# Lamb shift calculations

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**Abstract**

In this short note, we will present the derivation of the formula for calculating the Lamb shift and review the historical calculation of the self-energy diagram for this problem in the Furry picture.

## 1 Introduction

The Lamb shift is one of the most important effects both in theoretical and experimental quantum electrodynamics. In this brief note, we will explore various derivations of the formula used for calculating the Lamb shift from a mathematical perspective. Additionally, we will provide an overview of the classical calculation [Moh73] of the self-energy diagram in the Furry picture.

The Furry picture, on which our approach is based, is a way to partially solve part of the dynamics in a problem with an external potential by treating the external potential as part of the free Hamiltonian. We will not consider the bound state problem as a two-body problem in quantum electrodynamics, but restrict ourselves to the external potential approximation.

Motivation for this work is to understand whether Lamb shift calculations differ significantly in the case of electrodynamics with massive photons and in the case of bosonic matter.

## 2 Derivation of the formula for the energy shift

Consider the spinorial QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\Psi, \tag{2.1}$$

where $D_{\mu} = \partial_{\mu} - ieA_{\mu}$, $e$ is the electric charge, $\bar{\Psi} = \Psi^{\dagger}\gamma^{0}$, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The gamma matrices satisfy anticommutation relations $\{\gamma^{\mu}, \gamma^{\nu}\} = -2\eta^{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. This convention is used, for example, in the Srednicki book [Sre07]. We will use the notation $x = (t, \vec{x})$. Counterterms will be included later.

If we consider the above theory in a static background field $A^0_{cl}(x) = V(\vec{x})$, $A^i_{cl}(x) = 0$ for $i = 1, 2, 3$ then one can separate the electromagnetic potential $A^\mu(x) = A^\mu_{cl}(x) + A^\mu_{qu}(x)$ and rewrite the above Lagrangian as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Psi^\dagger\left(-i\partial_t + i\vec{\alpha}\cdot\vec{\nabla} - m\beta - eV(\vec{x})\right)\Psi + e\bar{\Psi}A^\mu_{qu}\gamma_\mu\Psi + F_{0i}\,\partial^i\,V(\vec{x}), \quad (2.2)$$

where we introduced $\vec{\alpha} = \gamma^0\vec{\gamma}$ and $\beta = \gamma^0$. Later, we will also write $\alpha^\mu = (1, \vec{\alpha})$. We have also omitted terms in the Lagrangian that do not depend on the dynamical variables. We have also redefined $F^{\mu\nu} = \partial^\mu A^\nu_{qu} - \partial^\nu A^\mu_{qu}$ and will write just $A^\mu$ instead of $A^\mu_{qu}$ in the rest of what follows.

The fermionic field will be quantized in the Furry (background field) picture. Let

$$\tilde{u}_N(x) = \tilde{u}_N(\vec{x})e^{-iE_N t} \quad (2.3)$$

be a positive frequency solution to the Dirac equation following from the Lagrangian 2.2 ($N$ is an index that goes through both bound and scattering states):

$$\left(i\vec{\alpha}\cdot\vec{\nabla} - m\beta - eV(\vec{x})\right)\tilde{u}_N(\vec{x}) = E_N\tilde{u}_N(\vec{x}). \quad (2.4)$$

Similarly for the negative frequency solutions[1]

$$v_N(x) = v_N(\vec{x})e^{iE_{N'}t}, \quad (2.5)$$

$$\left(i\vec{\alpha}\cdot\vec{\nabla} - m\beta - eV(\vec{x})\right)v_N(\vec{x}) = -E_{N'}v_N(\vec{x}). \quad (2.6)$$

This motivates us to define the Dirac operator:

$$\mathcal{D}f(\vec{x}) := \left(i\vec{\alpha}\cdot\vec{\nabla} - m\beta - eV(\vec{x})\right)f(\vec{x}) \quad (2.7)$$

for the suitably well-behaved potential $V(\vec{x})$ this operator is self-adjoint on the appropriate class of functions, so it has the spectral decomposition in terms of the spectral measure (which gives the completeness relation). All energies $E_N, E_{N'}$ are positive because we denote the negative eigenvalues of $\mathcal{D}$ by $-E_{N'}$, and for simplicity we will assume that potential is chosen such that the Dirac operator $\mathcal{D}$ has no zero eigenvalues (zero modes).

To simplify notation, will introduce $u_N$ as the unified symbol for the eigenfunctions of $\mathcal{D}$, regardless of the sign of their. This frequency will be denoted by $\omega_N$. (Index $N$ is now labeling all the states. Functions $u_N$ can be chosen to be orthonormal with respect to the $L^2(\mathbb{R}^3, \mathbb{C}^4)$ canonical scalar product $\langle\cdot|\cdot\rangle$

$$\langle u_N|u_M\rangle := \int \mathrm{d}\vec{x}\; u^\dagger_N(\vec{x})u_M(\vec{x}) = \delta_{NM}. \quad (2.8)$$

Symbols $\delta_{NM}$ should be everywhere interpreted as Kronecker deltas for the eigenfunctions in the discrete spectrum and the Dirac deltas with the appropriate factor needed for a relativistic normalization factor, for the scattering states.

[1] We denote negative eigenvalues by $-E_{N'}$, $|E_{N'}| > 0$ to avoid the potential confusion with the positive ones.

Let $b_N, d_N$ be annihilation operators of the positively and negatively charged states. Let $\tilde{b}_N$ be $b_N$ if $u_N$ is the positive frequency state, or $d_N^\dagger$ in the opposite case. We define the quantum field in the Furry picture as

$$\psi(x) := \sum_N b_N \tilde{u}_N(\vec{x}) e^{-iE_N t} + \sum_N d_N^\dagger v_N(\vec{x}) e^{iE_{N'} t} =: \sum_N B_N u_N e^{-i\omega_N t} \tag{2.9}$$

where the sum is written symbolically to denote summation over the bound states and integration with respect to the Lorentz invariant measure for the scattering states. We also have separated two sums, because there does not need to be any matching between positive and negative frequency states.

Let $\Omega_F$ be the vacuum of the theory in the Furry picture, that is the state annihilated by all $b_N$ and $d_N$ operators. By $(\cdot,\cdot)$, we will denote the scalar product in the Fock space built from eigenfunctions of $\mathcal{D}$.

We will define the fermionic propagator as follows:

$$S_0(x,y) := i\left(\Omega_F, T\psi(x)\bar{\psi}(y)\ \Omega_F\right), \tag{2.10}$$

$$-iS_0(x,y) = \theta(x^0-y^0)\sum_N \tilde{u}_N(\vec{x})\bar{\tilde{u}}_N(\vec{y})e^{-iE_N(x^0-y^0)} - \theta(y^0-x^0)\sum_N v_N(\vec{x})\bar{v}_N(\vec{y})e^{iE_{N'}(x^0-y^0)}. \tag{2.11}$$

We have written $S_0(x,y)$ instead of $S_0(x-y)$ because at this point we have not assumed any other symmetry of the potential, except time translation symmetry.

The propagator has the spectral representation[2]:

$$\begin{aligned} S_0(x,y) &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi}\left(\sum_N \frac{\tilde{u}_N(\vec{x})\bar{\tilde{u}}_N(\vec{y})}{E_N-\omega-i0} - \sum_N \frac{v_N(\vec{x})\bar{v}_N(\vec{y})}{E_{N'}+\omega-i0}\right) e^{-i\omega(x^0-y^0)} \\ &= \int_{C_F} \frac{\mathrm{d}\omega}{2\pi}\left(\sum_N \frac{\tilde{u}_N(\vec{x})\bar{\tilde{u}}_N(\vec{y})}{E_N-\omega} - \sum_N \frac{v_N(\vec{x})\bar{v}_N(\vec{y})}{E_{N'}+\omega}\right) e^{-i\omega(x^0-y^0)} \\ &= \int_{C_F} \frac{\mathrm{d}\omega}{2\pi}\sum_N \frac{u_N(\vec{x})\bar{u}_N(\vec{y})}{\omega_N-\omega} e^{-i\omega(x^0-y^0)}, \end{aligned} \tag{2.12}$$

where $C_F$ denotes the Feynman contour that is depicted on the figure 2.1 that goes below poles corresponding to the states belonging to the Dirac sea.

$$\begin{aligned} S_0(\vec{x},\vec{y};\omega) &:= \int \mathrm{d}(x^0-y^0)\ S_0(x,y)e^{i\omega(x^0-y^0)} \\ &= \sum_N \frac{u_N(\vec{x})\bar{u}_N(\vec{y})}{E_N-\omega-i0} - \sum_N \frac{v_N(\vec{x})\bar{v}_N(\vec{y})}{E_{N'}+\omega-i0} = \sum_N \frac{u_N(\vec{x})\bar{u}_N(\vec{y})}{\omega_N-\omega-i0\,\mathrm{sgn}\,\omega_N}. \end{aligned} \tag{2.13}$$

Now we will show how to systematically derive the formula for the correction to the energy of the given bound state using the pole structure of the propagator of the bounded

[2]We have used well-known formula for the step function $\theta(x) = (2\pi i)^{-1}\int_{-\infty}^{+\infty}\mathrm{d}z(z-i0)^{-1}e^{ixz}$

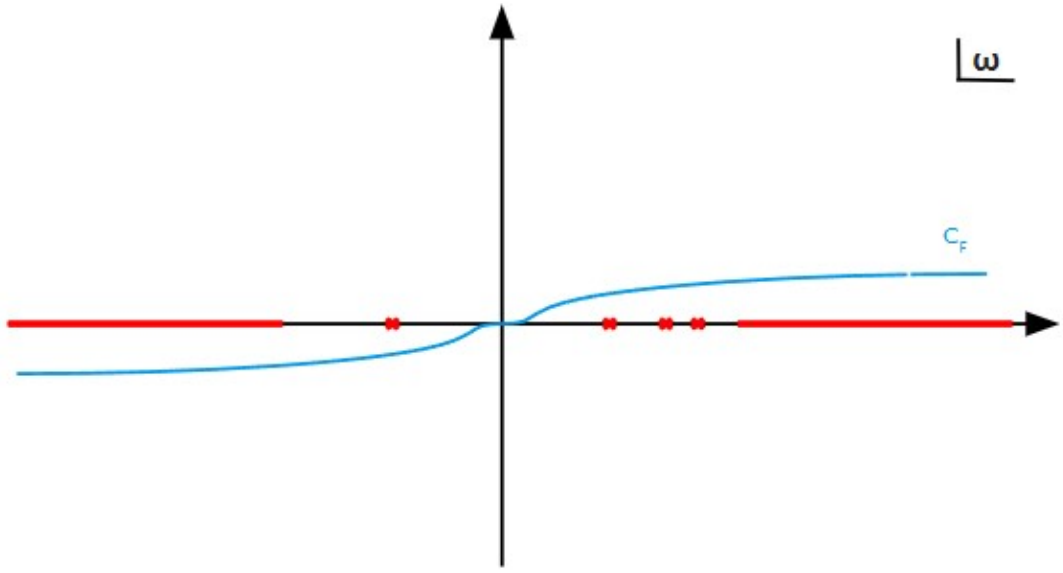


Figure 2.1: Feynman contour of integration in the $\omega$ plane. Red lines denote branch cuts coming from the scattering states, and red crosses depict poles coming from bound states.

particle. This derivation is valid regardless of whether the bounded particle is a boson or a fermion.

Let us denote the Heisenberg picture propagator of the particle moving in the potential by $S'(x,y)$. It is defined by the perturbation series from the given Lagrangian as a sum of all connected diagrams with two external legs, so it has the following form:

$$\begin{aligned} -iS(x,y) = -iS_0(x,y) &- \int \mathrm{d}z\ \mathrm{d}w\ \ (-iS_0(x,w))\,(i\Sigma(w,z))\,(-iS_0(z,y)) \\ &- i\int \mathrm{d}z\ \mathrm{d}w\ \mathrm{d}z'\mathrm{d}w'\ S_0(x,w)\Sigma(w,z)S_0(z,w')\Sigma(w',z')S_0(z',y) + ... \\ &= -iS_0(x,y) - i\int \mathrm{d}z\ \mathrm{d}w\ S_0(x,w)\Sigma(w,z)S(z,y), \end{aligned} \tag{2.14}$$

where $i\Sigma(w,z)$ is the sum of all "1-electron" irreducible diagrams with two external legs, fixed in the $w$ and $z$ spacetime points, with external propagators removed. $i\Sigma(w,z)$ is really a sum of "1-electron" irreducible diagrams (we cannot cut the photon lines), not of the one-particle irreducible diagrams, because we do not have a cancellation of tadpoles with the Furry propagators in the loops. Interactions with an external potential break translational symmetry needed for the tadpole cancellation. The Furry theorem does not apply either since it assumes charge conjugation symmetry of the theory, which is explicitly broken by the external potential. From the above equation, we see that $S$ is

the Green function satisfying:

$$\begin{aligned}(\mathcal{D}-\Sigma)\,S &= (\mathcal{D}-\Sigma)\,(S_0+S_0\Sigma S) = \mathbb{1}-\Sigma\;S_0+\Sigma\;S-\Sigma\;S_0\;\Sigma\;S \\ &= \mathbb{1}-\Sigma\;\;(S-S_0-S_0\;\Sigma\;S) = \mathbb{1}.\end{aligned} \tag{2.15}$$

We have used the operator notation $AB = \int \mathrm{d}z\; A(x,z)B(z,y)$, and used the fact that $\mathcal{D}S = \mathbb{1}$. We will therefore sometimes write shortly

$$S = (\mathcal{D}-\Sigma)^{-1}\,. \tag{2.16}$$

## 2.1 First derivation

In the standard textbook treatment due to Weinberg [Wei95], one assumes a certain spectral decomposition of $S$ and calculates its form using the perturbation theory. This assumption is ad hoc and unjustified. Below, we will show how to use Weinberg's method beyond first order. In the later discussion, we will show how to obtain the energy shift more directly. Considered energy shift $\delta\omega_N$ is a priori a complex number. Its imaginary part is related to the lifetime of the $N$'th excited state.

Assuming that the corrections to $S$ coming from the $\Sigma$ are small enough, we can write a spectral representation of the full propagator:

$$\begin{aligned}S(\vec{x},\vec{y};\omega) &= \sum_N \frac{\tilde{U}_N(\vec{x})\bar{\tilde{U}}_N(\vec{y})}{E_N+\delta E_N-\omega-i0} - \sum_N \frac{V_N(\vec{x})\bar{V}_N(\vec{y})}{E_{N'}+\delta E_{N'}+\omega-i0} \\ &=: \sum_N \frac{U_N(\vec{x})\bar{U}_N(\vec{y})}{\omega_N+\delta\omega_N-\omega-\operatorname{sgn}\omega\; i0}.\end{aligned} \tag{2.1}$$

Later, we will show that this assumption about the form of the decomposition is unnecessary to prove formulas for energy corrections. More importantly, it turns out that we do not even need to assume the existence of the $U_N$ functions and calculate them. We denoted by $\tilde{U}_N$ and $V_N$ functions that are heuristically thought of as solutions to the $(\mathcal{D}-\Sigma)\,f = 0$ equation. They will be calculated using perturbation expansion by the well-defined procedure that defines them.

We have combined the indexing of the energies and wavefunctions of both positive and negative frequency states to simplify the notation. The propagator of the scalar particle moving in the external potential will have the same final form as the propagator if we replace the Dirac adjoint by the Hermitian adjoint. This analysis will still be valid for the scalar case if we make this substitution in the final results.

We will do perturbation theory. Let us write $\delta\omega_N$, $U_N$ and $\Sigma$ as a Taylor series in the coupling constant $\alpha$:

$$\delta\omega_N = \sum_{n=1}^{\infty}\delta\omega_N^{(n)}\alpha^n,\quad \Sigma = \sum_{n=1}^{\infty}\Sigma^{(n)}\alpha^n,\quad \delta U_N(\vec{x}) = u_N(\vec{x}) + \sum_{n=1}^{\infty}\delta U_N^{(n)}(\vec{x})\alpha^n. \tag{2.2}$$

Next, we expand to the first order in $\alpha$ equations 2.1 and 2.14. Using the orthogonality of the wavefunctions and comparing the behavior of both sides of the equation in the

$\omega \to \omega_N$ limit we obtain:

$$
\begin{aligned}
-\frac{\delta\omega_N^{(1)}}{(\omega_N-\omega)^2} &= \frac{1}{(\omega_N-\omega)^2}\int \mathrm{d}\vec{x}\ \mathrm{d}\vec{y}\ \bar{u}_N(\vec{x})\left(\int \mathrm{d}(x^0-y^0)\Sigma^{(1)}(x,y)e^{i\omega(x^0-y^0)}\right)u_N(\vec{y}) \\
&=: \frac{1}{(\omega_N-\omega)^2}\int \mathrm{d}\vec{x}\ \mathrm{d}\vec{y}\ u_N(\vec{x})\Sigma^{(1)}(\vec{x},\vec{y};\omega)\bar{u}_N(\vec{y}).
\end{aligned}
\tag{2.3}
$$

The orthonormality is used to isolate a given term in the sum and pole structure comparison is needed to ensure that we are isolating a term proportional to the correction to the energy, not the one coming from the wavefunction correction. The final result is:

$$
\delta\omega_N^{(1)} = -\int \mathrm{d}\vec{x}\mathrm{d}\vec{y}\ \bar{u}_N(\vec{x})\Sigma^{(1)}(\vec{x},\vec{y};\omega)u_N(\vec{y}) = -\left\langle u_N\right|\gamma^0\Sigma^{(1)}(\omega)\left|u_N\right\rangle . \tag{2.4}
$$

Let us derive the formula for the second correction to the energy. Taking the matrix element of $S$ and expanding gives:

$$
\begin{aligned}
\left\langle u_N\right|S\gamma^0\left|u_N\right\rangle = \left\langle u_N\right| &(S_0+\alpha S_0\Sigma^{(1)}S_0+\alpha^2 S_0\Sigma^{(2)}S_0 \\
&+\alpha^2 S_0\Sigma^{(1)}S_0\Sigma^{(1)}S_0+...)\gamma^0\left|u_N\right\rangle
\end{aligned}
\tag{2.5}
$$

On the other hand, expansion of the spectral representation gives to the second order:

$$
\begin{aligned}
\left\langle u_N\right|S\gamma^0\left|u_N\right\rangle &= \frac{1+\alpha\left\langle\delta U_N^{(1)}|u_N\right\rangle+\alpha\left\langle u_N|\delta U_N^{(1)}\right\rangle}{\omega_N+\alpha\delta\omega_N^{(1)}+\alpha^2\delta\omega_N^{(2)}-\omega} \\
&= \frac{1+\alpha\left\langle\delta U_N^{(1)}|u_N\right\rangle+\alpha\left\langle u_N|\delta U_N^{(1)}\right\rangle}{\omega_N-\omega} \\
&- \frac{1+\alpha\left\langle\delta U_N^{(1)}|u_N\right\rangle+\alpha\left\langle u_N|\delta U_N^{(1)}\right\rangle}{(\omega_N-\omega)^2}\left(\alpha\delta\omega_N^{(1)}+\alpha^2\delta\omega_N^{(2)}\right) \\
&+ \frac{1}{(\omega_N-\omega)^3}\left(\alpha\delta\omega_N^{(1)}\right)^2+...
\end{aligned}
\tag{2.6}
$$

We see from the above expansion that $\delta\omega_N^{(n)}$ can be read off from the coefficient multiplying the $\alpha^n\left(\omega_N-\omega\right)^{-2}$ term in the perturbative series, provided that we know lower order correction to the energy and appropriate wavefunction corrections.

For instance, if we want to calculate the second-order correction, we need to consider the coefficient in front of $-\alpha^2\left(\omega_N-\omega\right)^{-2}$ that is equal to:

$$
\delta\omega_N^{(2)}+\delta\omega_N^{(1)}\left(\left\langle\delta U_N^{(1)}|u_N\right\rangle+\alpha\left\langle u_N|\delta U_N^{(1)}\right\rangle\right)
$$

Using in 2.5 spectral representation of the Furry propagator

$$
S_0 = \sum_N \frac{\left|u_N\right\rangle\left\langle u_N\right|\gamma^0}{\omega_N-\omega-\operatorname{sgn}\omega_N\ i0} \tag{2.7}
$$

we can collect in equation 2.5 terms with appropriate asymptotic behavior. This procedure is equivalent to removing two external leg propagators in the diagrams. We get the result

$$-\delta\omega_N^{(2)} = \langle u_N|\,\gamma^0(\Sigma^{(2)} + \Sigma^{(1)}S_{0\ red}\Sigma^{(1)})\,|u_N\rangle + \delta\omega_N^{(1)}\left(\langle\delta U_N^{(1)}|U_N\rangle + \langle U_N|\delta U_N^{(1)}\rangle\right) \quad (2.8)$$

where $S_{0\ red}$ is the reduced Furry picture propagator, that is, the Furry picture propagator with the term corresponding to the state to which we calculate the correction removed. We are omitting this term in the sum because it has a different behavior near the pole. We need to calculate $\langle\delta U_N^{(1)}|U_N\rangle + \langle U_N|\delta U_N^{(1)}\rangle$ .

Let us take a derivative of 2.5 with respect to $\omega$ and keep only first-order terms in $\alpha$:

$$\langle u_N|\frac{\mathrm{d}S_0}{\mathrm{d}\omega}\gamma^0|u_N\rangle = \alpha\,\langle u_N|\left(\frac{\mathrm{d}S}{\mathrm{d}\omega}\Sigma^{(1)}S_0 + S_0\frac{\mathrm{d}\Sigma^{(1)}}{\mathrm{d}\omega}S_0 + S_0\Sigma^{(1)}\frac{\mathrm{d}S_0}{\mathrm{d}\omega}\right)\gamma^0\,|u_N\rangle + O(\alpha^2). \quad (2.9)$$

Only the second term in the above equation has $\alpha\,(\omega_N-\omega)^{-2}$ behavior. If we take the derivative over $\omega$ of the equation 2.6 and keep only term proportional to $\alpha\,(\omega_N-\omega)^{-2}$ we will end up with:

$$\alpha\frac{\langle\delta U_N^{(1)}|\tilde{U}_N\rangle + \langle U_N|\delta U_N^{(1)}\rangle}{(\omega_N-\omega)^2}. \quad (2.10)$$

Hence, if we equate both sides and expand $S$, we get:

$$\langle\delta U_N^{(1)}|U_N\rangle + \langle U_N|\delta\tilde{U}_N^{(1)}\rangle = \langle u_N|\,\gamma^0\frac{\mathrm{d}\Sigma^{(1)}}{\mathrm{d}\omega}\,|u_N\rangle\,. \quad (2.11)$$

So if insert equation 2.11 into 2.8 we obtain:

$$\delta\omega_N^{(2)} = -\,\langle u_N|\,\gamma^0(\Sigma^{(2)} + \Sigma^{(1)}S_{red}\Sigma^{(1)})\,|u_N\rangle + \langle u_N|\gamma^0\Sigma^{(1)}|u_N\rangle\,\langle u_N|\,\gamma^0\frac{\mathrm{d}\Sigma^{(1)}}{\mathrm{d}\omega}\,|u_N\rangle\,. \quad (2.12)$$

## 2.2 Improved derivation

Now we will show a more rigorous approach to derive the above relations. We will do not As before, we will start with the fact that the position of the (isolated) pole of the full propagator $S(\vec{x},\vec{y};\omega)$ in the frequency domain, determines the energy (and average lifetime) of the given (bound) state. This is our initial assumption. If at some order of perturbation theory this assumption is invalid, we lose the interpretation that the singularity corresponds to the bound state.

The pole of the function that is the integral kernel of the operator

$$S(\omega) := (\mathcal{D} - \omega\mathbb{1} - \Sigma(\omega))^{-1}\,, \quad (2.1)$$

can be found by solving the operator equation:

$$(\mathcal{D} - \omega\mathbb{1} - \Sigma(\omega))\,f = 0. \quad (2.2)$$

This is just an eigenvalue equation for the inverse propagator.

The problem of solving this eigenvalue problem is hard due to the complicated form of $\Sigma$ that is calculated perturbatively. We want to solve equation 2.11 perturbatively. To

use perturbation theory in an effective manner, we will first project the problem onto the one-dimensional subspace of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ that is spanned by the state to which frequency $\omega_N$ corrections we want to calculate. Let us denote this considered state $u_N$ and the projection onto it by $P_N$. We can then use the algebraic formula:

$$\begin{aligned} &P_N S(\omega)\gamma^0 P_N \gamma^0 = |u_N\rangle \langle u_N| S(\omega)\gamma^0 |u_N\rangle \langle u_N| \gamma^0 \\ &= \big(\omega_N - \omega - \langle u_N|\gamma^0\Sigma(\omega)|u_N\rangle \\ &- \langle u_N|\gamma^0\Sigma(\omega)(\mathbb{1} - \gamma^0 P_N \gamma^0)S(\omega)(\mathbb{1} - P_N)\Sigma(\omega)|u_N\rangle\big)^{-1} |u_N\rangle \langle u_N| \gamma^0 \\ &=: \big(\omega_N - \omega - \langle u_N|\gamma^0\Sigma(\omega)|u_N\rangle - \langle u_N|\gamma^0\Sigma(\omega)S_{red}(\omega)\Sigma(\omega)|u_N\rangle\big)^{-1} |u_N\rangle \langle u_N| \gamma^0. \end{aligned} \tag{2.3}$$

The derivation of this formula is given in the appendix. The operator problem is reduced to the numerical problem because the equation 2.3 is the operator relation on the one-dimensional subspace.

$$\langle u_N| S(\omega)\gamma^0 |u_N\rangle = \big(\omega_N - \omega - \langle u_N|\gamma^0\Sigma(\omega)|u_N\rangle - \langle u_N|\gamma^0\Sigma(\omega)S_{red}(\omega)\Sigma(\omega)|u_N\rangle\big)^{-1}. \tag{2.4}$$

The left-hand side of the above equation is divergent when:

$$\omega_N - \omega - \langle u_N|\gamma^0\Sigma(\omega)|u_N\rangle - \langle u_N|\gamma^0\Sigma(\omega)S_{red}(\omega)\Sigma(\omega)|u_N\rangle = 0, \tag{2.5}$$

which gives:

$$\delta\omega_N = - \langle u_N|\gamma^0\Sigma(\omega)|u_N\rangle - \langle u_N|\gamma^0\Sigma(\omega)S_{red}(\omega)\Sigma(\omega)|u_N\rangle \,. \tag{2.6}$$

The above formula gives a clear, recursive derivation of the formula for the frequency shift of any state. Note that this derivation avoids considerations of the spectral form of $S(\omega)$ and complicated algebraic manipulations relating corrections to the wavefunctions with energy corrections at different orders of the perturbation theory.

Substituting expansions from 2.2, we obtain, to the first order in $\alpha$, the equation 2.4. Expanding to the second order and setting on the right-hand side $S_{red}(\omega) = S_0(\omega_N)$ gives:

$$\begin{aligned} \alpha\delta\omega_N^{(1)} + \alpha^2\delta\omega_N^{(2)} = &-\alpha \langle u_N|\gamma^0\Sigma^{(1)}|u_N\rangle - \alpha^2\delta\omega_N^{(1)} \langle u_N|\gamma^0\frac{\mathrm{d}\Sigma^{(1)}}{\mathrm{d}\omega}(\omega)|u_N\rangle - \alpha^2 \langle u_N|\gamma^0\Sigma^{(2)}(\omega)|u_N\rangle \\ &- \alpha^2 \langle u_N|\gamma^0\Sigma^{(1)}(\omega)S_{0\ red}(\omega)\Sigma^{(1)}(\omega)|u_N\rangle \,. \end{aligned} \tag{2.7}$$

Substituting $\delta\omega_N^{(1)}$ yields:

$$\begin{aligned} \delta\omega_N^{(2)} = &\langle u_N|\gamma^0\Sigma^{(1)}|u_N\rangle \langle u_N|\gamma^0\frac{\mathrm{d}\Sigma^{(1)}}{\mathrm{d}\omega}(\omega)|u_N\rangle - \langle u_N|\gamma^0\Sigma^{(2)}(\omega)|u_N\rangle \\ &- \langle u_N|\gamma^0\Sigma^{(1)}(\omega)S_{0\ red}(\omega)\Sigma^{(1)}(\omega)|u_N\rangle \,. \end{aligned} \tag{2.8}$$

# 3 Mohr's calculation

We will now briefly discuss the calculation of the energy shift due to the self-energy diagram done by Mohr [Moh73]. In this calculation, we will also see the standard procedure of dividing the integral into the low-energy part and the high-energy part, but in our

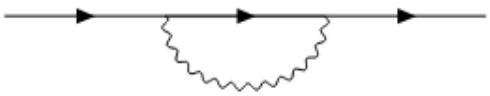

Figure 3.2: Self-energy diagram for spinorial electrodynamics.

opinion, in Mohr's treatment, this splitting is the most conceptually clean. Let us evaluate a 1PI diagram from the figure 3.2. Using well-known Feynman rules and Pauli-Villars regularization, we get:

$$i\Sigma_\Lambda^{SE}(x,y) = (ie)^2 \int \frac{\mathrm{d}k}{(2\pi)^4}\, \gamma^\mu(-iS_0(x,y))e^{ik(x-y)}\gamma^\nu \left(\frac{-i\eta_{\mu\nu}}{k^2-i0} - \frac{-i\eta_{\mu\nu}}{k^2+\Lambda^2-i0}\right). \quad (3.1)$$

After going to the energy domain and using $\alpha = e^2/4\pi$, we have:

$$\begin{aligned} i\Sigma_\Lambda^{SE}(\vec{x},\vec{y};\omega) = 4\pi\alpha \int_{C_F} \frac{\mathrm{d}k^0}{2\pi} \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3}\, \gamma^\mu S_0(\vec{x},\vec{y};\omega-k^0))\gamma_\mu \\ \times \left(\frac{1}{k^2-i0} - \frac{1}{k^2+\Lambda^2-i0}\right) e^{i\vec{k}(\vec{x}-\vec{y})}. \end{aligned} \quad (3.2)$$

We have explicitly written that the integral over $k^0$ is taken using the Feynman contour to emphasize the direction of going around poles. Let us expand $S_0$ using its spectral representation 2.12 and evaluate the matrix element of $i\Sigma_\Lambda^{SE}(\omega)$:

$$\begin{aligned} i\,\langle u_N|\gamma^0\Sigma_\Lambda^{SE}(\omega)|u_N\rangle = 2\alpha \int_{C_F} \mathrm{d}k^0 \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3}\, \langle u_N|\alpha^\mu S_0(\omega-k^0)\alpha_\mu|u_N\rangle \\ \times \left(\frac{1}{k^2-i0} - \frac{1}{k^2+\Lambda^2-i0}\right) e^{i\vec{k}(\vec{x}-\vec{y})}. \end{aligned} \quad (3.3)$$

The standard technique to evaluate similar integrals in quantum field theory is to use Feynman's parametrization and to perform a Wick rotation. However, in our case, because we do not have a simple expression for the propagator, momentum space integrals cannot be done that easily.

For the high-energy virtual photons, the contributions to the $S_0(\omega-k^0)$ propagator from the bound states are small, and we can express $S_0$ as the free-theory propagator dressed by the perturbative series in the external sources $V(\vec{x})$. The problem is reduced to the standard evaluation of the diagrams in quantum electrodynamics.

If $|k^0| \lesssim \omega$ the sum in the spectral representation 2.12 of $S_0(\omega-k^0)$ is expected to be dominated by contributions from the poles of the bound states. In this case, the integral over $k^0$ can be performed using the Sochocki formula[3] to isolate contributions for the intermediate bound states and reduce the problem to the old-fashioned perturbation theory[4].

[3] $\frac{1}{x\pm i\varepsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x)$.

[4] As it is done for example in chapter 13 of Weinberg's book [Wei95].

Here, we will do the opposite. We will integrate the momenta $\vec{k}$ of photons and leave the integral over $k^0$ for the end. It turns out that after the integration of the virtual photons, there is a natural split of the contributions from the high-energy and low-energy parts due to the analytical structure of the integrand. As we will see, the integration over the momenta will give us branch cuts in the complex $k^0$ plane.

Consider an integral:

$$\begin{aligned} I_z &= \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3} \frac{e^{iq(\vec{x}-\vec{y})}}{q^2 - z - i0} = \int \frac{\mathrm{d}q}{4\pi^2} \frac{e^{iq|\vec{x}-\vec{y}|} - e^{-iq|\vec{x}-\vec{y}|}}{iq|\vec{x}-\vec{y}|} \frac{q^2}{q^2 - z - i0} \\ &= \frac{1}{4\pi^2 i|\vec{x}-\vec{y}|} \int \mathrm{d}|\vec{k}| \, \frac{q e^{iq|\vec{x}-\vec{y}|}}{q^2 - z - i0}. \end{aligned} \tag{3.4}$$

In the second step, we performed integration over spherical coordinates $\theta, \varphi$ defined via $\vec{k} = q(\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ with $\vec{k} \cdot (\vec{x} - \vec{y}) = q|\vec{x} - \vec{y}| \cos\theta$. The above integral can be evaluated using the residue method. We close the contour from above and pick the residue at $q = \sqrt{z + i\varepsilon}$.

$$I_z = \frac{2\pi i}{4\pi^2 i|\vec{x}-\vec{y}|} \frac{\sqrt{z+i0} e^{i\sqrt{z+i0}|\vec{x}-\vec{y}|}}{2\sqrt{z+i0}} = \frac{e^{i\sqrt{z+i0}|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}. \tag{3.5}$$

To define this function, we need to suitably fix a branch cut of the square root function. Substituting $w = \omega - k^0$ and applying the formula (3.5) to evaluate the integrals with propagators $(k^2 - i0)^{-1}$, $(k^2 + \Lambda^2 - i0)^{-1}$ in (3.3) gives us

$$\begin{aligned} \langle u_N | \gamma^0 \Sigma^{SE}_\Lambda(\omega_N) | u_N \rangle = -\alpha \int_{C_F} \frac{\mathrm{d}w}{2\pi i} \int \mathrm{d}\vec{k} \; \langle u_N | \alpha^\mu S_0(w) \alpha_\mu | u_N \rangle \\ \times \left( \frac{e^{i\sqrt{(\omega_N - w)^2 + i0}|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} - \frac{e^{i\sqrt{(\omega_N - w)^2 - \Lambda^2 + i0}|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \right) = \Delta\omega_L + \Delta\omega_H. \end{aligned} \tag{3.6}$$

We choose such branches of the square root that it is real and positive on the integration contour $C_F$. The integration contour and the branch cuts are shown in the figure 3.3. Now, let us do the Wick rotation in the $w$ variable. We see that we can deform the contour $C_F$ to the new contour $C$. The portion of $C$ that is aligned on the imaginary axis $C_H$ gives rise to the high-energy part, and the part that encloses the branch cut $C_L$ gives us the low-energy part. Rotated contour is depicted on the figure 3.4.

The reason for this identification is as follows. If we make an approximation that propagators in $C_F$ can be represented as a series in the external sources $A^\mu_{cl}$, then the integral over $C_H$ reduces to the Wick rotated vertex correction diagram with external potential that can be evaluated using standard techniques. The low-energy part is given by an integral over the frequencies up to $\omega_N$, and it is insensitive to the UV physics (it is independent of $\Lambda$) due to the position of branch cuts:

$$\begin{aligned} \Delta\omega^L_N = \frac{\alpha}{\pi} \int_0^{\omega_N} \mathrm{d}w \int \mathrm{d}\vec{k} \; \langle u_N | \alpha^\mu S_0(w + i0) \alpha_\mu | u_N \rangle \\ \times \frac{\sin(\omega_N - w)|\vec{x}-\vec{y}|}{|\vec{x}-\vec{y}|}. \end{aligned} \tag{3.7}$$

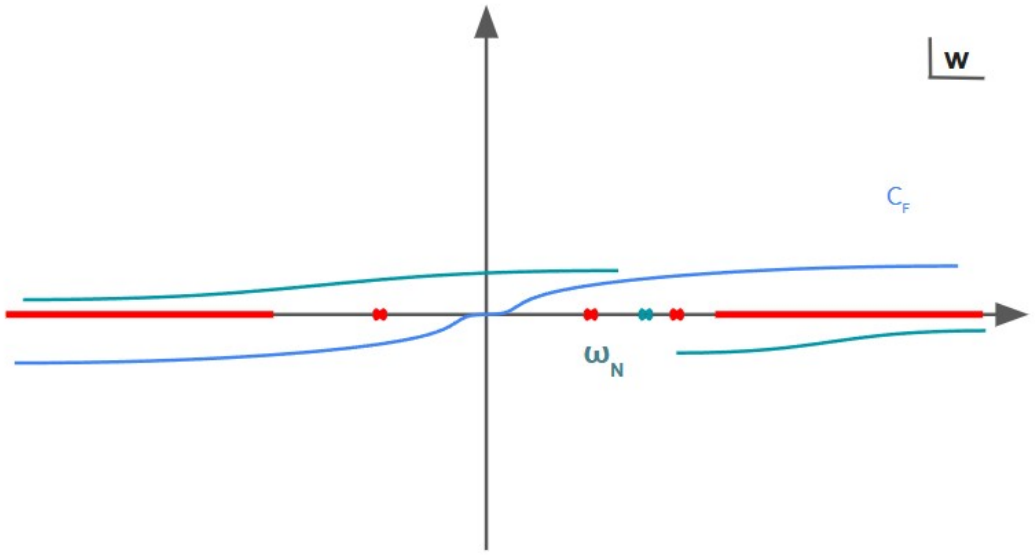


Figure 3.3: Contour of integration with branch cuts depicted.

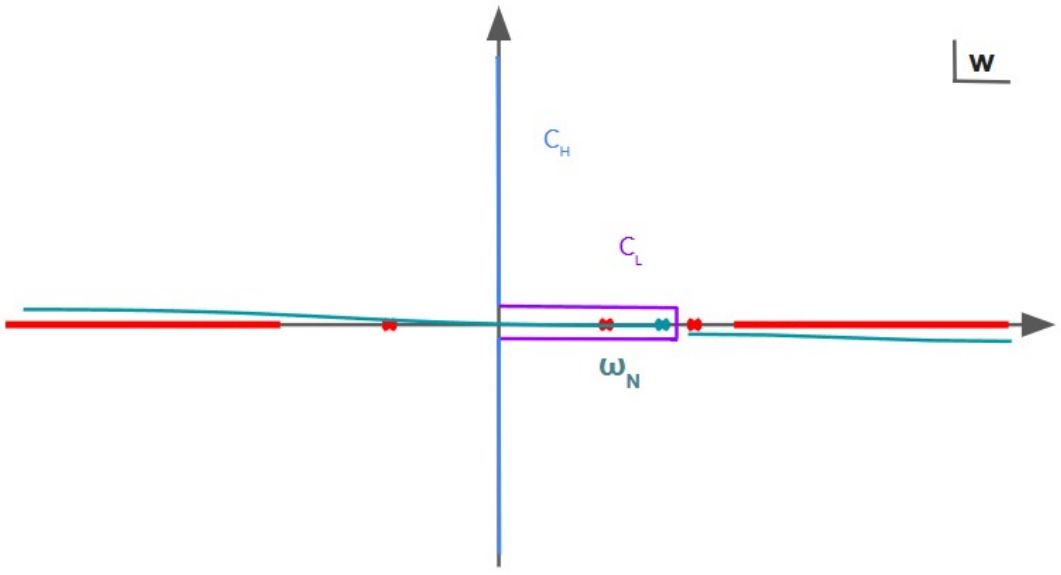


Figure 3.4: Wick rotated contour of integration split into $C_L$ and $C_H$..

Equation 3.7 is the final form of the formula for the non-perturbative contribution to the frequency shift of the state due to the self-energy diagram, to the first order.

# 4 Discussion

We will conclude by saying that the use of the Feschbach-Shur formula makes the derivation more transparent. Also, the splitting of a contour into the low-energy part and the high-energy part makes the separation of the perturbative and bound state contributions easily visible.

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# A Derivation of Feschbach-Schur formula

Consider an invertible operator $H : V \to W$. We make a decomposition of the domain $V = V_1 \oplus V_2$ and of its codomain $W = W_1 \oplus W_2$. Let us write $H$ in a matrix form compatible with this decomposition:

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.1}$$

The direct computation shows that $H$ can be written as a matrix product.

$$H = \begin{bmatrix} 1 & bd \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix}. \tag{1.2}$$

Inverting all matrices we get:

$$H^{-1} = \begin{bmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{bmatrix} \begin{bmatrix} (a - bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix} \begin{bmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{bmatrix}. \tag{1.3}$$

We want to obtain the restriction of the above operator to obtain an operator in $\mathrm{End}(V_1, W_1)$. Let $P_X$ be projection on the subspace $X$. Then we have the final result:

$$P_{V_1} H^{-1} P_{W_1} = \left(a - bd^{-1}c\right)^{-1}. \tag{1.4}$$

Substitution of $H = S(\omega)^{-1} = \mathcal{D} - \omega\ \mathbb{1} - \Sigma(\omega)$ and $V_1 = \mathrm{span}\{u\}$, $W_1 = \mathrm{span}\{\gamma^0 u\}$ gives the equation 2.3.