SCHRÖDINGER OPERATORS WITH SINGULAR BOUNDARY CONDITIONS

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Plan of the talk:

1. One-dimensional Schrödinger operators $-\partial_x^2 + V(x)$.
   J.D., V.Georgescu, also many classic authors.
2. Bessel operators $-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}$.
   L.Bruneau, J.D., V.Georgescu, S.Richard
3. Whittaker operators $-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2} - \frac{\beta}{x}$. J.D., J.Faupin, S.Richard, Q.N.Nguyen
4. Perturbed Bessel operators $-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2} + Q(x)$. J.D., J.Faupin

ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

Suppose that $\mathbb{R}_+ \ni x \mapsto V(x)$ is a function in $L^1_{\rm loc}(\mathbb{R}_+)$ bounded near infinity, possibly complex valued. Consider the one-dimensional Schrödinger operator

$$L := -\partial_x^2 + V(x).$$

We would like to describe closed (if possible self-adjoint) realizations of L on $L^2(\mathbb{R}_+)$. There are two obvious closed realizations: the minimal $L^{\min}$ and the maximal $L^{\max}$ with domains given by

$$\mathcal{D}(L^{\max}) := \big\{ f \in L^2(\mathbb{R}_+) \quad | \;\; Lf \in L^2(\mathbb{R}_+) \big\},$$
$$\mathcal{D}(L^{\min}) := \text{the closure of } \{ f \in \mathcal{D}(L^{\max}) \;\; | \;\; f = 0 \text{ near } 0 \},$$

the closure taken with respect to the graph norm of $L^{\max}$.

One can show that $\dim \mathcal{D}(L^{\max})/\mathcal{D}(L^{\min})$ is either 0 or 2.

In the latter case there exists a one-parameter family of operators $L^{\bullet}$ that satisfy $L^{\min} \subset L^{\bullet} \subset L^{\max}$ defined by boundary conditions (b.c.) near zero.

If $V \in L^1$ near zero, these b.c are easy to describe:

$$\mathcal{D}(L_\kappa) := \left\{ f \in \mathcal{D}(L^{\max}) \;\mid\; f(0) = \kappa f'(0) \right\}.$$

$\kappa = 0$ is called the Dirichlet b.c., $\kappa = \infty$ the Neumann b.c., the remaining are mixed or Robin b.c.. If $V \notin L^1$ near zero, the situation is more complicated.

In most of the literature it is assumed that V is real. Then the operator L with the domain $C_{\rm c}^\infty(\mathbb{R}_+)$ is Hermitian (symmetric), but not necessarily self-adjoint. The closure of L with this domain coincides with $L^{\min}$, its adjoint L^* coincides with $L^{\max}$. If $L^{\min} \neq L^{\max}$, then self-adjoint extensions of L are in between $L^{\min}$ and $L^{\max}$.

One can apply von Neumann's method of defining self-adjoint extentions. One looks for eigenvectors in $L^2(\mathbb{R}_+)$ of

$$L^* f_\pm = \pm {\rm i} f_\pm.$$

If $\dim \mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = 2$, Then

$$\mathcal{D}(L_\alpha) = \mathcal{D}(L^{\min}) + \mathbb{C}({\rm e}^{{\rm i}\alpha} f_+ + {\rm e}^{-{\rm i}\alpha} f_-)$$

gives all self-adjoint extensions of L. Not an optimal approach—it requires solving an unnecessarily difficult eigenvalue problem.

Let us go back to a possibly complex V.

Let $\mathcal{N}(L-\lambda)$ denote the space of eigenfunctions of L with eigenvalue $\lambda \in \mathbb{C}$ (not necessarily square integrable). One can show the alternative

$$\begin{aligned} &\dim \mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = 0 \\ \Leftrightarrow &\dim \left\{ f \in \mathcal{N}(L-\lambda) \ \middle|\ f \in L^2 \text{ near } 0 \right\} \leq 1, \quad \forall \lambda \in \mathbb{C}; \\ &\dim \mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = 2 \\ \Leftrightarrow &\dim \left\{ f \in \mathcal{N}(L-\lambda) \ \middle|\ f \in L^2 \text{ near } 0 \right\} = 2, \quad \forall \lambda \in \mathbb{C}. \end{aligned}$$

Usually, the following approach to define closed extensions of $L^{\min}$ is much more convenient than von Neumann's:

Fix λ (e.g. $\lambda = 0$). Choose a cutoff function $\xi \in C^\infty(\mathbb{R}_+)$ equal 1 near 0 and 0 near ∞. For $h \in \mathcal{N}(L-\lambda)$ set

$$\mathcal{D}(L_\bullet) := \mathcal{D}(L^{\min}) + \mathbb{C}\xi h.$$

Note that $L_\bullet$ does not depend on the cutoff ξ.

Introduce the Wronskian of two functions h_1 and h_2:

$$\mathcal{W}(h_1, h_2, x) = h_1(x)h_2'(x) - h_1'(x)h_2(x)$$

Note that for $h_1, h_2 \in \mathcal{N}(L - \lambda)$, the Wronskian $\mathcal{W}(h_1, h_2, x)$ does not depend on x. Thus it defines a symplectic form on $\mathcal{N}(L - \lambda)$.

Moreover, for $f, g \in \mathcal{D}(L^{\max})$ the Wronskian at 0 is well-defined:

$$\lim_{x \searrow 0} \mathcal{W}(f, g; x) =: \mathcal{W}(f, g; 0)$$

exists and defines a continuous bilinear form on $\mathcal{D}(L^{\max})$.

Here is an even better approach to defining close realizations of L:

$$\mathcal{D}(L_\bullet) = \{f \in \mathcal{D}(L^{\max}) \mid \mathcal{W}(h_0, f; 0) = 0\},$$

where h_0 is an approximate eigenfunction of L.

Later we will need a few integral kernels naturally associated with the operator L. Let $h_1, h_2 \in \mathcal{N}(L-\lambda)$ be linearly independent. The canonical bisolution of $L-\lambda$ defined by the integral kernel

$$G_{\leftrightarrow}(\lambda;x,y) = \frac{1}{\mathcal{W}(h_1,h_2)}\big(h_1(x)h_2(y) - h_2(x)h_1(y)\big),$$

does not depend on the choice of h_1, h_2. $G_{\leftrightarrow}(\lambda)$ is usually unbounded on $L^2(\mathbb{R}_+)$. It satisfies

$$\big(-\partial_x^2 + V(x) - \lambda\big)G_{\leftrightarrow}(\lambda;x,y) = 0.$$

We will use the term Green's operator as a synonym for a right inverse of $L-\lambda$ (not necessarily bounded). In other words, the integral kernel $G_{\bullet}(\lambda;x,y)$ of Green's operator $G_{\bullet}(\lambda)$ satisfies

$$\big(-\partial_x^2 + V(x) - \lambda\big)G_{\bullet}(\lambda;x,y) = \delta(x-y).$$

We have various types of Green's operators:

1. the forward Green's operator

$$G_{\rightarrow}(\lambda; x, y) := \theta(x-y)G_{\leftrightarrow}(\lambda; x, y),$$

2. the backward Green's operator

$$G_{\leftarrow}(\lambda; x, y) := -\theta(y-x)G_{\leftrightarrow}(\lambda; x, y).$$

3. Choose $h_1, h_2 \in \mathcal{N}(L-\lambda)$. The two-sided Green's operator corresponding to b.c. near 0 given by h_1, resp. near ∞ given by h_2:

$$G_{\bullet}(\lambda; x, y) := \frac{h_1(x)h_2(y)\theta(x-y) + h_2(x)h_1(y)\theta(y-x)}{\mathcal{W}(h_1, h_2)}$$

Suppose we have two Schrödinger operators

$$L^0 = -\partial_x^2 + V^0(x),$$
$$L = -\partial_x^2 + V(x) = L^0 + Q(x).$$

Suppose we know an eigenfunction of the unperturbed operator $u^0 \in \mathcal{N}(L^0 - \lambda)$. Then one can try to construct an eigenfunction of the perturbed operator $u \in \mathcal{N}(L - \lambda)$ by applying

$$u = \left(\mathbb{1} + G^0_\bullet(\lambda)Q\right)^{-1} u^0,$$

where $G^0_\bullet(\lambda)$ is one of the Green's operators of $L^0 - \lambda$. This is a generalization of the Lippmann-Schwinger equation well-known from Quantum Mechanics.

BESSEL OPERATORS

One of the most important families of exactly solvable 1-dimensional Schrödinger operators is the family of Bessel operators

$$-\partial_x^2 + \frac{\mathrm{c}}{x^2}.$$

We will allow c to be complex.

As is well-known, it is convenient to set $\mathrm{c} = m^2 - \frac{1}{4}$, so that the Bessel operator is rewritten as

$$L_{m^2}^0 := -\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2}.$$

We will often assume that $\mathrm{Re}(m) \geq 0$, because $L_{m^2}^0$ depends only on m^2.

Many operators in mathematics and physics can be reduced to Bessel operators:

1. the usual Laplacian on the halfline, Dirichlet $m = \frac{1}{2}$, Neumann $m = -\frac{1}{2}$;
2. the usual Laplacian in dimension $d \geq 3$, $m = \frac{d}{2} - 1 + \ell$, $\ell \in \mathbb{N}_0$;
3. the 2d Aharonov-Bohm Hamiltonian with flux θ, $m = \frac{\theta}{2\pi} + n$, $n \in \mathbb{Z}$,
4. the Laplacian on a conical surface of angle α, $m = \frac{2\pi n}{\alpha}$, $n \in \mathbb{Z}$;
5. the Laplacian on a wedge of angle α with Dirichlet or Neumann b.c., $m = \frac{\pi n}{\alpha}$, $n \in \mathbb{Z}$;
6. perturbed Bessel operators with m complex define Regge poles,
7. three-body systems with contact interactions.
8. generators of $sl(2, \mathbb{R})$.

The zero energy eigenvalue problem $L^0_{m^2} f = 0$ is easy:

$$x^{\frac{1}{2}+m}, \quad x^{\frac{1}{2}-m}, \quad m \neq 0; \qquad x^{\frac{1}{2}}, \quad x^{\frac{1}{2}}\ln(x), \quad m = 0.$$

One can distinguish 3 regimes with a different behavior of eigensolutions:

1. $\mathrm{Re}(m) > 0$. Eigensolutions of $L^0_{m^2}$ can be divided into principal, and non-principal, all the others. Principal solutions behave as $x^{\frac{1}{2}+m}$ and are more regular than non-principal ones, which behave as $x^{\frac{1}{2}-m}$.
2. $\mathrm{Re}(m) = 0$, $m \neq 0$. Eigensolutions of $L^0_{m^2}$ are spanned by eigensolutions with a comparable oscillating behavior $x^{\frac{1}{2}+m}$ and $x^{\frac{1}{2}-m}$ near zero.
3. $m = 0$. Eigensolutions of $L^0_{m^2}$ are spanned by a principal solution behaving like x^0. All others are non-principal, behave like $x^0 \ln(x)$, and are less regular.

Let us now sketch the theory of closed realizations of $L_{m^2}^0$ on $L^2(\mathbb{R}_+)$. First of all, the minimal and maximal realization of $L_{m^2}^0$ denoted by $L_{m^2}^{0,\min}$ resp. $L_{m^2}^{0,\max}$, satisfy

$$
\begin{aligned}
|\mathrm{Re}(m)| \geq 1 \text{ implies} \quad & L_{m^2}^{0,\min} = L_{m^2}^{0,\max}, \\
|\mathrm{Re}(m)| < 1 \text{ implies} \quad & \dim \mathcal{D}(L_{m^2}^{0,\max})/\mathcal{D}(L_{m^2}^{0,\min}) = 2.
\end{aligned}
$$

Thus for $|\mathrm{Re}(m)| < 1$ there exists a 1-parameter family of closed realisations of $L_{m^2}^0$ between $L_{m^2}^{0,\min}$ and $L_{m^2}^{0,\max}$ defined by b.c. at zero. To describe all closed realizations of Bessel operators one can introduce the following three holomorphic families of operators:

$$
\begin{aligned}
\{-1 < \mathrm{Re}(m)\} \ni m &\mapsto H_m^0, \\
\{-1 < \mathrm{Re}(m) < 1\} \times \big(\mathbb{C} \cup \{\infty\}\big) \ni (m,\kappa) &\mapsto H_{m,\kappa}^0, \\
\big(\mathbb{C} \cup \{\infty\}\big) \ni \nu &\mapsto H_0^{0,\nu}.
\end{aligned}
$$

The family H_m^0 is the most basic one. It is called homogeneous. For $1 \leq \mathrm{Re}(m)$ it is the unique closed realization of $L_{m^2}^0$. Then it is extended to the strip $-1 < \mathrm{Re}(m) < 1$ by analytic continuation. Its domain is defined by the boundary condition $\sim x^{\frac{1}{2}+m}$ at zero.

The operator $H_{m,\kappa}^0$ is defined by

$$\begin{aligned}\mathcal{D}(H_{m,\kappa}^0) &= \mathcal{D}(L_{m^2}^{\min}) + \mathbb{C}(x^{\frac{1}{2}+m} + \kappa x^{\frac{1}{2}-m})\xi \\ &= \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathcal{W}(f, x^{\frac{1}{2}+m} + \kappa x^{\frac{1}{2}-m}; 0) = 0\}.\end{aligned}$$

It is holomorphic except for a singularity at $(m, \kappa) = (0, -1)$.

Finally, for the special case $m = 0$, $H_0^{0,\nu}$ is defined by

$$\begin{aligned}\mathcal{D}(H_m^{0,\nu}) &= \mathcal{D}(L_{m^2}^{\min}) + \mathbb{C}(x^{\frac{1}{2}}\ln(x) + \nu x^{\frac{1}{2}})\xi \\ &= \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathcal{W}(f, x^{\frac{1}{2}}\ln(x) + \nu x^{\frac{1}{2}}; 0) = 0\}.\end{aligned}$$

Let U_τ be the group of dilations:

$$(U_\tau f)(x) = \mathrm{e}^{\tau/2} f(\mathrm{e}^\tau x).$$

For any m with $|\mathrm{Re}(m)| < 1$ and any $\kappa, \nu \in \mathbb{C} \cup \{\infty\}$, we have

$$U_\tau H_{m,\kappa} U_{-\tau} = \mathrm{e}^{-2\tau} H_{m,\mathrm{e}^{-2\tau m}\kappa},$$
$$U_\tau H_0^\nu U_{-\tau} = \mathrm{e}^{-2\tau} H_0^{\nu+\tau}.$$

In particular, only

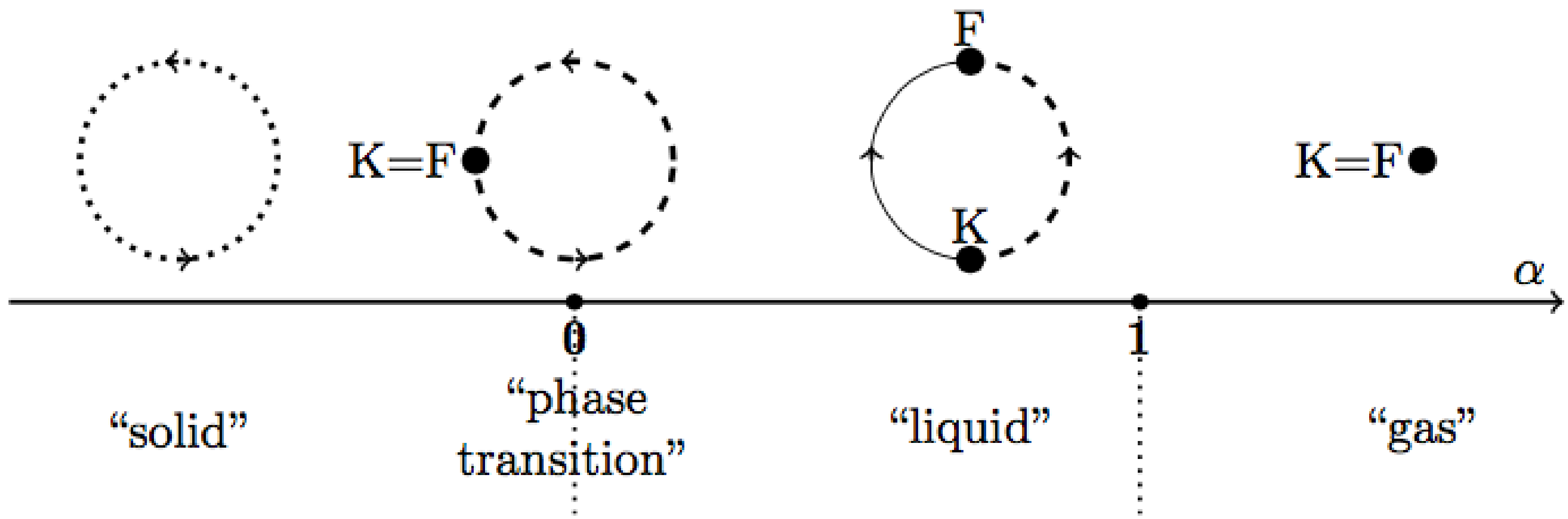
$H_{m,0} = H_m, \quad H_{m,\infty} = H_{-m}, \quad H_0^\infty = H_0$ are homogeneous (of degree 2).

If A is an operator, then the transformation $R_\tau(A) := \mathrm{e}^{2\tau} U_\tau A U_{-\tau}$. will be called the renormalization group action. Operator is homogeneous iff it is its fixed point.

Self-adjoint extensions of the Hermitian operator

$$L_\alpha = -\partial_x^2 + \big(-\frac{1}{4} + \alpha\big)\frac{1}{x^2}.$$

K—Krein, F—Friedrichs, dashed line—single bound state, dotted line—infinite sequence of bound states.



For $k \neq 0$ the eigenvalue problem $L_{m^2}^0 f = k^2 f$ reduces to the Bessel equation. $\mathcal{N}(L_{m^2}^0 + k^2)$ is spanned by

$$u_m^0(x,k) := \left(\frac{2}{k}\right)^m \sqrt{x} I_m(kx), \quad u_m^0(x) \simeq \frac{x^{\frac{1}{2}+m}}{\Gamma(m+1)} \quad \text{near } 0;$$
$$v_m^0(x,k) := \left(\frac{k}{2}\right)^m \sqrt{x} K_m(kx), \qquad \text{decaying exponentially}$$

where I_m is the modified Bessel function and K_m the Macdonald function.

u_m^0 is principal for $\mathrm{Re}(m) > 0$ and for $m = 0$.

v_m^0 is always non-principal.

u_m^0 for $\mathrm{Re}(m) < 0$ and $m \neq -1, -2, \dots$ is a distinguished non-principal eigenfunction. (For $m = -1, -2, \dots$ it is proportional to u_{-m}^0).

We have

$$u_m^0(x,0) = \frac{x^{\frac{1}{2}+m}}{\Gamma(m+1)},$$

$$v_m^0(x,0) = \frac{\Gamma(m)x^{\frac{1}{2}-m}}{2}, \quad \mathrm{Re}(m) \geq 0,\ m \neq 0.$$

It is convenient to introduce another eigensolution

$$w_m^0(x,k) = w_{-m}^0(x,k) \sim \mathrm{e}^{-kx} \qquad x \to \infty,$$

which differs from $v_m^0(x,k)$ only by a different normalization:

$$w_m^0(x,k) = \sqrt{\frac{2k}{\pi}}\left(\frac{2}{k}\right)^m v_m^0(x,k) = \sqrt{\frac{2xk}{\pi}} K_m(kx).$$

The Bessel operator for $m = 0$ often needs a separate treatment. This case is actually very important – it corresponds to the 2-dimensional Laplacian in the s-wave sector.

For instance, $v_0^0(\cdot, k)$ does not have a limit at $k = 0$. To treat the case $m = 0$ in a satisfactory way it is useful to introduce a family of non-principal eigenfunctions of L_0^0:

$$p_0^0(x,k) := -v_0^0(x,k) - \Big(\ln\Big(\frac{k}{2}\Big) + \gamma\Big) u_0^0(x,k),$$

where γ denotes Euler's constant. At $k = 0$ it coincides with the logarithmic solution:

$$p_0^0(x,0) = x^{\frac{1}{2}} \ln(x).$$

As explained in the previous subsection, with $L^0_{m^2} + k^2$ one can associate various Green's operators: The most important are

1. the forward Green's operator $G^0_{m^2,\rightarrow}(-k^2)$;
2. the backward Green's operator $G^0_{m^2,\leftarrow}(-k^2)$;
3. the two-sided Green's operator with homogeneous boundary conditions, $G^0_{m^2,\bowtie}(-k^2)$, using u^0_m and v^0_m, for brevity often called two-sided;
4. for $m = 0$, additionally, the two-sided Green's operator logarithmic near zero $G^0_{0,\diamond}(-k^2)$, using u^0_0, p^0_0, for brevity called logarithmic.

For $\mathrm{Re}(m) > -1$, $\mathrm{Re}(k) > 0$ the two-sided Green's operator with homogeneous b.c. is bounded on $L^2(\mathbb{R}_+)$ and is with the resolvent of H^0_m:

$$G^0_{m^2,\bowtie}(-k^2) = \left(H^0_m + k^2\right)^{-1}.$$

However, the integral kernel $G^0_{m^2,\bowtie}(-k^2; x, y)$ is well defined and useful also for other values of k and m, when it does not define a bounded operator.

THE WHITTAKER OPERATOR

The radial Schrödinger operator with the Coulomb potential

$$L_{\beta,m^2} := -\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} - \frac{\beta}{x}$$

will be called the Whittaker operator. It is an example of an perturbed Bessel operator.

It has a distinguished family of closed realizations

$$\mathbb{C} \times \{m \in \mathbb{C} \mid \mathrm{Re}(m) > -1\} \ni (\beta, m) \mapsto H_{\beta,m}\,,$$

holomorphic except for a singularity at $(0, -\frac{1}{2})$. We will call them pure. They are generalizations of the homogeneous family H_m to the Whittaker case. In particular, for $\beta = 0$ they coincide with Bessel operators:

$$H_{0,m} = H_m.$$

If $\mathrm{Re}(m) \geq 1$ the boundary condition is not needed. For $\mathrm{Re}(m) > -1$ $H_{\beta,m}$ can be defined by analytic continuation.

Alternatively, if $\mathrm{Re}(m) \geq -\frac{1}{2}$, we can use a simplified boundary condition:

$$\mathcal{D}(H_{\beta,m}) = \{f \in \mathcal{D}(L_{\beta,m^2}^{\max}) \mid \mathcal{W}(f, x^{\frac{1}{2}+m}; 0) = 0\}.$$

For all $\mathrm{Re}(m) > -1$ pure boundary conditions are defined by

$$\mathcal{D}(H_{\beta,m}) = \Big\{f \in \mathcal{D}(L_{\beta,m^2}^{\max}) \mid \mathcal{W}\Big(f, x^{\frac{1}{2}+m}\big(1 - \tfrac{\beta}{1+2m}x\big); 0\Big) = 0\Big\}.$$

The singularity at $(\beta, m) = (0, -\frac{1}{2})$ is quite curious: it is invisible when we consider just the variable m. In fact, the Bessel operator

$$\{\mathrm{Re}(m) > -1\} \ni m \mapsto H_m = H_{0,m}$$

is holomorphic.

$H_{-\frac{1}{2}}$ is the Laplacian with the Neumann boundary condition;
$H_{\frac{1}{2}}$ is the Laplacian with the Dirichlet boundary condition.
Thus one has

$$H_{0,-\frac{1}{2}} \neq H_{0,\frac{1}{2}}.$$

If we introduce the Coulomb potential, then

$$\text{whenever } \beta \neq 0, \quad H_{\beta,-\frac{1}{2}} = H_{\beta,\frac{1}{2}}.$$

The function

$$(\beta, m) \mapsto H_{\beta,m} \qquad \text{is holomorphic around } (0, \tfrac{1}{2}).$$

In particular, in the sense of the strong resolvent limit

$$\lim_{\beta \to 0} H_{\beta,\frac{1}{2}} = H_{0,\frac{1}{2}}.$$

But

$$\lim_{\beta \to 0} H_{\beta,-\frac{1}{2}} = H_{0,\frac{1}{2}} \neq H_{0,-\frac{1}{2}}$$

Thus $\quad (\beta, m) \mapsto H_{\beta,m} \qquad$ is not even continuous near $(0, -\frac{1}{2})$.

PERTURBED BESSEL OPERATORS

Consider now $m \in \mathbb{C}$ and complex $Q \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$

$$L_{m^2} := -\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} + Q(x).$$

Proposition 1. *Let* $\mathrm{Re}(m) \geq 0$, $k \in \mathbb{C}$ *and suppose that*

$$\int_0^1 x|Q(x)|\mathrm{d}x < \infty, \quad \textit{if } m \neq 0;$$

$$\int_0^1 x(1 + |\ln(x)|)|Q(x)|\mathrm{d}x < \infty, \quad \textit{if } m = 0.$$

Suppose that $g^0 \in \mathcal{N}(L^0_{m^2} + k^2)$ *such that* $g^0(x) = \mathcal{O}(x^{\frac{1}{2}+\mathrm{Re}(m)})$ *near* 0. *Then, there exists a unique* $g \in \mathcal{N}(L_{m^2} + k^2)$ *such that,*

$$g(x) - g^0(x) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}),$$

$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)}), \quad x \to 0.$$

If we want to well approximate all unperturbed solutions, including the more singular ones, we need to strengthen the assumption on the perturbation.

Proposition 2. *Let* $\mathrm{Re}(m) \geq 0$, $k \in \mathbb{C}$ *and suppose that*

$$\int_0^1 x^{1-2\mathrm{Re}(m)} |Q(x)| \mathrm{d}x < \infty, \quad \text{if } m \neq 0;$$
$$\int_0^1 x\big(1 + (\ln(x))^2\big) |Q(x)| \mathrm{d}x < \infty, \quad \text{if } m = 0.$$

Suppose that $g^0 \in \mathcal{N}(L^0_{m^2} + k^2)$. *Then, there exists a unique* $g \in \mathcal{N}(L_{m^2} + k^2)$ *such that*

$$g(x) - g^0(x) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}),$$
$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)}), \quad x \to 0.$$

Here are consequences of Propositions 1 and 2:

Corollary 3. *Let $m \in \mathbb{C}$, $k \in \mathbb{C}$ and suppose that*

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x < \infty, \quad \varepsilon \geq 0, \quad \mathrm{Re}(m) \geq -\frac{\varepsilon}{2}, \quad m \neq 0;$$

$$\int_0^1 x(1+|\ln(x)|)|Q(x)|\mathrm{d}x < \infty, \quad m = 0.$$

Then there exists a unique $u_m(\cdot, k) \in \mathcal{N}(L_{m^2} + k^2)$ that satisfies

$$u_m(x,k) - u_m^0(x,k) = o(x^{\frac{1}{2}+|\mathrm{Re}(m)|}),$$

$$\partial_x u_m(x,k) - \partial_x u_m^0(x,k) = o(x^{-\frac{1}{2}+|\mathrm{Re}(m)|}), \quad x \to 0.$$

Note that if $|Q(x)| \lesssim |x|^\alpha$ near 0, then the above condition is satisfied for $\alpha > -2 + \varepsilon$.

Corollary 4. *Let* $k \in \mathbb{C}$ *and suppose that*

$$\int_0^1 x(1 + (\ln(x))^2)|Q(x)|\mathrm{d}x < \infty, \quad m = 0.$$

Then there exists a unique $p_0(\cdot, k) \in \mathcal{N}(L_{m^2} + k^2)$ *such that*

$$\begin{aligned} p_0(x, k) - p_0^0(x, k) &= o(x^{\frac{1}{2}}), \\ \partial_x p_0(x, k) - \partial_x p_0^0(x, k) &= o(x^{-\frac{1}{2}}), \quad x \to 0. \end{aligned}$$

Conditions of Proposition 1 are the minimal assumptions near zero for our purposes. They guarantee the existence of distinguished eigenfunctions u_m with $\mathrm{Re}(m) \geq 0$, principal for $\mathrm{Re}(m) > 0$ and $m = 0$. They also imply that the behavior near zero of non-principal eigenfunctions is roughly as in the unperturbed case:

Proposition 5. *Let* $\mathrm{Re}(m) \geq 0$, $\mathrm{Re}(k) \geq 0$. *Under the assumptions of Proposition 1 for all* $g \in \mathcal{N}(L_{m^2} + k^2)$, *we have*

$$g(x) = \mathcal{O}(x^{\frac{1}{2} - \mathrm{Re}(m)}), \qquad \partial_x g(x) = \mathcal{O}(x^{-\frac{1}{2} - \mathrm{Re}(m)}),$$
$$g(x) = \mathcal{O}(x^{\frac{1}{2}} \ln(x)), \qquad \partial_x g(x) = \mathcal{O}(x^{-\frac{1}{2}} \ln(x)), \qquad x \to 0.$$

It seems that to have distinguished non-principal eigenfunctions one needs to impose stronger conditions on Q. In particular, under the conditions of Corollary 3 u_m is constructed only in the region $\mathrm{Re}(m) \geq -\frac{\varepsilon}{2}$. This suggests the following question:

Open Problem 6. *Let Q satisfy the condition of Corollary 3. Does it imply that the function $m \mapsto u_m(\cdot, k)$ extends holomorphically (or at least meromorphically) to the whole $\mathbb{C}$? (This is true for the Coulomb potential, see S.Richard, J.D.).*

Let us now consider the behavior near infinity. To prove the existence of solutions well approximating exponentially decaying solutions, called Jost solutions, we need the so-called short-range condition on the potential.

Proposition 7. *Let $m \in \mathbb{C}$. Suppose that*

$$\int_1^\infty |Q(x)| \mathrm{d}x < \infty.$$

Let $k \neq 0$ be such that $\mathrm{Re}(k) \geq 0$. Then there exists a unique $w_m(\cdot, k) = w_{-m}(\cdot, k) \in \mathcal{N}(L_{m^2} + k^2)$ such that

$$\begin{aligned} w_m(x,k) - w_m^0(x,k) &= o(\mathrm{e}^{-x\mathrm{Re}(k)}), \\ \partial_x w_m(x,k) - \partial_x w_m^0(x,k) &= o(\mathrm{e}^{-x\mathrm{Re}(k)}), \qquad x \to \infty. \end{aligned}$$

Proposition 7 does not cover the zero energy, that is, $k = 0$.

Proposition 8. *Let $m \in \mathbb{C}$. Suppose that*

$$\int_1^\infty x^\delta |Q(x)| \mathrm{d}x < \infty, \qquad \textit{if } m \neq 0, \quad \textit{with } \delta = 1 + 2\max\big(\mathrm{Re}(m), 0\big);$$

$$\int_1^\infty x(1 + \ln(x))|Q(x)| \mathrm{d}x < \infty, \qquad \textit{if } m = 0.$$

Then there exists a unique $q_m \in \mathcal{N}(L_{m^2})$ such that

$$q_m(x) - x^{\frac{1}{2}+m} = o(x^{\frac{1}{2}-\mathrm{Re}(m)}),$$

$$\partial_x q_m(x) - \partial_x x^{\frac{1}{2}+m} = o(x^{-\frac{1}{2}-\mathrm{Re}(m)}), \quad x \to \infty.$$

Proposition 9. *Let $m = 0$. Suppose that*

$$\int_1^\infty x(1 + (\ln(x))^2)|Q(x)|\mathrm{d}x < \infty.$$

Then there exists a unique $q_{0,\ln} \in \mathcal{N}(L_0)$ such that

$$\begin{aligned} q_{0,\ln}(x) - x^{\frac{1}{2}}\ln(x) &= o(x^{\frac{1}{2}}), \\ \partial_x q_{0,\ln}(x) - \partial_x x^{\frac{1}{2}}\ln(x) &= o(x^{-\frac{1}{2}}), \quad x \to \infty. \end{aligned}$$

The zero energy eigenequation near infinity is equivalent to the zero energy eigenequation near zero:

$$-\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} + Q(x) = y^3\Big(- \partial_y^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{y^2} + \tilde{Q}(y)\Big)y,$$

$$y = \frac{1}{x}, \qquad \tilde{Q}(y) := y^{-4}Q(y^{-1}).$$

Note also a simple relationship between the integral conditions near zero on Q and near infinity on $\tilde{Q}$:

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x = \int_1^\infty y^{1+\varepsilon}|\tilde{Q}(y)|\mathrm{d}y,$$

$$\int_0^1 x(1 + |\ln(x)|^\alpha)|Q(x)|\mathrm{d}x = \int_1^\infty y(1 + |\ln(y)|^\alpha)|\tilde{Q}(y)|\mathrm{d}y.$$

Thus one can derive Propositions 8 and 9 from the $k = 0$ case of Corollaries 3 and 4.

The main tools used in the construction of eigenfunctions are various Green's operators for the unperturbed Bessel operator. The forward Green's operator is used in Propositions 1, 2 and their corollaries. For instance,

$$u_m(\cdot,k) = \big(\mathbb{1} + G^0_{m^2,\rightarrow}(-k^2)Q\big)^{-1} u^0_m(\cdot,k),$$
$$p_0(\cdot,k) = \big(\mathbb{1} + G^0_{0,\rightarrow}(-k^2)Q\big)^{-1} p^0_0(\cdot,k).$$

The backward Green's operator is used in Propositions 7, 8 and 9:

$$w_m(\cdot,k) = \big(\mathbb{1} + G^0_{m^2,\leftarrow}(-k^2)Q\big)^{-1} w^0_m(\cdot,k),$$
$$q_m = \big(\mathbb{1} + G^0_{m^2,\leftarrow}(0)Q\big)^{-1} u^0_m(\cdot,0),$$
$$q_{0,\ln} = \big(\mathbb{1} + G^0_{0,\leftarrow}(0)Q\big)^{-1} p^0_0(\cdot,0).$$

In quantum physics the equation for the Jost solution $w(\cdot,k)$ is called the Lippmann–Schwinger Equation.

If assumptions of Proposition 1 holds and $\frac{\varepsilon}{2} < \mathrm{Re}(m)$, then Corollary 3 guarantees the existence only of $u_m(\cdot, k)$, but not of $u_{-m}(\cdot, k)$. Therefore, in this case it is more complicated to describe non-principal solutions. One way to do this is to use compressed two-sided Green's operators:

Proposition 10. *Suppose the assumptions of Proposition 1 hold. If a is small enough, the following functions are well defined and belong to $\mathcal{N}(L_{m^2} + k^2)$ on $]0, a[$:*

$$u_{-m}^{\bowtie(a)}(\cdot, k) := \left(\mathbb{1} + G_{m,\bowtie}^{0(a)}(-k^2)Q\right)^{-1} u_{-m}^{0}(\cdot, k),$$
$$p_0^{\diamond(a)}(\cdot, k) := \left(\mathbb{1} + G_{0,\diamond}^{0(a)}(-k^2)Q\right)^{-1} p_0^{0}(\cdot, k).$$

Here,

$$G_\bullet^{(a)}(x, y) = \mathbb{1}_{[0,a]}(x) G_\bullet(x, y) \mathbb{1}_{[0,a]}(y).$$

Unfortunately, the construction of Proposition 10 involves inverting a complicated integral operator. Alternatively, choose a non-negative integer n, expand the denominator into a power series retaining n first terms, fix $a = 1$ (quite arbitrarily) and set

$$u_{-m}^{0[n]}(x,k) = \sum_{j=0}^{n} (-G_{\bowtie}^{0(1)}(0)Q)^j u_{-m}^0(x,k).$$

Proposition 11. *Let* $\mathrm{Re}(k) \geq 0$. *Let* $n \in \mathbb{N}$ *such that*

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x < \infty, \quad \varepsilon \geq 0,$$

is satisfied for $-\frac{\varepsilon}{2}(n+1) \leq \mathrm{Re}(-m) \leq 0$. *Then there exists a unique* $u_{-m}^{[n]}(\cdot,k) \in \mathcal{N}(L_{m^2}+k^2)$ *such that*

$$u_{-m}^{[n]}(x,k) - u_{-m}^{0[n]}(x,k) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}),$$
$$\partial_x u_{-m}^{[n]}(x,k) - \partial_x u_{-m}^{0[n]}(x,k) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)}), \quad x \to 0.$$

Boundary conditions determined by $u_{-m}^{0[n]}(\cdot, k)$ still have an unpleasant feature – they depend on k. If we want to have boundary conditions independent of k we need to assume that $|\mathrm{Re}(m)| < 1$. Then it is reasonable to choose $k = 0$, which we do setting

$$u_{-m}^{0[n]}(x) := u_{-m}^{0[n]}(x, 0). \tag{1}$$

In particular, under the condition $|\mathrm{Re}(m)| < 1$ in Proposition 11 we can replace $u_{-m}^{0[n]}(\cdot, k)$ with $u_{-m}^{0[n]}(\cdot)$.

We have seen the condition $|\mathrm{Re}(m)| < 1$ already in the L^2 theory of Bessel operators.

An important object of our analysis is the Jost function $\mathcal{W}_m(k)$, that is the Wronskian of the two main solutions $u_m(\cdot, k)$ and $v_m(\cdot, k)$.

Proposition 12. *Assume* $\mathrm{Re}(m) > -1$, *as well as assumptions of Corollary 3. Then*

$$\lim_{|k|\to\infty} \mathcal{W}_m(k) = 1, \quad \mathrm{Re}(k) \geq 0. \tag{2}$$

Note the assumption $\mathrm{Re}(m) > -1$ that appears in the above proposition—again anticipating the basic condition needed in the L^2 analysis.

Let us now dscribe close realizations of perturbed Bessel operators. As usual, we can introduce the minimal and maximal Bessel perturbed Bessel operator $L_{m^2}^{\min}$ and $L_{m^2}^{\max}$. Under the assumptions of Propositions 1 the basic picture is the same as in the unperturbed case:

$$|\mathrm{Re}(m)| \geq 1 \text{ implies } \quad L_{m^2}^{\min} = L_{m^2}^{\max},$$
$$|\mathrm{Re}(m)| < 1 \text{ implies } \quad \dim \mathcal{D}(L_{m^2}^{\max})/\mathcal{D}(L_{m^2}^{\min}) = 2.$$

In particular, for $|\mathrm{Re}(m)| < 1$, beside the minimal and maximal realizations, there exists a 1-parameter family of closed realizations of L_{m^2} defined by b.c. at zero. They can be fixed by specifying continuous linear functionals on $\mathcal{D}(L_{m^2}^{\max})$ vanishing on $\mathcal{D}(L_{m^2}^{\min})$, called boundary functionals, forming the boundary space

$$\mathcal{B}_{m^2} := \big(\mathcal{D}(L_{m^2}^{\max})/\mathcal{D}(L_{m^2}^{\min})\big)', \qquad \text{where the prime denotes the dual.}$$

As we discussed in the general theory, boundary functionals can be efficiently described by the Wronskian at zero with an eigenfunctions of L_{m^2}. Zero-energy eigenfunctions are the simplest. In practice we can use approximate ones.

One can ask about distinguished bases of the boundary space. Under the assumptions of Proposition 1 we have the principal boundary functional, which for $0 \leq \mathrm{Re}(m) < 1$ can be defined as $\mathcal{W}(x^{\frac{1}{2}+m}, \cdot; 0)$. There are also non-principal boundary functionals, which lead to boundary conditions roughly of the type $x^{\frac{1}{2}-m}$ for $m \neq 0$, or $x^{\frac{1}{2}}\ln(x)$ for $m = 0$.

Let us now impose the assumption

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x < \infty.$$

If $2 > \varepsilon > 0$, then for $0 \leq \mathrm{Re}(m) \leq \varepsilon/2$ we have a distinguished non-principal boundary functional given by $\mathcal{W}(x^{\frac{1}{2}-m}, \cdot; 0)$ if $m \neq 0$ and $\mathcal{W}(x^{\frac{1}{2}}\ln(x), \cdot; 0)$ if $m = 0$. We obtain three families of perturbed Bessel operators

$$\Big\{ -\frac{\varepsilon}{2} < \mathrm{Re}(m) \Big\} \ni m \mapsto H_m,$$

$$\Big\{ |\mathrm{Re}(m)| < \frac{\varepsilon}{2} \Big\} \times \big(\mathbb{C} \cup \{\infty\}\big) \ni (m, \kappa) \mapsto H_{m,\kappa},$$

$$\mathbb{C} \cup \{\infty\} \ni \nu \mapsto H_0^{\nu},$$

analogous to the families of the unperturbed case. All three families are holomorphic except for a singularity at $(m, \kappa) = (0, -1)$.

They are defined as the restrictions of L_{m^2} to the domains:

$$
\begin{aligned}
\mathcal{D}(H_m) &:= \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathcal{W}(x^{\frac12+m}, f; 0) = 0 \Big\}, \\
\mathcal{D}(H_{m,\kappa}) &:= \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathcal{W}\big(x^{\frac12+m} + \kappa x^{\frac12-m}, f; 0\big) = 0 \Big\}, \quad \kappa \in \mathbb{C}, \\
\mathcal{D}(H_{m,\infty}) &:= \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathcal{W}\big(x^{\frac12-m}, f; 0\big) = 0 \Big\}, \\
\mathcal{D}(H_0^{\nu}) &:= \Big\{ f \in \mathcal{D}(L_0^{\max}) \mid \mathcal{W}\big(\nu x^{\frac12} + x^{\frac12}\ln(x), f; 0\big) = 0 \Big\}, \quad \nu \in \mathbb{C}, \\
\mathcal{D}(H_0^{\infty}) &:= \mathcal{D}(H_0).
\end{aligned}
$$

Open Problem 13. *Under the conditions of Proposition 1, does the family $m \mapsto H_m$ extend meromorphically from $\{\mathrm{Re}(m) > 1\}$ to $\{\mathrm{Re}(m) > -1\}$?*

Let us now consider a nonnegative integer n. Under the assumptions of Proposition 11 we can use the function $u_{-m}^{0[n]}$. Then every non-principal boundary functional can be written as

$$\mathcal{W}(\Gamma(1-m)u_{-m}^{0[n]} + \kappa x^{\frac{1}{2}+m}, \cdot; 0)$$

for some $\kappa \in \mathbb{C}$. Clearly, it is proportional to $\mathcal{W}(x^{\frac{1}{2}-m} + \kappa x^{\frac{1}{2}+m}, \cdot; 0)$ for $n = 0$. For $n \geq 1$ it is less canonical. The set of non-principal boundary conditions can be viewed as a 1-dimensional affine space, where we can use $\mathcal{W}(\Gamma(1-m)u_{-m}^{0[n]}, \cdot; 0)$ as a possible "reference point".

Physicists often prefer to fix the operator not by a b.c. bear zero, but by the behavior of the zero-energy eigenfunction near infinity. The behavior of zero energy eigenfunctions at large distances is responsible for large scale properties of quantum systems. It is described by a parameter called the scattering length, which at least in dimension 2 and 3 is popular in physics.

THANK YOU FOR YOUR ATTENTION