} = \int d\alpha \left|\Psi_\alpha^\pm\right\rangle \left\langle\Psi_\alpha^\pm\right|, \quad \langle \Psi_\alpha^\pm | \Psi_\beta^\pm \rangle = \delta(\alpha, \beta). \tag{A.4}$$

For $\Lambda^\mu{}_\nu = \delta^\mu_\nu$, $b^\mu = (t, \vec{0})$ we have $U(\Lambda, a) = e^{-itH}$, where H is the Hamiltonian of the theory. Eq. (A.2) implies:

$$e^{-itH} \left|\Psi_\alpha^\pm\right\rangle = e^{-iE_\alpha t} \left|\Psi_{\alpha^\pm}\right\rangle, \tag{A.5}$$

and hence,

$$H \left|\Psi_\alpha^\pm\right\rangle = E_\alpha \left|\Psi_\alpha^\pm\right\rangle, \tag{A.6}$$

where $E_\alpha = p_1^0 + p_2^0 + \cdots + p_n^0$. Note that we are working in the Heisenberg picture, so state vectors do not evolve in time. We do not say that $|\Psi_\alpha^\pm\rangle$ are some limits of the same $|\Psi(t)\rangle$ with $t \to \pm\infty$.

Take $H = H_0 + V$, where H_0 is a "free theory" Hamiltonian, with known eigenstates $|\Phi\rangle_\alpha$, and the same spectrum $\{E_\alpha\}$ as Hamiltonian H. In particular, masses in H_0 must be the physical masses, which are actually measured, not necessarily equal to "bare" masses of H. We think of V as a small correction to H_0.

A.1 Definition of in and out states

"In" states $|\Psi_\alpha^-\rangle$ are defined via:

$$H\left|\Psi_\alpha^-\right\rangle = E_\alpha\left|\Psi_\alpha^-\right\rangle \tag{A.7}$$

$$e^{-iHt}\left|\Psi_\alpha^-\right\rangle \overset{t\to-\infty}{\sim} \int d\alpha\, e^{-iE_\alpha t} f(\alpha)\left|\Phi_\alpha\right\rangle = e^{-iH_0 t}\int d\alpha f(\alpha)\left|\Phi_\alpha\right\rangle \tag{A.8}$$

for any well-beaved smearing function $f(\alpha)$. Similarly, we define "out" states via:

$$H\left|\Psi_\alpha^+\right\rangle = E_\alpha\left|\Psi_\alpha^+\right\rangle \tag{A.9}$$

$$e^{-iHt}\left|\Psi_\alpha^-\right\rangle \overset{t\to\infty}{\sim} \int d\alpha\; e^{-iE_\alpha t} f(\alpha)\left|\Phi_\alpha\right\rangle = e^{-iH_0 t}\int d\alpha f(\alpha)\left|\Phi_\alpha\right\rangle. \tag{A.10}$$

The second condition can be rewritten as

$$\left|\Psi_\alpha^-\right\rangle = \Omega(\pm\infty)\left|\Phi_\alpha\right\rangle, \tag{A.11}$$

where

$$\Omega(t) = e^{iHt}e^{-iH_0 t}. \tag{A.12}$$

$\Omega(t)$ is defined in a distributional sense, the above equation is true only when smeared with $f(\alpha)$.

A.2 S-matrix

Define S-matrix as:

$$S_{\alpha\beta} = \langle\Psi_\beta^+|\Psi_\alpha^-\rangle. \tag{A.13}$$

Completeness of "in" and "out" states implies:

$$\delta(\alpha,\beta) = \int d\gamma\langle\Psi_\beta^+|\Psi_\gamma^-\rangle\langle\Psi_\gamma^-|\Psi_\alpha^+\rangle = \int d\gamma S_{\beta\gamma}S^*_{\alpha\gamma} = \left\langle\Psi_\beta^+\right| SS^\dagger\left|\Psi_\alpha^+\right\rangle, \tag{A.14}$$

$$\delta(\alpha,\beta) = \int d\gamma\langle\Psi_\beta^-|\Psi_\gamma^+\rangle\langle\Psi_\gamma^+|\Psi_\alpha^-\rangle = \int d\gamma S^*_{\gamma\beta}S_{\gamma\alpha} = \left\langle\Psi_\beta^-\right| S^\dagger S\left|\Psi_\alpha^-\right\rangle, \tag{A.15}$$

for all α, β. This means that the S-matrix is unitary:

$$SS^\dagger = S^\dagger S = \mathbb{1}. \tag{A.16}$$

It is convenient to encode the information about scattering amplitudes using free theory eigenstates $|\Phi_\alpha\rangle$. To this end, we define "S-operator" $\hat{S}$ via:

$$\langle\Phi_\beta|\,\hat{S}\,|\Phi_\alpha\rangle = S_{\beta\alpha}. \tag{A.17}$$

Since $|\Psi_\alpha^\pm\rangle = \Omega(\pm\infty)\left|\Phi_\alpha\right\rangle$, the $\hat{S}$-operator is:

$$\hat{S} = \Omega^\dagger(+\infty)\Omega(-\infty) \equiv U(+\infty,-\infty), \tag{A.18}$$

where

$$U(t,t') = \Omega^\dagger(t)\Omega(t) = e^{iH_0 t}e^{-iHt}e^{iHt'}e^{-iH_0 t'} = e^{iH_0 t}e^{-iH(t-t')}e^{-iH_0 t'}. \tag{A.19}$$

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