Bessel equation

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1 Introduction

Consider the Laplace operator on $\mathbb{R}^2$

$$\Delta_2 = \partial_x^2 + \partial_y^2$$

and the Helmholtz

$$-\Delta_2 F(x,y) = EF(x,y). \tag{1.1}$$

Introduce the polar coordinates

$$x = r\cos\phi, \quad y = r\sin\phi,$$

$$r = \sqrt{x^2+y^2}, \quad \phi = \arctan\frac{y}{x}.$$

The 2-dimensional Laplacian in polar coordinates is

$$-\Delta_2 = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\phi^2. \tag{1.2}$$

We make an ansatz

$$F(x,y) = v(r)u(\phi).$$

The equation (1.1) becomes

$$\Big(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\phi^2\Big)v(r)u(\phi) = Ev(r)u(\phi). \tag{1.3}$$

We divide both sides by $v(r)u(\phi)$ and multiply by r^2. We put functions depending on r on one side and on ϕ on the other side. We obtain

$$\frac{\big(r^2\partial_r^2 + r\partial_r + Er^2\big)v(r)}{v(r)} = -\frac{\partial_\phi^2 u(\phi)}{u(\phi)} = C, \tag{1.4}$$

where C does not depend on r or ϕ. We obtain an equation for u:

$$\partial_\phi^2 u(\phi) = -Cu(\phi), \tag{1.5}$$

solved by linear combinations of $\mathrm{e}^{\mathrm{i}m\phi}$ and $\mathrm{e}^{-\mathrm{i}m\phi}$ with $C = m^2$. Condition $u(\phi) = u(\phi + 2\pi)$ implies that $m \in \mathbb{Z}$. Thus we obtain

$$\big(r^2\partial_r^2 + r\partial_r - m^2 + Er^2\big)v(r) = 0. \tag{1.6}$$

Now if $E < 0$, then we change the variables $z = \sqrt{-E}r$ obtaining the *modified Bessel equation*

$$\left(z^2\partial_z^2 + z\partial_z - m^2 - z^2\right) v = 0. \tag{1.7}$$

Now if $E > 0$, then we change the variables $z = \sqrt{E}r$ obtaining the *(standard) Bessel equation*

$$\left(z^2\partial_z^2 + z\partial_z - m^2 + z^2\right) v = 0. \tag{1.8}$$

Certain distinguished solutions of (1.7) are denoted I_m, K_m, and of (1.8) are denoted J_m and $H_m^\pm$. We call them jointly *the Bessel family.* They are probably the best known and the most widely used special functions in mathematics and its applications.

The parameter m in the above analysis was an integer. However, if we consider the Laplacian on the surface of the cone described by $0 \leq \phi < \alpha$, then the condition $u(\phi) = u(\phi + \alpha)$ leads to $\frac{m\alpha}{2\pi} \in \mathbb{Z}$. Hence non-integer values of the parameter m are also relevant in applications.

We will see that the Helmholtz equation in any dimension d

$$-\Delta_d F = EF \tag{1.9}$$

leads in spherical coordinates to the Bessel equation. The d-dimensional Laplacian in spherical coordinates is given by

$$-\Delta_d = -\partial_r^2 - \frac{d-1}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathrm{S}^{d-1}},$$

where r is the radial coordinate and $\Delta_{\mathrm{S}^{d-1}}$ is the *Laplace-Beltrami operator on the sphere* S^{d-1}. Eigenvalues of $-\Delta_{\mathrm{S}^{d-1}}$ for $d = 2, 3, \dots$ are

$$l(l + d - 2), \quad l = 0, 1, 2, \dots \tag{1.10}$$

where l corresponds to the order of spherical harmonics. Setting $F = v(r)u(\Omega)$, where Ω are the angular coordinates, we obtain the radial part of the Helmholtz equation

$$\big(r^2\partial_r^2 + (d-1)r\partial_r - l(l + d - 2) + Er^2\big)v(r). \tag{1.11}$$

By the same scaling argument as above, we can reduce ourselves to the case $E = \pm 1$. We will see that (1.11) is equivalent to (1.7) or (1.8).

The operator in (1.7)/(1.8) can be transformed as

$$r^{1-\frac{d}{2}}\left(r^2\partial_r^2 + r\partial_r \mp r^2 - m^2\right)r^{-1+\frac{d}{2}} \tag{1.12}$$

$$= r^2\partial_r^2 + (d-1)r\partial_r \mp r^2 - \Big(m - \frac{d}{2} + 1\Big)\Big(m + \frac{d}{2} - 1\Big). \tag{1.13}$$

Setting

$$m := l + \frac{d}{2} - 1, \tag{1.14}$$

we can rewrite (1.13) as

$$r^2\partial_r^2 + (d-1)r\partial_r \mp r^2 - l(l+d-2), \tag{1.15}$$

which is the radial part of the Helmholtz equation in dimension d for spherical harmonics of order l, see (1.11). (1.15) is sometimes called the *d-dimensional Bessel equation.*

Note that in even dimensions the parameter m takes integer values and in odd dimensions it takes half-integer values, see (1.14).

Note special cases of (1.13):

$$\begin{aligned} & r^{-m}\left(r^2\partial_r^2 + r\partial_r \mp r^2 - m^2\right)r^m \\ = \; & r^2\partial_r^2 + (1+2m)r\partial_r \mp r^2, \end{aligned} \tag{1.16}$$

$$\begin{aligned} & r^{\frac{1}{2}}\left(r^2\partial_r^2 + r\partial_r \mp r^2 - m^2\right)r^{-\frac{1}{2}} \\ = \; & r^2\left(\partial_r^2 \mp 1 + \left(1/4 - m^2\right)\frac{1}{r^2}\right). \end{aligned}$$

Here are some other operators related to the Bessel equation: Set $r := t^\delta$, so that $\partial_r = \frac{1}{\delta}t^{1-\delta}\partial_t$. Then

$$r^{\frac{1}{2\delta}}\left(r^2\partial_r^2 + r\partial_r \mp r^2 - m^2\right)r^{-\frac{1}{2\delta}} \tag{1.17}$$

$$= t^{\frac{1}{2}}\Big(\delta^{-2}t^2\partial_t^2 + \delta^{-2}t\partial_t \mp t^{2\delta} - m^2\Big)t^{-\frac{1}{2}} \tag{1.18}$$

$$= \delta^{-2}t^2\Big(\partial_t^2 \mp (\delta t^{\delta-1})^2 + \Big(\frac{1}{4} - m^2\delta^2\Big)\frac{1}{t^2}\Big). \tag{1.19}$$

If we set $r = \mathrm{e}^t$, so that $r\partial_r = \partial_t$, then

$$r^2\partial_r^2 + r\partial_r \mp r^2 - m^2 = \partial_t^2 \mp \mathrm{e}^{2t} - m^2. \tag{1.20}$$

2 Modified Bessel equation

2.1 Integral representations

The modified Bessel equation is given by the operator

$$\mathcal{I}_m(z, \partial_z) := z^2\partial_z^2 + z\partial_z - z^2 - m^2.$$

Theorem 2.1 Bessel–Schläfli type representations *Let* $]0,1[\ni\tau\overset{\gamma}{\mapsto}t(\tau)$ *be a contour such that*

$$\left(\frac{z}{2}(t-t^{-1})+m\right)\exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m}\Big|_{t(0)}^{t(1)}=0, \tag{2.21}$$

Then

$$\int_\gamma \exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m-1}\mathrm{d}t \tag{2.22}$$

is a solution of the modified Bessel equation

Proof. We differentiate the integral with respect to the parameter z:

$$\begin{aligned}
&(z^2\partial_z^2+z\partial_z-z^2-m^2)\int_\gamma \exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m-1}\mathrm{d}t\\
=\ &\int_\gamma\left(\left(\frac{z}{2}\right)^2\left(t+t^{-1}\right)^2+\frac{z}{2}\left(t+t^{-1}\right)-z^2-m^2\right)\exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m-1}\mathrm{d}t\\
=\ &\int_\gamma\left(\left(\frac{z}{2}\right)^2\left(t-t^{-1}\right)^2+\frac{z}{2}\left(t+t^{-1}\right)-m^2\right)\exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m-1}\mathrm{d}t\\
=\ &\int_\gamma\partial_t\left(\left(\frac{z}{2}(t-t^{-1})+m\right)\exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m}\right)\mathrm{d}t\\
=\ &\left(\frac{z}{2}(t-t^{-1})+m\right)\exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m}\Big|_{t(0)}^{t(1)}\quad=\quad 0.
\end{aligned}$$

□

Theorem 2.2 Poisson type representations. *Let* $]0,1[\ni\tau\overset{\gamma}{\mapsto}t(\tau)$ *be a contour such that*

$$(1-t^2)^{m+\frac12}\mathrm{e}^{zt}\Big|_{t(0)}^{t(1)}=0.$$

Then

$$z^m\int_\gamma(1-t^2)^{m-\frac12}\mathrm{e}^{zt}\mathrm{d}t$$

is a solution of the modified Bessel equation.

Proof. We use (1.16).

$$\begin{aligned}
&\left(z\partial_z^2+(1+2m)\partial_z-z\right)\int_\gamma(1-t^2)^{m-\frac12}\mathrm{e}^{zt}\mathrm{d}t\\
=\ &\int_\gamma(1-t^2)^{m-\frac12}(zt^2+(1+2m)t-z)\mathrm{e}^{zt}\mathrm{d}t\\
=\ &-\int_\gamma\partial_t\left((1-t^2)^{m+\frac12}\mathrm{e}^{zt}\right)\mathrm{d}t\quad=\quad 0.
\end{aligned}$$

2.2 Modified Bessel function

The modified Bessel equation has a regular-singular point at 0 with the indicial equation

$$\lambda(\lambda-1)+\lambda-m^2=0.$$

Its indices at 0 are equal to $\pm m$.

Therefore, we should look for a solution of the modified Bessel equation in the form

$$0=\Big(z^2\partial_z^2+z\partial_z-z^2-m^2\Big)\sum_{n=0}^{\infty}c_n z^{m+n} \tag{2.23}$$

$$=\sum_{n=0}^{\infty}c_n\Big((m+n)^2(m+n-1)+m+n-m^2\Big)z^n-\sum_{n=0}^{\infty}c_n z^{n+2}. \tag{2.24}$$

This leads to the recurrence relation

$$c_n(2m+n)n=c_{n-2}. \tag{2.25}$$

The initial condition $c_{-1}=0$ together with (2.25) implies that $c_n=0$ for n odd. For even subscripts, we can rewrite (2.25) in the form

$$c_{2n}(2m+2n)2n=c_{2(n-1)}. \tag{2.26}$$

With $c_0=1$, this is solved by

$$c_{2n}=\frac{1}{2^{2n}(m+1)\cdots(m+n)n!}.$$

Multiplying this with $\frac{1}{\Gamma(m+1)}$, we obtain

$$c_{2n}=\frac{1}{2^{2n}\Gamma(m+n+1)n!}.$$

The resulting function is called the *modified Bessel function*:

$$I_m(z)=\sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2n+m}}{n!\Gamma(m+n+1)}.$$

It is a solution of the modified Bessel equation with the parameter $\pm m$. Note that $\frac{1}{\Gamma(m+1)}\neq 0$ for $m\neq -1,-2,\dots$ For $m\neq -1,-2,\dots$ the function I_m is the unique solution of the modified Bessel equation satisfying

$$I_m(z)\sim\left(\frac{z}{2}\right)^m\frac{1}{\Gamma(m+1)},\ z\sim 0,$$

which can be treated as a definition of the modified Bessel function. (By $f(z)\sim g(z)$, $z\sim 0$, we understand that $\frac{f(z)}{g(z)}$ is analytic around zero and at zero equals 1).

If $m \notin \mathbb{Z}$, then $I_{-m}(z)$ and $I_m(z)$ are linearly independent and span the space of solutions of the modified Bessel equation.

We have

$$\begin{aligned} I_m(\mathrm{e}^{\pm \mathrm{i}\pi} z) &= \mathrm{e}^{\pm \mathrm{i}\pi m} I_m(z), && (2.27)\\ \overline{I_m(z)} &= I_{\overline{m}}(\overline{z}). && (2.28) \end{aligned}$$

In particular, $I_m(x)$ is real for $x > 0$, $m \in \mathbb{R}$.

2.3 Integral representations of modified Bessel function

Theorem 2.3 (Bessel-Schläfli-type representations.) *Let* $\mathrm{Re} z > 0$. *Then*

$$\begin{aligned} I_m(z) &= \frac{1}{2\pi \mathrm{i}} \int_{]-\infty,0^+,-\infty[} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t && (2.29)\\ &= \frac{1}{2\pi \mathrm{i}} \left(\frac{z}{2}\right)^m \int_{]-\infty,0^+,-\infty[} \exp\left(s + \frac{z^2}{4s}\right) s^{-m-1}\mathrm{d}s. && (2.30) \end{aligned}$$

We also have

$$I_{-m}(z) = \frac{1}{2\pi \mathrm{i}} \int_{[(0-0)^+]} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t. \qquad (2.31)$$

Here, the contour starts at 0 *from the negative side on the lower sheet, encircling* 0 *in the positive direction and ends at* 0 *from the negative side on the upper sheet.*

One of concrete realizations of (2.29) is the Schläfli representation*:*

$$I_m(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{z\cos\phi} \cos(m\phi)\mathrm{d}\phi - \frac{1}{\pi}\sin(m\pi) \int_0^\infty \mathrm{e}^{-z\cosh\beta - m\beta}\mathrm{d}\beta. \qquad (2.32)$$

Proof. To see this note that by Thm 2.1, the RHS of (2.29) is a solution of the modified Bessel equation. Besides,

$$\frac{1}{2\pi \mathrm{i}} \left(\frac{z}{2}\right)^m \int_{]-\infty,0^+,-\infty[} \exp\left(s + \frac{z^2}{4s}\right) s^{-m-1}\mathrm{d}s$$

is holomorphic around zero. By the Hankel identity

$$\frac{1}{\Gamma(m+1)} = \frac{1}{2\pi \mathrm{i}} \int_{]-\infty,0^+,-\infty[} \exp(s)\, s^{-m-1}\mathrm{d}s.$$

Therefore, the RHS of (2.29) behaves at zero as $\sim \left(\frac{z}{2}\right)^m \frac{1}{\Gamma(m+1)}$. Therefore, it coincides with $I_m(z)$, at least for $m \notin \{\dots,-2,-1\}$. By continuity, it is I_m also for $m \in \{\dots,-2,-1\}$.

To see (2.31), we make a substitution $t = s^{-1}$ in (2.29) noting that $\mathrm{d}t = -s^{-2}\mathrm{d}s$, and then we change the orientation of the contour.

To see (2.32), we take a contour consisting of three pieces: $-\mathrm{e}^{-\beta}: \ \beta \in]-\infty, 0]$, $\mathrm{e}^{\mathrm{i}\phi}: \ \phi \in [-\pi, \pi]$, $-\mathrm{e}^{-\beta}: \ \beta \in [0, -\infty[$. □

Theorem 2.4 (Poisson-type representations.) *We have the* Poisson representation

$$I_m(z) = \frac{1}{\sqrt{\pi}\Gamma\Big(m+\frac{1}{2}\Big)}\Big(\frac{z}{2}\Big)^m \int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\mathrm{e}^{zt}\mathrm{d}t, \qquad m > -\frac{1}{2}. \tag{2.33}$$

The following representation is due to Hankel*:*

$$I_m(z) = \frac{\Gamma(\frac{1}{2}-m)}{2\pi\mathrm{i}\sqrt{\pi}}\Big(\frac{z}{2}\Big)^m \int\limits_{[1,-1^-,1^+]} (t-1)^{m-\frac{1}{2}}(t+1)^{m-\frac{1}{2}}\mathrm{e}^{zt}\mathrm{d}t. \tag{2.34}$$

Proof. By Thm 2.2, the RHS of (2.33) satisfies the modified Bessel equation. Then we use the fact that

$$\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\mathrm{d}t = \frac{\Gamma(m+\frac{1}{2})\sqrt{\pi}}{\Gamma(m+1)}, \tag{2.35}$$

to see that the RHS of (2.33) behaves at zero as $\sim \left(\frac{z}{2}\right)^m \frac{1}{\Gamma(m+1)}$.

Similarly, to show (2.34) we use

$$\frac{1}{2\pi\mathrm{i}}\int_{[1,-1^-,1^+]}(t-1)^{m-\frac{1}{2}}(t+1)^{m-\frac{1}{2}}\mathrm{d}t = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-m)\Gamma(m+1)}. \tag{2.36}$$

□

2.4 Modified Bessel function for integral parameters

For $m \in \mathbb{Z}$ the Bessel-type integrals (2.29), (2.30) and (2.32) simplify:

Theorem 2.5 *For $m \in \mathbb{Z}$ we have*

$$I_m(z) = I_{-m}(z).$$

$$\begin{aligned} I_m(z) &= \frac{1}{2\pi\mathrm{i}}\int_{[0^+]}\exp\Big(\frac{z}{2}(t+t^{-1})\Big)\frac{\mathrm{d}t}{t^{m+1}} && (2.37)\\ &= \frac{1}{2\pi\mathrm{i}}\Big(\frac{z}{2}\Big)^m\int_{[0^+]}\exp\Big(s+\frac{z^2}{4s}\Big)\frac{\mathrm{d}s}{s^{m+1}} && (2.38)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{z\cos\phi}\cos(m\phi)\mathrm{d}\phi. && (2.39) \end{aligned}$$

Proof. We will give two proofs.

The first is based on the power series. It is enough to assume that $m = 0, 1, \dots$.

$$I_m(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+m}}{n!(n+m)!} \tag{2.40}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2(n+m)-m}}{(n+m)!(n+m-m)!} \tag{2.41}$$

$$= \sum_{n=m}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n-m}}{n!(n-m)!} \tag{2.42}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n-m}}{n!\Gamma(n-m+1)} = I_{-m}(z). \tag{2.43}$$

The second uses the contour integrals (2.29) and (2.31): we note that if $m \in \mathbb{Z}$, one can be deformed into the other. □

Theorem 2.6 (Generating function.)

$$\exp\left(\frac{z}{2}(t+t^{-1})\right) = \sum_{m=-\infty}^{\infty} t^m I_m(z). \tag{2.44}$$

Proof. Again, we will give two proofs.

The first uses power series:

$$\sum_{m=-\infty}^{\infty} t^m I_m(z) = \sum_{n\geq 0,\ n+m\geq 0} \frac{t^m (z/2)^{m+2n}}{n!(n+m)!}$$

$$= \sum_{n\geq 0} \sum_{n+m\geq 0} \frac{(z/2t)^n (tz/2)^{m+n}}{n!(n+m)!}$$

$$= \mathrm{e}^{z/2t}\mathrm{e}^{tz/2}.$$

The second notes that (2.44) is the Laurent series in t of a function holomorphic in $\mathbb{C}\backslash\{0\}$, and (2.37) is just the formula for the coefficient in the Laurent series. □

2.5 MacDonald function

We define the *Macdonald function*:

$$K_m(z) := \frac{1}{2}\int_0^{\infty} \exp\left(-\frac{z}{2}(s+s^{-1})\right) s^{\pm m-1}\mathrm{d}s. \tag{2.45}$$

Other names: the *Basset function* or the *modified Bessel function of the second kind*. The integral (2.45) is absolutely convergent. Substitution $s = t^{-1}$ shows that m can be replaced by $-m$ (and thus $K_m = K_{-m}$).

Theorem 2.7 *K_m solves the modified Bessel equation. We have*

$$\overline{K_m(z)} = K_{\overline{m}}(\overline{z}).$$

$K_m(x)$ is real for $x > 0$, and $m \in \mathbb{R}$ or $m \in \mathrm{i}\mathbb{R}$.

Proof. We can write

$$K_m(z) = \frac{\mathrm{e}^{\mathrm{i}\pi m}}{2} \int_{]-\infty-\mathrm{i}0,0]} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t \tag{2.46}$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\pi m}}{2} \int_{]-\infty+\mathrm{i}0,0]} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t, \tag{2.47}$$

which are integrals satisfying Theorem 2.1. (Note that the contours lie on the boundary of the Riemann surface of the principal branch of t^{-m-1}, each projecting onto $]-\infty, 0]$, the first is on the lower sheet and the second on the upper sheet). □

Theorem 2.8

$$K_{-m}(z) = K_m(z) = \frac{\pi}{2\sin\pi m}(I_{-m}(z) - I_m(z)). \tag{2.48}$$

Proof. (**typos**) We add the appropriate multiples of (2.46) and (2.47):

$$\begin{aligned}
-4\mathrm{i}\sin(\pi m)K_m(z) &= 2\mathrm{e}^{-\mathrm{i}\pi m}K_m(z) - 2\mathrm{e}^{\mathrm{i}\pi m}K_m(z) \\
&= \int_{]-\infty-\mathrm{i}0,0]} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t \\
&\quad + \int_{]0,-\infty+\mathrm{i}0]} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t \\
&= \int_{]-\infty,0^+,-\infty[} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t \\
&\quad - \int_{(0-0)^+} \exp\left(\frac{z}{2}(t+t^{-1})\right) t^{-m-1}\mathrm{d}t \\
&= 2\pi\mathrm{i}(I_m(z) - I_{-m}(z)).
\end{aligned}$$

where, as we recall, $(0-0)^+$ is the contour starting at 0 from the negative side on the lower sheet, encircling 0 in the positive direction and ending at 0 from the negative side on the upper sheet. This proves (2.48). □

It is useful to note that for positive z the contour in (2.45) can be turned by $\frac{\pi}{2}$, losing however its absolute convergence at 0 or ∞. We thus obtain

$$K_m(z) := \frac{\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}m}}{2} \int_0^\infty \exp\left(-\frac{z\mathrm{i}}{2}(s-s^{-1})\right) s^{\pm m-1}\mathrm{d}s. \tag{2.49}$$

Theorem 2.9 *Setting $s = \mathrm{e}^{\theta}$ in (2.45) and $t = \mathrm{e}^{\phi}$ in 2.49) we obtain*

$$
\begin{aligned}
K_m(z) &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-z\cosh\theta}\mathrm{e}^{-m\theta}\mathrm{d}\theta, && (2.50)\\
&= \frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}m}}{2}\int_{-\infty}^{\infty} \mathrm{e}^{\mp\mathrm{i}z\sinh\phi}\mathrm{e}^{-m\phi}\mathrm{d}\phi. && (2.51)
\end{aligned}
$$

2.6 Analytic continuation of the MacDonald function

Theorem 2.10

$$
I_m(z) = \frac{1}{\pi}\big(\mp\mathrm{i}K_m(\mathrm{e}^{\mp\mathrm{i}\pi}z) \pm \mathrm{i}\mathrm{e}^{\mathrm{i}\pi m}K_m(z)\big). \tag{2.52}
$$

Proof. Write

$$
K(\mathrm{e}^{-\mathrm{i}\pi}z) = \frac{\pi}{2\sin\pi m}\big(\mathrm{e}^{\mathrm{i}\pi m}I_{-m}(z) - \mathrm{e}^{-\mathrm{i}\pi m}I_m(z)\big)
$$

We subtract from this $\mathrm{e}^{\mathrm{i}\pi m}$ times (2.48) obtaining

$$
K(\mathrm{e}^{-\mathrm{i}\pi}z) - \mathrm{e}^{\mathrm{i}\pi m}K(z) = \frac{\pi}{2\sin\pi m}\big(-\mathrm{e}^{-\mathrm{i}\pi m}I_m(z) + \mathrm{e}^{\mathrm{i}\pi m}I_m(z)\big) = \mathrm{i}\pi I_m(z).
$$

This proves (2.52). □

The function $K_m(z)$ is exponentially decaying whereas $\mp\mathrm{i}K_m(\mathrm{e}^{\mp\mathrm{i}\pi}z)$ are exponentially exploding. It will be useful to have such "exploding partners" for $K_m(z)$. Unfortunately, the natural domain for $\mp\mathrm{i}K_m(\mathrm{e}^{\mp\mathrm{i}\pi}z)$ is $\mathbb{C}\backslash[0,\infty[$, which is inconvenient if we are primarily interested in $[0,\infty[$. It seems useful to introduce the following function (which has no name and apparently does not appear in the literature):

$$
X_m(z) := \frac{\mathrm{i}}{2}\big(K_m(\mathrm{e}^{\mathrm{i}\pi}z) - K_m(\mathrm{e}^{-\mathrm{i}\pi}z)\big). \tag{2.53}
$$

Theorem 2.11 *We have the identities*

$$
\begin{aligned}
I_m(z) &= \frac{1}{\pi}\big(X_m(z) - \sin(\pi m)K_m(z)\big), && (2.54)\\
X_m(z) &= \frac{\pi}{2}\big(I_{-m}(z) + I_{-m}(z)\big). && (2.55)
\end{aligned}
$$

2.7 Asymptotics of the MacDonald function

Theorem 2.12 *For $|\arg z| < \pi - \epsilon$,*

$$
\lim_{|z|\to\infty} \frac{K_m(z)}{\frac{\mathrm{e}^{-z}\sqrt{\pi}}{\sqrt{2z}}} = 1.
$$

Proof. We use the *steepest descent method.* Set $\phi(t) := -\frac{1}{2}(t + t^{-1})$. We compute

$$\phi'(t) = -\frac{1}{2}(1 - t^{-2}), \quad \phi''(t) = -t^{-3}.$$

Hence ϕ has a critical point at $t_0 = 1$ with $\phi(t_0) = -1$ and $\phi''(t_0) = -1$. Thus

$$\begin{aligned}
K_m(z) &= \frac{1}{2}\int_0^\infty t^{-m-1}\exp(z\phi(t))\mathrm{d}t \\
&\simeq \frac{1}{2}\int_{-\infty}^\infty \exp\left(z\phi(t_0) + z\frac{\phi''(t_0)}{2}(t-t_0)^2\right)\mathrm{d}t \\
&= \frac{1}{2}\mathrm{e}^{-z}\int_{-\infty}^\infty \exp\left(\frac{z}{2}(t-1)^2\right)\mathrm{d}t \;=\; \frac{1}{2}\mathrm{e}^{-z}\frac{\sqrt{2\pi}}{\sqrt{z}}.
\end{aligned}$$

□

Corollary 2.13 *As $x \to \infty$ we have*

$$I_m(x) \quad \sim \quad \frac{1}{\sqrt{2\pi x}}\mathrm{e}^x. \tag{2.56}$$

Next we derive the precise asymptotics of the MacDonald function

Theorem 2.14

$$K_m(z) \sim \sqrt{\pi}\mathrm{e}^{-z}\sum_{n=0}^\infty (-1)^n \frac{(\frac{1}{2}-m)_n(\frac{1}{2}+m)_n}{n!(2z)^{n+\frac{1}{2}}}. \tag{2.57}$$

Proof. To derive (2.57), at least formally, we first we transform the equation:

$$\mathrm{e}^z z^{\frac{1}{2}}\left(z^2\partial_z^2 + z\partial_z - z^2 - m^2\right)z^{-\frac{1}{2}}\mathrm{e}^{-z} \tag{2.58}$$

$$= \quad z^2\partial_z^2 - 2z^2\partial_z - m^2 + \frac{1}{4}. \tag{2.59}$$

Acting with (2.59) on

$$\sum_{n=0}^\infty c_n z^{-n} \tag{2.60}$$

we obtain

$$\sum_{n=0}^\infty \Big(n(n+1)c_n z^{-n} + 2nc_n z^{-n+1} - \Big(m^2 - \frac{1}{4}\Big)c_n z^{-n}\Big). \tag{2.61}$$

This yields the recurrence relation

$$2nc_n \quad = \quad -\Big((n-1)n - m^2 + \frac{1}{4}\Big)c_{n-1} \tag{2.62}$$

$$= \quad -\Big(n - \frac{1}{2} - m\Big)\Big(n - \frac{1}{2} + m\Big)c_{n-1}. \tag{2.63}$$

Therefore,

$$c_n = (-1)^n\frac{(\frac{1}{2}-m)_n(\frac{1}{2}+m)_n}{n!2^n}. \tag{2.64}$$

□

2.8 More integral representations

Theorem 2.15 (Poisson-type representations.)

$$\begin{aligned}
&K_m(z)\\
= \;& \left(\frac{z}{2}\right)^{\pm m} \frac{\sqrt{\pi}\Gamma(\mp m+\frac12)}{2\pi\mathrm{i}} \int\limits_{]-\infty,-1^+,-\infty[} \mathrm{e}^{zt}(1-t^2)^{\pm m-\frac12}\mathrm{d}t. && (2.65)\\
= \;& \left(\frac{z}{2}\right)^{m} \frac{\sqrt{\pi}}{\Gamma(m+\frac12)} \int_1^\infty \mathrm{e}^{-sz}(s^2-1)^{m-\frac12}\mathrm{d}s, \quad m>-\frac12; && (2.66)\\
= \;& \left(\frac{z}{2}\right)^{-m} \frac{\Gamma(m+\frac12)}{2\sqrt{\pi}} \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}sz}(s^2+1)^{-m-\frac12}\mathrm{d}s, \quad m>0. && (2.67)
\end{aligned}$$

Proof. The RHS of (2.65) solves the modified Bessel equation by Thm 2.2. Let us check its behavior for $\mathrm{Re} z\to\infty$. We set $t=\frac{s}{z}-1$:

$$\begin{aligned}
& \frac{1}{2\pi\mathrm{i}} \int\limits_{]-\infty,-1^+,-\infty[} \mathrm{e}^{zt}(t+1)^{\pm m-\frac12}(-t+1)^{\pm m-\frac12}\mathrm{d}t && (2.68)\\
\sim\; & \left(\frac{2}{z}\right)^{m-\frac12}\frac{\mathrm{e}^{-z}}{z}\frac{1}{2\pi\mathrm{i}} \int\limits_{]-\infty,-1^+,-\infty[} \mathrm{e}^{-s}s^{\pm m-\frac12}\mathrm{d}s && (2.69)\\
=\; & \left(\frac{2}{z}\right)^{\pm m}\frac{\mathrm{e}^{-z}}{\sqrt{2z}\Gamma(\frac12\mp m)}. && (2.70)
\end{aligned}$$

Therefore, the RHS of (2.65) behaves as $\frac{\mathrm{e}^{-z}\sqrt{\pi}}{\sqrt{2z}}$. Therefore, it coincides with $K_m(z)$.

(2.67) follows from (2.65) by setting $t=\mathrm{i}s$. □

Here are various integral representations in an exponential parametrization:

$$\begin{aligned}
K_m(z) &= \frac12\int_{-\infty}^\infty \mathrm{e}^{-z\cosh\phi}\mathrm{e}^{\pm m\phi}\mathrm{d}\phi && (2.71)\\
&= \frac12\int_{-\infty}^\infty \mathrm{e}^{-z\cosh\phi}\cosh(m\phi)\mathrm{d}\phi && (2.72)\\
&= \left(\frac{z}{2}\right)^{m}\frac{\sqrt{\pi}}{\Gamma(m+\frac12)}\int_0^\infty \mathrm{e}^{-z\cosh\phi}(\sinh\phi)^{2m}\mathrm{d}\phi, \quad m>-\frac12; && (2.73)\\
&= \left(\frac{z}{2}\right)^{-m}\frac{\Gamma(m+\frac12)}{2\sqrt{\pi}}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}z\sinh\phi}(\cosh\phi)^{-2m}\mathrm{d}\phi, \quad m>0. && (2.74)
\end{aligned}$$

((2.72) is an immediate consequence of (2.71)). In order to appreciate how different these representations are, let us consider (2.72), (2.73) and (2.74) in the case $m = \pm\frac{1}{2}$:

$$K_{\pm\frac{1}{2}}(z) = \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} \mathrm{e}^{-z} = \int_{-\infty}^{\infty} \mathrm{e}^{-z(1+2t^2)}\mathrm{d}t, \qquad t = \sinh\frac{\phi}{2}; \tag{2.75}$$

$$= \Big(\frac{z\pi}{2}\Big)^{\frac{1}{2}} \int_{1}^{\infty} \mathrm{e}^{-zt}\mathrm{d}t, \qquad t = \cosh\phi; \tag{2.76}$$

$$= \frac{1}{(2\pi z)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}zt}\mathrm{d}t}{(t^2+1)}, \qquad t = \sinh\phi. \tag{2.77}$$

2.9 MacDonald function for integer parameters

Theorem 2.16 *Set $H_n := \sum_{k=1}^{n} \frac{1}{k}$. Then for $m = 0, 1, 2, \dots$*

$$K_m(z) = (-1)^{m+1}\Big(\log\frac{z}{2} + \gamma\Big) I_m(z)$$
$$+\frac{1}{2}\sum_{k=0}^{m-1}(-1)^k\Big(\frac{z}{2}\Big)^{2k-m}\frac{(m-k-1)!}{k!} + \frac{(-1)^m}{2}\sum_{k=0}^{\infty}\frac{H_k + H_{m+k}}{k!(m+k)!}\Big(\frac{z}{2}\Big)^{2k+m}.$$

Proof. Set

$$\phi(z) := \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\Gamma(z)} = -\frac{1}{\Gamma(z)}\partial_z \log\Gamma(z).$$

Then

$$\phi(-n) = (-1)^n n!, \quad n = 0, 1, 2, \dots,$$

$$\phi(n+1) = \frac{\gamma - H_n}{n!}, \quad n = 0, 1, 2, \dots$$

Besides,

$$\partial_m I_m(z) = \log\Big(\frac{z}{2}\Big) I_m(z) + \sum_{k=0}^{\infty}\frac{\phi(m+k+1)}{k!}\Big(\frac{z}{2}\Big)^{m+2k}.$$

Hence for $m = 0, 1, 2, \dots$

$$\partial_n I_n(z)\Big|_{n=m} = \Big(\log\frac{z}{2} + \gamma\Big) I_m(z) - \sum_{k=0}^{\infty}\frac{H_{m+k}}{(m+k)!k!}\Big(\frac{z}{2}\Big)^{m+2k}, \tag{2.78}$$

$$\partial_n I_n(z)\Big|_{n=-m} = \Big(\log\frac{z}{2} + \gamma\Big) I_{-m}(z) + \sum_{k=0}^{m-1}(-1)^{m-k-1}\frac{(m-k-1)!}{k!}\Big(\frac{z}{2}\Big)^{2k-m} \tag{2.79}$$

$$-\sum_{k=m}^{\infty}\frac{H_{-m+k}}{(-m+k)!k!}\Big(\frac{z}{2}\Big)^{-m+2k}. \tag{2.80}$$

The last sum can be written as

$$-\sum_{k=m}^{\infty}\frac{H_k}{k!(k+m)!}\Big(\frac{z}{2}\Big)^{m+2k}.$$

We use the De L'Hopital rule:

$$\begin{aligned} K_m(z) &= \frac{\pi}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}m}(I_{-m}-I_m(z))}{\frac{\mathrm{d}}{\mathrm{d}m}\sin\pi m} \\ &= \frac{(-1)^m}{2}\Big(-\frac{\mathrm{d}}{\mathrm{d}n}I_n(z)\Big|_{n=-m}-\frac{\mathrm{d}}{\mathrm{d}n}I_n(z)\Big|_{n=m}\Big) \end{aligned}$$

□

Corollary 2.17 *As $x \to 0$, we have*

$$I_m(x) \sim \frac{1}{\Gamma(m+1)}\Big(\frac{x}{2}\Big)^m, \qquad m \neq -1,-2,\dots; \tag{2.81}$$

$$K_m(x) \sim \begin{cases} -\ln\left(\frac{x}{2}\right)-\gamma & \text{if} \quad m=0, \\ \frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m & \text{if} \quad \mathrm{Re}m \geq 0,\ m\neq 0; \\ \frac{\Gamma(-m)}{2}\left(\frac{x}{2}\right)^m & \text{if} \quad \mathrm{Re}m \leq 0,\ m \neq 0. \end{cases} \tag{2.82}$$

Clearly, for integer m

$$I_m(z) = \frac{1}{\pi}Z_m(z). \tag{2.83}$$

2.10 Relationship to hypergeometric type functions

The modified Bessel function and the hypergeometric functions ${}_0F_1$ and ${}_1F_1$ are closely related:

$$\begin{aligned} I_m(z) &= \frac{1}{\Gamma(m+1)}\Big(\frac{z}{2}\Big)^m {}_0F_1\Big(1+m;\frac{z^2}{4}\Big) \\ &= \frac{1}{\Gamma(m+1)}\Big(\frac{z}{2}\Big)^m \mathrm{e}^{-z}{}_1F_1\Big(m+\frac{1}{2};2m+1;2z\Big). \end{aligned}$$

The MacDonald function is closely related to the hypergeometric function ${}_2F_0$:

$$K_m(z) = \frac{\sqrt{\pi}}{\sqrt{2z}}\mathrm{e}^{-z}{}_2F_0\Big(\frac{1}{2}+m,\frac{1}{2}-m;-;-\frac{1}{2z}\Big).$$

2.11 Recurrence relations

Theorem 2.18

$$\begin{aligned} 2\partial_z I_m(z) &= I_{m-1}(z)+I_{m+1}(z), & (2.84) \\ 2mI_m(z) &= zI_{m-1}(z)-zI_{m+1}(z). & (2.85) \end{aligned}$$

$$\begin{aligned} 2\partial_z K_m(z) &= -K_{m-1}(z) - K_{m+1}(z), && (2.86) \\ 2mK_m(z) &= -zK_{m-1}(z) + zK_{m+1}(z). && (2.87) \end{aligned}$$

Proof. Recall that

$$2\partial_z I_m(z) = \int_\gamma \exp\Big(\frac{z}{2}(t+t^{-1})\Big)(t^{-m}+t^{-m-2})\mathrm{d}t.$$

$$\begin{aligned} 0 &= 2\int_\gamma \partial_t \exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m}\mathrm{d}t \\ &= -2m\int_\gamma \exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m-1}\mathrm{d}t \\ &\quad +z\int_\gamma \exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m}\mathrm{d}t - z\int_\gamma \exp\left(\frac{z}{2}(t+t^{-1})\right)t^{-m-2}\mathrm{d}t. \end{aligned}$$

By (2.46), $\mathrm{e}^{\mathrm{i}\pi m}K_m$ satisfy the same recurrence relations. □

Corollary 2.19

$$\partial_z\left(z^m I_m(z)\right) = z^m I_{m-1}(z), \ \ \textit{or} \ \left(\partial_z + \frac{m}{z}\right)I_m(z) = I_{m-1}(z),$$

$$\partial_z\left(z^{-m} I_m(z)\right) = z^{-m} I_{m+1}(z), \ \ \textit{or} \ \left(\partial_z - \frac{m}{z}\right)I_m(z) = I_{m+1}(z).$$

Hence

$$\left(\frac{1}{z}\partial_z\right)^n z^m I_m(z) = z^{m-n}I_{m-n}(z), \tag{2.88}$$

$$\left(\frac{1}{z}\partial_z\right)^n z^{-m} I_m(z) = z^{-m-n}I_{m+n}(z). \tag{2.89}$$

Analoguous identities hold for $K_m(z)$.

2.12 Half-integral parameters

For $m = \frac{1}{2}$ we have

$$z^{\frac{1}{2}}\mathcal{I}_{\frac{1}{2}}z^{-\frac{1}{2}} = z^2\Big(\partial_z^2 - 1\Big).$$

Hence, $z^{-\frac{1}{2}}\mathrm{e}^{\pm z}$ is annihilated by $\mathcal{I}_{\frac{1}{2}}$.

$$\begin{aligned} I_{\frac{1}{2}}(z) &= \left(\frac{z}{2}\right)^{\frac{1}{2}}\frac{1}{\Gamma(1+\frac{1}{2})}\frac{\sinh z}{z} = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sinh z && (2.90) \\ I_{-\frac{1}{2}}(z) &= \left(\frac{z}{2}\right)^{-\frac{1}{2}}\frac{1}{\Gamma(1-\frac{1}{2})}\cosh z = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cosh z, && (2.91) \\ K_{\pm\frac{1}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}}\mathrm{e}^{-z}, && (2.92) \\ X_{\pm\frac{1}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}}\mathrm{e}^{z}. && (2.93) \end{aligned}$$

One way to derive (2.90) and (2.91) is to use

$$\begin{aligned}
2^{2n+\frac12}n!\Gamma(1/2+n+1) &= \sqrt{\frac{\pi}{2}}2^n n!2^{n+1}(1/2)_{n+1} = \sqrt{\frac{\pi}{2}}(2n+1)!,\\
2^{2n-\frac12}n!\Gamma(-1/2+n+1) &= \sqrt{\frac{\pi}{2}}2^n n!2^{n}(1/2)_{n} = \sqrt{\frac{\pi}{2}}(2n)!.
\end{aligned}$$

For $k=0,1,2,\dots$ we have

$$\begin{aligned}
I_{\frac12+k}(z) &= \Big(\frac{2}{\pi}\Big)^{\frac12} z^{\frac12+k}\Big(\frac1z\partial_z\Big)^k\frac{\sinh z}{z} \qquad (2.94)\\
&= \Big(\frac{2}{\pi}\Big)^{\frac12}(-2)^k\sum_{j=0}^{\infty} z^{\frac12+2j-k}\frac{(-j)_k}{(2j+1)!}\\
&= \frac{1}{\pi}\Big(Z_{\frac12+k}(z)-(-1)^kK_{\frac12+k}(z)\Big),\\
I_{-\frac12-k}(z) &= \Big(\frac{2}{\pi}\Big)^{\frac12} z^{\frac12+k}\Big(\frac1z\partial_z\Big)^k\frac{\cosh z}{z} \qquad (2.95)\\
&= \Big(\frac{2}{\pi}\Big)^{\frac12}(-2)^k\sum_{j=0}^{\infty} z^{-\frac12+2j-k}\frac{(\frac12-j)_k}{(2j)!}\\
&= \frac{1}{\pi}\Big(Z_{\frac12+k}(z)+(-1)^kK_{\frac12+k}(z)\Big),\\
K_{\pm(\frac12+k)}(z) &= \Big(\frac{\pi}{2}\Big)^{\frac12} z^{\frac12+k}\Big(-\frac1z\partial_z\Big)^k\frac{\mathrm{e}^{-z}}{z} \qquad (2.96)\\
&= \Big(\frac{\pi}{2}\Big)^{\frac12}2^k\sum_{n=0}^{\infty} z^{-\frac12-k+n}(-1)^n\frac{(\frac12-\frac n2)_k}{n!},\\
&= (-1)^k\frac{\pi}{2}\Big(I_{-\frac12-k}(z)-I_{\frac12+k}(z)\Big),\\
X_{\pm(\frac12+k)}(z) &= \Big(\frac{\pi}{2}\Big)^{\frac12} z^{\frac12+k}\Big(\frac1z\partial_z\Big)^k\frac{\mathrm{e}^{z}}{z} \qquad (2.97)\\
&= \Big(\frac{\pi}{2}\Big)^{\frac12}(-2)^k\sum_{n=0}^{\infty} z^{-\frac12-k+n}\frac{(\frac12-\frac n2)_k}{n!}\\
&= \frac{\pi}{2}\Big(I_{-\frac12-k}(z)+I_{\frac12+k}(z)\Big).
\end{aligned}$$

Note that in the sum for (2.94) the terms with $j=0,\dots,k-1$ vanish.

2.13 Wronskians

Recall that the Wronskian is defined as $W(f,g) := fg' - f'g$. The Wronskian of two solutions of the modified Bessel equation satisfies

$$\Big(\partial_z+\frac1z\Big)W(z)=0.$$

Hence $W(z)$ is proportional to $\frac{1}{z}$. Using

$$I_{\pm m}(z) \sim \frac{1}{\Gamma(\pm m+1)}\Big(\frac{z}{2}\Big)^{\pm m}, \qquad I'_{\pm m}(z) \sim \frac{1}{2\Gamma(\pm m)}\Big(\frac{z}{2}\Big)^{\pm m-1},$$

we can compute the Wronskian of I_m and I_{-m}, and then other Wronskians:

$$W(I_m, I_{-m}) = -\frac{2\sin \pi m}{\pi z}, \tag{2.98}$$

$$W(K_m, I_m) = \frac{1}{z}, \tag{2.99}$$

$$W(K_m, X_m) = \frac{2}{z}. \tag{2.100}$$

3 Standard Bessel equation

3.1 Bessel equation

Replacing z with $\pm \mathrm{i}z$ in the modified Bessel equation leads to the standard *Bessel equation*, given by the operator

$$\mathcal{J}_m(z, \partial_z) := z^2\partial_z^2 + z\partial_z + z^2 - m^2.$$

3.2 Integral representations

Theorem 3.1 Bessel–Schläfli representations *Let γ be a contour satisfying*

$$\Big(\frac{z}{2}(t+t^{-1})+m\Big)\exp\Big(\frac{z}{2}(t-t^{-1})\Big)\frac{1}{t^m}\Big|_{\gamma(0)}^{\gamma(1)} = 0, \tag{3.101}$$

Then

$$C\int_\gamma \exp\Big(\frac{z}{2}(t-t^{-1})\Big)\frac{\mathrm{d}t}{t^{m+1}} \tag{3.102}$$

is a solution of the Bessel equation.

Theorem 3.2 Poisson type representations *Let γ be a contour satisfying*

$$(1-t^2)^{m+\frac{1}{2}}\mathrm{e}^{\mathrm{i}zt}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$z^m\int_\gamma (1-t^2)^{m-\frac{1}{2}}\mathrm{e}^{\mathrm{i}zt}\mathrm{d}t$$

is a solution of the Bessel equation.

3.3 Bessel function

The *Bessel function* is defined as

$$\begin{aligned} J_m(z) &= \mathrm{e}^{\pm \mathrm{i}\pi\frac{m}{2}} I_m(\mp \mathrm{i}z) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+m}}{n!\Gamma(m+n+1)} \\ &= \frac{1}{\mathrm{i}\pi}\left(\mathrm{e}^{-\mathrm{i}\pi\frac{m}{2}} K(-\mathrm{i}z) - \mathrm{e}^{\mathrm{i}\pi\frac{m}{2}} K(\mathrm{i}z)\right) \end{aligned}$$

$$J_m(\mathrm{e}^{\pm \mathrm{i}\pi} z) = \mathrm{e}^{\pm \mathrm{i}m\pi} J_m(z).$$

Theorem 3.3 *Let* $\mathrm{Re}z > 0$. **(Bessel-Schläfli-type representations)**

$$\begin{aligned} J_m(z) &= \frac{1}{2\pi \mathrm{i}} \int_{]-\infty,0^+,-\infty[} \exp\left(\frac{z}{2}(t - t^{-1})\right) \frac{\mathrm{d}t}{t^{m+1}} \\ &= \frac{1}{2\pi \mathrm{i}} \left(\frac{z}{2}\right)^m \int_{]-\infty,0^+,-\infty[} \exp\left(s - \frac{z^2}{4s}\right) \frac{\mathrm{d}s}{s^{m+1}}. \end{aligned}$$

(Poisson-type representations)

$$\begin{aligned} J_m(z) &= \frac{1}{\sqrt{\pi}\Gamma\left(m+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^m \int_{-1}^{1} (1-t^2)^{m-\frac{1}{2}} \mathrm{e}^{\mathrm{i}zt} \mathrm{d}t, \qquad m > -\frac{1}{2}, \\ J_m(z) &= \frac{1}{2\pi \mathrm{i}\sqrt{\pi}} \Gamma\left(\frac{1}{2} - m\right) \left(\frac{z}{2}\right)^m \int_{[1,-1^-,1^+]} (t-1)^{m-\frac{1}{2}} (t+1)^{m-\frac{1}{2}}. \end{aligned}$$

3.4 Relationship to hypergeometric type functions

$$\begin{aligned} J_m(z) &= \frac{1}{\Gamma(m+1)} \left(\frac{z}{2}\right)^m {}_0F_1\left(1+m; -\frac{z^2}{4}\right) \\ &= \frac{1}{\Gamma(m+1)} \left(\frac{z}{2}\right)^m \mathrm{e}^{-\mathrm{i}z} {}_1F_1\left(m+\frac{1}{2}; 2m+1; 2\mathrm{i}z\right). \end{aligned}$$

3.5 Bessel function for integer parameters

Let $m \in \mathbb{Z}$.

Theorem 3.4

$$J_m(z) = (-1)^m J_{-m}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+m}}{n!(n+m)!}.$$

Theorem 3.5

$$\begin{aligned} J_m(z) &= \frac{1}{2\pi \mathrm{i}} \int_{[0^+]} \exp\left(\frac{z}{2}(t - t^{-1})\right) \frac{\mathrm{d}t}{t^{m+1}} \\ &= \frac{1}{2\pi \mathrm{i}} \left(\frac{z}{2}\right)^m \int_{[0^+]} \exp\left(s - \frac{z^2}{4s}\right) \frac{\mathrm{d}s}{s^{m+1}} \\ &= \frac{1}{\pi} \int_0^\pi \cos(z \sin\phi - m\phi) \mathrm{d}\phi. \end{aligned}$$

3.6 Hankel functions

There are two *Hankel functions.* Both are analytic continuations of the MacDonald function – one to the lower and the other to the upper part of the complex plane:

$$H_m^\pm(z) = \frac{2}{\pi} \mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2}(m+1)} K_m(\mp \mathrm{i} z).$$

In the literature the usual (and less natural) notation for Hankel functions is

$$H_m^{(1)}(z) = H_m^+(z), \quad H_m^{(2)}(z) = H_m^-(z). \tag{3.103}$$

Note the identities

$$K_m(z) = \frac{\pi}{2} \mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2}(m+1)} H^\pm(\pm \mathrm{i} z), \tag{3.104}$$

$$H_{-m}^\pm(z) = \mathrm{e}^{\pm m\pi \mathrm{i}} H_m^\pm(z), \tag{3.105}$$

$$J_m(z) = \frac{1}{2}\left(H_m^+(z) + H_m^-(z)\right), \tag{3.106}$$

$$J_{-m}(z) = \frac{1}{2}\left(\mathrm{e}^{m\pi \mathrm{i}} H_m^+(z) + \mathrm{e}^{-m\pi \mathrm{i}} H_m^-(z)\right), \tag{3.107}$$

$$H_m^\pm(z) = \pm \frac{\mathrm{i}\mathrm{e}^{\mp m\pi \mathrm{i}} J_m(z) - \mathrm{i} J_{-m}(z)}{\sin m\pi}. \tag{3.108}$$

.

Theorem 3.6 *The following asymptotic formulas are true for* $-\pi \pm \frac{\pi}{2} + \delta < \arg z < \pi \pm \frac{\pi}{2} - \delta$, $\delta > 0$*:*

$$\lim_{z \to \infty} \frac{H_m^\pm(z)}{\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{\pm \mathrm{i} z} \mathrm{e}^{\mp \frac{\mathrm{i} m\pi}{2} \mp \frac{\mathrm{i}\pi}{4}}} = 1,$$

Here is a more precise asymptotics:

$$H_m^\pm(z) \sim \frac{\sqrt{2}}{\sqrt{\pi z}} \mathrm{e}^{\pm \mathrm{i} z \mp \mathrm{i}\frac{\pi m}{2} \mp \mathrm{i}\frac{\pi}{4}} \sum_{n=0}^\infty \frac{(\frac{1}{2} - m)_n (\frac{1}{2} + m)_n}{n! (\pm 2\mathrm{i} z)^n}. \tag{3.109}$$

3.7 Integral representations of Hankel functions

Let us first consider Bessel-Schläfli-type representations. For $\mathrm{Re}z > 0$,

$$H_m^+(z) = -\frac{1}{\pi \mathrm{i}} \int_{]-\infty,(0+1\cdot 0)^-[} \exp\left(\frac{z}{2}(t - t^{-1})\right) \frac{\mathrm{d}t}{t^{m+1}},$$

$$H_m^-(z) = \frac{1}{\pi \mathrm{i}} \int_{]-\infty,(0+1\cdot 0)^+[} \exp\left(\frac{z}{2}(t - t^{-1})\right) \frac{\mathrm{d}t}{t^{m+1}}.$$

By $]-\infty,(0+1\cdot 0)^-[$ we understand the contour starting at $-\infty$, encircling 0 clockwise and reaching zero from the positive direction. The contour is located on the upper halfplane.

Similarly, by $]-\infty,(0+1\cdot 0)^+[$ we understand the contour starting at $-\infty$, encircling 0 counterclockwise and reaching zero from the positive direction. The contour is located on the lower halfplane.

Note that

$$\lim_{t\to 0+1\cdot 0} \left(\frac{z}{2}(t + t^{-1}) + m\right) \exp\left(\frac{z}{2}(t - t^{-1})\right) \frac{1}{t^m} = 0,$$

where by $t \to 0+1\cdot 0$ we denote the convergence to zero through positive values of t (sometimes denoted by $t \to 0^+$). Hence the contours $]-\infty,(0+1\cdot 0)^+[$ and $]-\infty,(0+1\cdot 0)^-[$ satisfy (3.101).

If $0 < \arg z < \pi$, then a good contour in the representation of H_m^+ is $[\mathrm{i}\infty, 0]$. If $-\pi < \arg < 0$, then for H_m^- one can use $[-\mathrm{i}\infty, 0]$. This leads to the representations valid for $\pm\mathrm{Im}z \geq 0$:

$$\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}m} H_m^\pm(z) = \mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}m} H_{-m}^\pm(z) = \pm\frac{1}{\pi\mathrm{i}} \int_0^\infty \exp\left(\pm\mathrm{i}\frac{z}{2}(s + s^{-1})\right) \frac{\mathrm{d}s}{s^{m+1}} \quad (3.110)$$

$$= \pm\frac{1}{\pi\mathrm{i}} \int_{-\infty}^\infty \exp\left(\pm\mathrm{i}z\cosh(t) - mt\right) \mathrm{d}t. \quad (3.111)$$

Poisson type integral representations valid for $\mathrm{Im}z \geq 0$ and all m:

$$H_m^\pm(z) = \frac{\Gamma(\frac{1}{2} - m)}{\pi\mathrm{i}\sqrt{\pi}} \left(\frac{z}{2}\right)^m \int_{]\mathrm{i}\infty,\mp 1^\mp,\mathrm{i}\infty[} \mathrm{e}^{\mathrm{i}zt} (t-1)^{m-\frac{1}{2}} (t+1)^{m-\frac{1}{2}} \mathrm{d}t.$$

Poisson type representations valid for $m \geq -\frac{1}{2}$.

$$\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}m} H_m^\pm(z) = \pm\left(\frac{z}{2}\right)^m \frac{2}{\mathrm{i}\sqrt{\pi}\Gamma(\frac{1}{2} + m)} \int_1^\infty \mathrm{e}^{\pm\mathrm{i}sz} (s^2 - 1)^{m-\frac{1}{2}} \mathrm{d}s.$$

3.8 Hankel functions for integer parameters

$$H_m^\pm(z) = J_m(z) \pm \frac{\mathrm{i}2}{\pi}\left(\log\frac{z}{2} + \gamma\right) J_m(z) \quad (3.112)$$

$$\mp \frac{\mathrm{i}}{\pi} \sum_{k=0}^{m-1} \left(\frac{z}{2}\right)^{2k-m} \frac{(m-k-1)!}{k!} \mp \frac{\mathrm{i}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (H_k + H_{m+k})}{k!(m+k)!} \left(\frac{z}{2}\right)^{2k+m}. \quad (3.113)$$

3.9 Neumann function

Neumann function is defined as

$$\begin{aligned} Y_m(z) &= \tfrac{1}{2\mathrm{i}}\big(H_m^+(z) - H_m^-(z)\big) \\ &= \tfrac{\cos \pi m J_m(z) - J_{-m}(z)}{\sin \pi m}. \end{aligned}$$

We then have

$$H_m^+(z) = J_m(z) + \mathrm{i}Y_m(z), \qquad H_m^-(z) = J_m(z) - \mathrm{i}Y_m(z).$$

Theorem 3.7 *For $m \in \mathbb{Z}$ we have*

$$\begin{aligned} Y_m(z) &= \tfrac{2}{\pi}\Big(\log(\tfrac{z}{2}) + \gamma\Big)J_m(z) \\ &\quad -\tfrac{1}{\pi}\sum_{k=0}^{m-1} \tfrac{(m-k-1)!}{k!}(\tfrac{z}{2})^{2k-m} - \tfrac{1}{\pi}\sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!(m+k)!}(\tfrac{z}{2})^{m+2k}\big(H_k + H_{m+k}\big). \end{aligned}$$

3.10 Recurrence relations

Theorem 3.8 *We have the identities*

$$2\partial_z J_m(z) = J_{m-1}(z) - J_{m+1}(z),$$

$$2mJ_m(z) = zJ_{m-1}(z) + zJ_{m+1}(z).$$

Sometimes, more convenient are the following forms of the recurrence relations:

Corollary 3.9

$$\partial_z\left(z^m J_m(z)\right) = z^m J_{m-1}(z), \text{ or } \left(\partial_z + \frac{m}{z}\right) J_m(z) = J_{m-1}(z),$$

$$-\partial_z\left(z^{-m} J_m(z)\right) = z^{-m} J_{m+1}(z), \text{ or } \left(-\partial_z + \frac{m}{z}\right) J_m(z) = J_{m+1}(z).$$

Besides,

$$\left(\frac{1}{z}\partial_z\right)^n z^m J_m(z) = z^{m-n} J_{m-n}(z),$$

$$\left(-\frac{1}{z}\partial_z\right)^n z^{-m} J_m(z) = z^{-m-n} J_{m+n}(z).$$

Analogous identities hold for $H_m^{\pm}(z)$, and $Y_m(z)$.

3.11 Half-integer parameters

$$\begin{aligned} J_{\frac12}(z) &= = \left(\frac{2}{\pi z}\right)^{\frac12} \sin z, \\ J_{-\frac12}(z) &= \left(\frac{2}{\pi z}\right)^{\frac12} \cos z, \\ H^{\pm}_{\frac12}(z) &= \mp \mathrm{i} \left(\frac{2}{\pi z}\right)^{\frac12} \mathrm{e}^{\pm \mathrm{i} z}, \\ H^{\pm}_{-\frac12}(z) &= \left(\frac{2}{\pi z}\right)^{\frac12} \mathrm{e}^{\pm \mathrm{i} z} \end{aligned}$$

For $k = 0, 1, 2, \ldots$ we have

$$\begin{aligned} J_{\frac12+k}(z) &= \left(\frac{2}{\pi}\right)^{\frac12} z^{\frac12+k} \left(-\frac{1}{z}\partial_z\right)^k \frac{\sin z}{z} \qquad (3.114) \\ &= \left(\frac{2}{\pi}\right)^{\frac12} 2^k \sum_{j=k}^{\infty} (-1)^j z^{\frac12+2j-k} \frac{(-j)_k}{(2j+1)!}, \end{aligned}$$

$$\begin{aligned} J_{-\frac12-k}(z) &= \left(\frac{2}{\pi}\right)^{\frac12} z^{\frac12+k} \left(\frac{1}{z}\partial_z\right)^k \frac{\cos z}{z} \qquad (3.115) \\ &= \left(\frac{2}{\pi}\right)^{\frac12} (-2)^k \sum_{j=0}^{\infty} (-1)^j z^{-\frac12+2j-k} \frac{(\frac12 - j)_k}{(2j)!}, \end{aligned}$$

$$\begin{aligned} H^{\pm}_{\frac12+k}(z) &= \mp \mathrm{i} \left(\frac{2}{\pi}\right)^{\frac12} z^{\frac12+k} \left(-\frac{1}{z}\partial_z\right)^k \frac{\mathrm{e}^{\pm \mathrm{i} z}}{z} \qquad (3.116) \\ &= \mp \mathrm{i} \left(\frac{2}{\pi}\right)^{\frac12} 2^k \sum_{n=0}^{\infty} (\pm \mathrm{i})^n z^{-\frac12-k+n} \frac{(\frac12 - \frac{n}{2})_k}{n!}, \end{aligned}$$

$$\begin{aligned} H^{\pm}_{-\frac12-k}(z) &= \left(\frac{2}{\pi}\right)^{\frac12} z^{\frac12+k} \left(\frac{1}{z}\partial_z\right)^k \frac{\mathrm{e}^{\pm \mathrm{i} z}}{z} \qquad (3.117) \\ &= \left(\frac{2}{\pi}\right)^{\frac12} (-2)^k \sum_{n=0}^{\infty} (\pm \mathrm{i})^n z^{-\frac12-k+n} \frac{(\frac12 - \frac{n}{2})_k}{n!}. \end{aligned}$$

3.12 Wronskians of solutions of the Bessel equation

The Wronskian of two solutions of the Bessel equation satisfies

$$\left(\partial_z + \frac{1}{z}\right) W(z) = 0.$$

Hence $W(z)$ is proportional to $\frac{1}{z}$. Using

$$J_{\pm m}(z) \sim \tfrac{1}{\Gamma(\pm m+1)} (\tfrac{z}{2})^{\pm m}, \qquad J'_{\pm m}(z) \sim \tfrac{1}{\Gamma(\pm m)} (\tfrac{z}{2})^{\pm m-1},$$

we can compute the Wronskian of $J_m(z)$ and $J_{-m}(z)$:

$$W(J_m, J_{-m}) = -\frac{2\sin\pi m}{\pi z}, \qquad W(H_m^-, H_m^+) = -\frac{4\mathrm{i}}{\pi z}, \qquad W(J_m, Y_m) = \frac{2}{\pi z}.$$

3.13 Putting together the Hankel and Macdonald function

K_m and $H_m^\pm$ are naturally analytic functions on the Riemann's surface of the logarithm. However, it is sometimes convenient to treat them together, as functions of $\sqrt{w}$, resp. $\sqrt{-w}$, where $w \in \mathbb{R}$. It is important to indicate precisely how the analytic continuation of the square root is performed—whether we bypass the branch point at zero from above or from below. This is encoded by adding $\pm\mathrm{i}0$ to the variable. More precisely, we will use the following notation:

$$K_m\big(\sqrt{w \mp \mathrm{i}0}\big) := \begin{cases} K_m\big(\sqrt{w}\big), & w > 0, \\ K_m\big(\mp\mathrm{i}\sqrt{-w}\big) = \pm\mathrm{i}\frac{\pi}{2}\mathrm{e}^{\pm\mathrm{i}\pi m}H_m^\pm\big(\sqrt{-w}\big), & w < 0; \end{cases} \tag{3.118a}$$

$$H_m^\pm\big(\sqrt{-w \pm \mathrm{i}0}\big) := \begin{cases} H_m^\pm\big(\pm\mathrm{i}\sqrt{w}\big) = \mp\mathrm{i}\frac{2}{\pi}\mathrm{e}^{\mp\mathrm{i}\pi m}K_m\big(\sqrt{w}\big), & w > 0, \\ H_m^\pm\big(\sqrt{-w}\big), & w < 0. \end{cases} \tag{3.118b}$$

Note the functions (3.118a) and (3.118b) should be understood as distributions, possibly with a singularity at 0.

4 Helmholtz equation and the group $\mathbb{R}^2 \times SO(2)$

4.1 Action of a group on functions

Let G be a group with the neutral element $e \in G$. Let X be a set. Let $\mathrm{Bij}(X)$ denote the set of bijections of X. Clearly, $\mathrm{Bij}(X)$ is a group. A homomorphism $\pi : G \to \mathrm{Bij}(X)$ is called the action of G on X. Thus

$$\pi(e) = \mathrm{Id}, \quad \pi(gh) = \pi(g)\pi(h). \tag{4.1}$$

In other words

$$\pi(g)\big(\pi(h)x\big) = \big(\pi(g)\pi(h)\big)x, \qquad g, h \in G, \quad x \in X. \tag{4.2}$$

Suppose $F(X)$ denotes the set of functions in a certain favorite space (typically $\mathbb{C}$). Then we have the action of G on $F(X)$:

$$\big(\pi^*(g)f\big)(x) := f\big(\pi^{-1}(g)x\big). \tag{4.3}$$

Let us check that this is a homomorphism of G:

$$\Big(\big(\pi^*(g)\pi^*(h)\big)f\Big)(x) = \Big(\pi^*(g)\big(\pi^*(h)f\big)\Big)(x) \tag{4.4}$$

$$= \big(\pi^*(h)f\big)\big(\pi(g)^{-1}x\big) = f\big(\pi(h)^{-1}\pi(g)^{-1}x\big) \tag{4.5}$$

$$= f\big(\pi(h^{-1}g^{-1})x\big) = f\Big(\pi\big((gh)^{-1}\big)x\Big). \tag{4.6}$$

4.2 Translation operator

$\mathbb{R}$ acts on $\mathbb{R}$ by translations:

$$u_t x = x + t, \qquad x, t \in \mathbb{R}. \tag{4.7}$$

The corresponding action on functions on $\mathbb{R}$ will be denoted U_t:

$$(U_t f)(x) := f(x - t).$$

It satisfies $U_t U_s = U_{t+s}$, $U_t = 1$. U_t can be understood as a family of operators acting on various spaces of functions on $\mathbb{R}$. For instance, U_t can be interpreted as operators on $L^2(\mathbb{R})$, in which case they are unitary.

We also have the operator ∂_x. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t f = -\partial_x U_t f.$$

Therefore, we write

$$U_t = \mathrm{e}^{-t\partial_x}$$

and call ∂_x the generator of translations. In quantum mechanics customarily instead of ∂_x one uses the momentum operator $p = \frac{\hbar}{\mathrm{i}}\partial_x$, which is Hermitian in the sense of $L^2(\mathbb{R})$.

4.3 Rotation operator on $L^2(\mathbb{R}^2)$

Let $SO(2)$ denote the group of rotations of the plane $\mathbb{R}^2$. It can be parametrized by $\theta \in [0, 2\pi[$ and acts on $\mathbb{R}^2$ as follows:

$$r_\theta(x, y) := (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y). \tag{4.8}$$

The corresponding action on functions on $\mathbb{R}^2$ will be denoted R_θ and is given be

$$(R_\theta f)(x, y) := f(\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y).$$

Again, we have $R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$, $R_0 = \mathrm{Id}$. In particular, understood as operators on $L^2(\mathbb{R}^2)$, they are unitary.

Define

$$L = x\partial_y - y\partial_x.$$

We will show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} R_\theta f = -L R_\theta f. \tag{4.9}$$

Introduce notation

$$\tilde{x} := x\cos\theta + y\sin\theta, \qquad \tilde{y} := -x\sin\theta + y\cos\theta.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}R_\theta f(x,y) &= (-x\sin\theta + y\cos\theta)\partial_{\tilde{x}} f(\tilde{x},\tilde{y}) \\
&\quad +(-x\cos\theta - y\sin\theta)\partial_{\tilde{y}} f(\tilde{x},\tilde{y}), \\
LR_\theta f(x,y) &= x(\sin\theta\partial_{\tilde{x}} + \cos\theta\partial_{\tilde{y}}) f(\tilde{x},\tilde{y}) \\
&\quad -y(\cos\theta\partial_{\tilde{x}} - \sin\theta\partial_{\tilde{y}}) f(\tilde{x},\tilde{y}),
\end{aligned}$$

which shows (4.9). Therefore, we write

$$R_\theta = \mathrm{e}^{-\theta L}.$$

In quantum mechanics customarily one uses the angular momentum operator $\frac{\hbar}{\mathrm{i}}L$, which is Hermitian in the sense of $L^2(\mathbb{R}^2)$.

4.4 The group $\mathbb{R}^2 \rtimes SO(2)$

Consider translations and rotations on $\mathbb{R}^2$:

$$u_{(x_1,y_1)}(x,y) = (x + x_1, y + y_1), \tag{4.10}$$

$$r_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \tag{4.11}$$

Transformations of the form $(x_1, y_1, \theta) := u_{(x_1,y_1)} r_\theta$ form a group:

$$\begin{aligned}
&(x_2, y_2, \theta_2)(x_1, y_1, \theta_1) \\
=&(x_2 + \cos\theta_2 x_1 - \sin\theta_2 y_1, y_2 + \sin\theta_2 x_1 + \cos\theta_2 y_1, \theta_2 + \theta_1)
\end{aligned} \tag{4.12}$$

We have an obvious complex form of this group, where we write $z_i = x_i + \mathrm{i}y_i$, $w_i = \mathrm{e}^{\mathrm{i}\theta_i}$. Then (4.12) corresponds to

$$(z_2, w_2)(z_1, w_1) = (z_2 + w_2 z_1, w_2 w_1). \tag{4.13}$$

For a function f on $\mathbb{R}^2$ set

$$U_{(x_1,y_1)} f(x,y) = f\Big(u^{-1}_{(x_1,y_1)}(x,y)\Big), \tag{4.14}$$

$$R_\theta f(x,y) = f\big(r_\theta^{-1}(x,y)\big) \tag{4.15}$$

We have

$$(U_{(x_1,y_1)} f)(x) := f(x - x_1, y - y_1), \tag{4.16}$$

$$(R_\theta f)(x,y) := f(\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y). \tag{4.17}$$

These operators can be understood on various spaces of functions on $\mathbb{R}^2$, e.g. on $L^2(\mathbb{R}^2)$. We have the representation

$$\mathbb{R}^2 \rtimes SO(2) \ni (x,y,\theta) \mapsto U(x,y,\theta) := U(x,y)R(\theta) \in U(L^2(\mathbb{R}^2).$$

The Laplacian Δ is an invariant operator:

$$U(x,y,\theta)\Delta U(x,y,\theta)^{-1} = \Delta. \tag{4.18}$$

The operators ∂_x, ∂_y and L form a Lie algebra commuting with Δ:

$$[\partial_x, L] = \partial_y, \tag{4.19}$$

$$[\partial_y, L] = -\partial_x. \tag{4.20}$$

Introduce

$$\begin{aligned} A^+ : &= \partial_x + \mathrm{i}\partial_y, \\ A^- : &= \partial_x - \mathrm{i}\partial_y. \end{aligned}$$

Note the relations

$$[A^+, A^-] \;=\; 0, \tag{4.21}$$

$$[A^\pm, L] \;=\; \mp\mathrm{i}A^\pm, \tag{4.22}$$

$$\Delta \;=\; A^+A^- = A^-A^+, \tag{4.23}$$

$$[\Delta, L] \;=\; [\Delta, A^\pm] \;=\; 0. \tag{4.24}$$

4.5 Polar coordinates

Introduce in $\mathbb{R}^2$ polar coordinates

$$x = r\cos\phi, \quad y = r\sin\phi,$$

$$r = \sqrt{x^2+y^2}, \quad \phi = \arctan\frac{y}{x}.$$

Remark 4.1 *Change from Cartesian coordinates to polar cordinates can be interpreted as a unitary transformation* $U : L^2(\mathbb{R}^2) \to L^2([0,\infty[\times[0,2\pi],\ r\mathrm{d}r\mathrm{d}\phi)$ *defined as*

$$(Uf)(r,\phi) := f(r\cos\phi, r\sin\phi).$$

It is also convenient to introduce another variety of polar coordinates, setting $w = \mathrm{e}^{\mathrm{i}\phi}$:

$$\begin{aligned} x &= \tfrac{r}{2}(w + w^{-1}), \\ y &= \tfrac{r}{2\mathrm{i}}(w - w^{-1}). \end{aligned}$$

In the polar coordinates we have

$$\begin{aligned} \partial_x &= \cos\phi\partial_r - r^{-1}\sin\phi\partial_\phi, \\ \partial_y &= \sin\phi\partial_r + r^{-1}\cos\phi\partial_\phi, \\ A^+ &= \mathrm{e}^{\mathrm{i}\phi}(\partial_r + \mathrm{i}r^{-1}\partial_\phi) = w(\partial_r - r^{-1}w\partial_w), \\ A^- &= \mathrm{e}^{-\mathrm{i}\phi}(\partial_r - \mathrm{i}r^{-1}\partial_\phi) = w^{-1}(\partial_r + r^{-1}w\partial_w), \\ (R_\theta f)(r,\phi) &= f(r, \phi-\theta), \quad R_\theta f(r,w) = f(r, w\mathrm{e}^{-\mathrm{i}\theta}), \\ L &= \partial_\phi \;=\; \mathrm{i}w\partial_w. \end{aligned}$$

Here is the Laplacian in polar coordinates:

$$\begin{aligned} \Delta &= \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\phi^2 \\ &= \partial_r^2 + r^{-1}\partial_r - r^{-2}(w\partial_w)^2. \end{aligned}$$

4.6 Helmholtz equation

Suppose Ψ, a distribution in $\mathcal{S}'(\mathbb{R}^2)$, solves the Helmholtz equation

$$(\Delta + 1)\Psi = 0. \tag{4.25}$$

Let $\hat{\Psi}$ be the Fourier transform of Ψ. We have

$$\Psi(x, y) = \frac{1}{(2\pi)^2} \int\int \hat{\Psi}(\xi, \eta) \mathrm{e}^{\mathrm{i}(x\xi + y\eta)} \mathrm{d}\xi \mathrm{d}\eta. \tag{4.26}$$

(4.25) and (4.26) yield

$$(-\xi^2 - \eta^2 + 1)\hat{\Psi}(\xi, \eta) = 0,$$

so that $\mathrm{supp}\hat{\Psi} \subset \{(\xi, \eta) \mid \xi^2 + \eta^2 = 1\}$. Thus the Helmholtz equation is solved by

$$\Psi(g)(x, y) = \frac{1}{2\pi} \int \mathrm{e}^{\mathrm{i}(x \cos\psi + y \sin\psi)} g(\psi) \mathrm{d}\psi. \tag{4.27}$$

g is an arbitrary distribution on the unit circle $\mathbb{T}$, that is $g \in \mathcal{S}'(\mathbb{T})$

We will denote by $\mathcal{F}\mathcal{S}'(\mathbb{T})$ the Hilbert space of functions on $\mathbb{R}^2$ of the form (4.27) with $g \in \mathcal{S}'(\mathbb{T})$. Inside this space there is a Hibert space $\mathcal{F}L^2(\mathbb{T})$ with $g \in L^2(\mathbb{T})$ with the scalar product

$$\big(\Psi(g_1)|\Psi(g_2)\big) := \int_0^{2\pi} \overline{g_1(\psi)} g_2(\psi) \mathrm{d}\psi.$$

4.7 Plane waves and circular waves

Let $\delta_\psi(\phi) := \delta(\phi - \psi)$ denote the deltafunction on the circle at the point $\psi \in \mathbb{T}$. $\delta_\psi(\phi)$ is the integral kernel of the identity transformation:

$$g(\phi) = \int_{\mathbb{T}} g(\psi)\delta_\psi(\phi)\mathrm{d}\psi. \tag{4.28}$$

Moreover, $\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}$ is an o.n. basis of $L^2(\mathbb{T})$. Therefore,

$$\delta_\psi(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi - \psi)}, \tag{4.29}$$

which is the Fourier decomposition of δ_ψ.

Define the *plane wave* in the direction ψ, which we write in Cartesian and polar coordinates:

$$f_\psi := \Psi(2\pi\delta_\psi), \tag{4.30}$$

$$f_\psi(x, y) = \mathrm{e}^{\mathrm{i}(x \cos\psi + y \sin\psi)}, \tag{4.31}$$

$$f_\psi(r, \phi) = \mathrm{e}^{\mathrm{i}r \cos(\phi - \psi)}. \tag{4.32}$$

Define also the the *mth circular wave* with $g_m := \mathrm{i}^{-m}\mathrm{e}^{\mathrm{i}m\phi}$:

$$f_m := \Psi(g_m), \tag{4.33}$$

$$f_m(x,y) = \frac{1}{2\pi}\mathrm{i}^{-m}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(x\cos\psi + y\sin\psi)}\mathrm{e}^{\mathrm{i}m\psi}\mathrm{d}\psi, \tag{4.34}$$

$$\begin{aligned} f_m(r,\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}r\cos(\psi-\phi)}\mathrm{e}^{\mathrm{i}m(\psi-\frac{\pi}{2})}\mathrm{d}\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}r\sin\xi}\mathrm{e}^{\mathrm{i}m(\phi-\xi)}\mathrm{d}\xi. \end{aligned} \tag{4.35}$$

Using (4.28) with $g = g_m$ for (4.36) and (4.29) for (4.37) we see that one can pass from the plane waves to circular waves and back::

$$f_m = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{i}^{-m} f_\psi \mathrm{e}^{\mathrm{i}m\psi}\mathrm{d}\psi, \tag{4.36}$$

$$f_\psi = \sum_{m=-\infty}^{\infty} \mathrm{i}^m f_m \mathrm{e}^{-\mathrm{i}m\psi}. \tag{4.37}$$

Recall now the formulas

$$J_m(r) = \frac{1}{2\pi\mathrm{i}}\int_{[0^+]} \exp\Big(\frac{r}{2}(t-t^{-1})\Big)t^{-m-1}\mathrm{d}t, \tag{4.38}$$

$$\mathrm{e}^{\frac{r}{2}(t-t^{-1})} = \sum_{m=-\infty}^{\infty} t^m J_m(r). \tag{4.39}$$

Setting $t = \mathrm{e}^{\mathrm{i}\phi}$ we obtain

$$J_m(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}r\sin\phi}\mathrm{e}^{-\mathrm{i}m\phi}\mathrm{d}\phi, \tag{4.40}$$

$$\mathrm{e}^{\mathrm{i}r\sin\phi} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\phi} J_m(r). \tag{4.41}$$

Thus

$$f_m(r,\phi) = J_m(r)\mathrm{e}^{\mathrm{i}m\phi}, \tag{4.42}$$

(4.36) is essentially equivalent to (4.40), the Bessel integral representation of J_m and (4.37) corresponds to the formula for the generating function (4.41).

4.8 Action of translations and rotations on solutions of the Helmholtz equation

The operators $U_{(x,y)}$ preserve $\mathcal{F}\mathcal{S}'(\mathbb{T})$ and $\mathcal{F}\mathrm{L}^2(\mathbb{T})$. They are unitary on $\mathcal{F}\mathrm{L}^2(\mathbb{T})$:

$$U_{(x_0,y_0)}\Psi(g) = \Psi\big(\mathrm{e}^{-\mathrm{i}\cos\psi x_0 - \mathrm{i}\sin\psi y_0}g\big), \tag{4.43}$$

$$\partial_x\Psi(g) = \Psi(\mathrm{i}\cos\psi g), \tag{4.44}$$

$$\partial_y\Psi_g = \Psi(\mathrm{i}\sin\psi g) \tag{4.45}$$

In other words, plane waves diagonalize translations:

$$U_{x_0,y_0} f_\psi = \mathrm{e}^{-\mathrm{i}x_0\cos\psi - \mathrm{i}y_0\sin\psi)} f_\psi.$$

$$\partial_x f_\psi = \mathrm{i}\cos\psi f_\psi, \qquad \partial_y f_\psi = \mathrm{i}\sin\psi f_\psi. \tag{4.46}$$

The operators R_θ preserve the space $\mathcal{F}L^2(\mathbb{T})$ and are unitary:

$$R_\theta \Psi(g) = \Psi(g(\cdot - \theta)), \tag{4.47}$$

$$L\Psi(g) = \Psi(\partial_\psi g). \tag{4.48}$$

Thus circular waves diagonalize rotations:

$$R_\theta f_m = \mathrm{e}^{-\mathrm{i}m\theta} f_m, \tag{4.49}$$

$$L f_m = \mathrm{i}m f_m. \tag{4.50}$$

By (4.22) $A^\pm$ raises/lowers L by i. Indeed, let $L\Psi = \mathrm{i}m$. Then

$$LA^\pm\Psi = [L, A^\pm]\Psi + A^\pm L\Psi = \mathrm{i}(m\pm 1)A^\pm\Psi. \tag{4.51}$$

More precisely, we compute

$$A^+ f_m = -f_{m+1}, \qquad A^- f_m = f_{m-1}. \tag{4.52}$$

Indeed,

$$\begin{aligned} A^\pm f_m(x,y) &= (\partial_x \pm \mathrm{i}\partial_y)\frac{\mathrm{i}^{-m}}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(x\cos\psi + y\sin\psi)}\mathrm{e}^{\mathrm{i}m\psi}\mathrm{d}\psi \\ &= \mathrm{e}^{\pm\mathrm{i}\psi}\mathrm{i}\frac{\mathrm{i}^{-m}}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(x\cos\psi + y\sin\psi)}\mathrm{e}^{\mathrm{i}m\psi}\mathrm{d}\psi = \mp f_{m\pm1}(x,y). \end{aligned} \tag{4.53}$$

Now

$$\begin{aligned} A^\pm f_m(r,\phi) &= \mathrm{e}^{\pm\mathrm{i}\phi}(\partial_r \pm \mathrm{i}r^{-1}\partial_\phi)\mathrm{e}^{\mathrm{i}m\phi}J_m(r) \\ &= (\partial_r \mp m r^{-1})\mathrm{e}^{\mathrm{i}(m\pm1)\phi}J_{m\pm1}(r). \end{aligned} \tag{4.54}$$

Comparing (4.53) and (4.54) we obtain a new proof of the recurrence relations

$$\Big(\partial_r - \frac{m}{r}\Big)J_m = -J_{m+1}, \qquad \Big(\partial_r + \frac{m}{r}\Big)J_m = J_{m-1}. \tag{4.55}$$

4.9 Graf addition formula

Theorem 4.2 *Assume that R, r, ρ and Φ, ϕ, ψ are related as*

$$R = \sqrt{(r\mathrm{e}^{\mathrm{i}\phi} + \rho\mathrm{e}^{\mathrm{i}\psi})(r\mathrm{e}^{-\mathrm{i}\phi} + \rho\mathrm{e}^{-\mathrm{i}\psi})}, \qquad \mathrm{e}^{\mathrm{i}\Phi} = \sqrt{\frac{r\mathrm{e}^{\mathrm{i}\phi} + \rho\mathrm{e}^{\mathrm{i}\psi}}{r\mathrm{e}^{-\mathrm{i}\phi} + \rho\mathrm{e}^{-\mathrm{i}\psi}}}.$$

Then

$$J_m(R)\mathrm{e}^{\mathrm{i}m\Phi} = \sum_{n=-\infty}^{\infty} J_n(\rho)\mathrm{e}^{\mathrm{i}n\psi} J_{m-n}(r)\mathrm{e}^{\mathrm{i}(m-n)\phi}.$$

If $m \in \mathbb{Z}$, then there are no restrictions on the parameters in the formula. If m is nonintegral and all variables real, then one has to assume that $\rho < r$ (or, equivalently, $|\Phi - \phi| < \frac{\pi}{2}$). We can then replace the Bessel function in $J_m(R)$ and $J_{m-n}(r)$ with $H_m^{(i)}$ or Y_m

Proof. We put $\tilde{\psi} = \psi - \phi$, $\tilde{\Phi} = \Phi - \phi$. Then the problem is reduced to the case $\phi = 0$.

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} J_n(\rho)\mathrm{e}^{\mathrm{i}n\psi} J_{m-n}(r) \\
= & \frac{1}{2\pi\mathrm{i}} \sum_{n=-\infty}^{\infty} \int_\gamma \exp\big(\frac{r}{2}(t - t^{-1})\big) t^{-m-1} J_n(\rho)(t\mathrm{e}^{\mathrm{i}\psi})^n \\
= & \frac{1}{2\pi\mathrm{i}} \int_\gamma \exp\Big(\frac{r}{2}(t - t^{-1})\big) + \frac{\rho}{2}\big(t\mathrm{e}^{\mathrm{i}\psi} - (t\mathrm{e}^{\mathrm{i}\psi})^{-1}\big)\Big) t^{-m-1}\mathrm{d}t \\
= & \frac{1}{2\pi\mathrm{i}} \int_\gamma \exp\big(\frac{R}{2}(s - s^{-1})\big) s^{-m-1}\mathrm{d}s\mathrm{e}^{\mathrm{i}m\Phi} = \mathrm{e}^{\mathrm{i}m\Phi} J_m(R).
\end{aligned}$$

In the first step we used the integral representation of $J_{m-n}(r)$, in the second step we used the generating function to sum up $J_n(\rho)$, finally, in the last step we used $r + \rho\mathrm{e}^{\mathrm{i}\psi} = R\mathrm{e}^{\mathrm{i}\Phi}$ and turned the contour. □

Substituting

$$\begin{aligned}
x_1 &= r\cos\phi, \qquad & y_1 &= r\sin\phi, \\
x_2 &= \rho\cos\psi, \qquad & y_2 &= \rho\sin\psi, \\
x &= R\cos\Phi, \qquad & y &= R\sin\Phi,
\end{aligned}$$

we obtain

$$(x_1, y_1) + (x_2, y_2) = (x, y)$$

and the addition formula can be rewritten as

$$\begin{aligned}
& J_m(\sqrt{x^2+y^2})\Big(\frac{x + \mathrm{i}y}{\sqrt{x^2+y^2}}\Big)^m \qquad (4.56) \\
= & \sum_{n\in\mathbb{Z}} J_{m-n}(\sqrt{x_2^2+y_2^2})\Big(\frac{x_2 + \mathrm{i}y_2}{\sqrt{x_2^2+y_2^2}}\Big)^{m-n} J_n(\sqrt{x_1^2+y_1^2})\Big(\frac{x_1 + \mathrm{i}y_1}{\sqrt{x_1^2+y_1^2}}\Big)^n.
\end{aligned}$$

Below we will interprete the Graf addition formula for $m, n \in \mathbb{Z}$ in terms of the representation of the translation group. Let us compute the matrix elements of the translation $U(x,y)$ in the basis f_m. Let $(x,y)=(r\cos\psi, r\sin\psi)$:

$$U_{nm}(x,y) = \frac{1}{2\pi}\big(f_n|U(x,y)f_m\big) \tag{4.57}$$
$$= \frac{1}{2\pi}\int \mathrm{e}^{-\mathrm{i}r\cos(\psi-\phi)}\mathrm{e}^{\mathrm{i}(m-n)\phi}\mathrm{i}^{-m+n}\mathrm{d}\phi \tag{4.58}$$
$$= J_{m-n}(r)\mathrm{e}^{-\mathrm{i}(m-n)\psi}. \tag{4.59}$$

Now, by definition of matrix elements, and then by the group property,

$$\big(U(-x,-y)f_m\big)(x_1,y_1) = f_m(x_1+x, y_1+y) \tag{4.60}$$
$$= \sum_{n=-\infty}^{\infty} f_n(x_1,y_1)U_{nm}(-x,-y),$$
$$U_{k,m}\big(-x_2-x_1,-y_2-y_1\big) = \sum_{n=-\infty}^{\infty} U_{k,n}(-x_2,-y_2)U_{n,m}(-x_1,-y_1). \tag{4.61}$$

Both (4.60) and (4.61) are interpretations of the Graf addition formula.

4.10 Helmholtz equation on a disk

Consider the disc $K=\{x^2+y^2<1\}$ and the Helmholtz equation on K with the Dirichlet

$$\Delta_{\mathrm{D}}F+\omega^2F=0,\ (x,y)\in K;\quad F(x,y)=0,\ (x,y)\in\partial K; \tag{4.62}$$

and Neumann boundary conditions

$$\Delta_{\mathrm{N}}F+\omega^2F=0,\ (x,y)\in K;\quad (x\partial_x+y\partial_y)F(x,y)=0,\ (x,y)\in\partial K. \tag{4.63}$$

We are looking for $F\in L^2(K)$ and $\omega\geq 0$ that solve these eigenvalue problems.

One can show if $F\in L^2(K)$ satisfies the Helmholtz equation, it has to be smooth.

Recall that

$$\Delta = \partial_r^2+\frac{1}{r}\partial_r+\frac{\partial_\phi^2}{r^2}. \tag{4.64}$$

Solutions of the radial equation are spanned by $J_m(\omega r)$ and $H_m^{\pm}(\omega r)$, $m=0,1,2,\dots$. It is easy to see that $J_m(\omega r)\mathrm{e}^{\mathrm{i}m\phi}$, $m\in\mathbb{Z}$, are smooth.

The functions $H_m^{\pm}(\omega r)\mathrm{e}^{\mathrm{i}m\phi}$ are not smooth, at zero, hence they are discarded. Actually, $H_m^{\pm}(\omega r)\sim r^{-m}$, $m=1,2,\dots$, therefore $H_m^{\pm}(\omega r)\mathrm{e}^{\mathrm{i}m\phi}$ are even not square integrable for such m.

The function $H_0^{\pm}(\omega r)\sim\log(r)$, is square integrable. It should be discarded as well—it corresponds to a “Dirac delta charge” at the origin. In fact, if we apply the Gauss law and compute the flux through the circle around 0 we obtain

$$\int_0^{2\pi} r\mathrm{d}\phi\partial_r\ln(r) = \frac{1}{r}2\pi r = 2\pi.$$

In other words

$$\Delta \ln(r) = 2\pi\delta_0, \quad \text{hence} \quad (\Delta + \omega^2)H_0^{\pm}(\omega r) = 2\pi\delta_0. \tag{4.65}$$

Hence acceptable solutions are given by

$$F(r,\phi) = \mathrm{e}^{\mathrm{i}m\phi} J_m(\omega r), \quad m \in \mathbb{Z}. \tag{4.66}$$

The boundary conditions give

$$J_m(\omega) = 0, \quad \text{(Dirichlet)} \tag{4.67}$$

$$J_m'(\omega) = 0, \quad \text{(Neumann)}. \tag{4.68}$$

The Helmholtz equation can be derived from the wave equation

$$(-\partial_t^2 + \Delta)f = 0 \tag{4.69}$$

by setting $F(t,x,y) = \mathrm{e}^{\mathrm{i}\omega t}F(x,y)$. Thus the frequencies of a disc-shaped drum are the zeros of the Bessel function J_m, and the frequencies of a disc-shaped cymbal are the zeros of its derivative J_m'.

5 Distributions in $d = 1$

5.1 Homogeneous distributions of order -1 and 0

The function $\frac{1}{x}$ is not in L^1_{loc}, therefore it does not define a regular distribution. However, it can be naturally interpreted as a distribution as follows

$$\mathcal{P}\int \frac{1}{x}\phi(x)\mathrm{d}x := \Big(\int_{-\infty}^{a} + \int_{a}^{\infty}\Big)\frac{1}{x}\phi(x)\mathrm{d}x + \int_{-a}^{a}\frac{1}{x}\big(\phi(x) - \phi(0)\big)\mathrm{d}x.$$

(We are writing $\mathcal{P}$ to indicate that it is not the usual integral). Equivalently,

$$\frac{1}{x} := \frac{1}{2}\Big(\frac{1}{(x+\mathrm{i}0)} + \frac{1}{(x-\mathrm{i}0)}\Big).$$

The Sochocki formula is relationship between three kinds of order -1 distributions:

$$\frac{1}{x \pm \mathrm{i}0} = \frac{1}{x} \mp \mathrm{i}\pi\delta(x).$$

$$\int e^{-ixk}dx = 2\pi\delta(k), \tag{5.70}$$

$$\int \theta(\pm x)e^{-ixk}dx = \frac{\mp i}{k \mp i0}, \tag{5.71}$$

$$\int \mathrm{sgn}(x)e^{-ixk}dx = -2i\frac{1}{k}, \tag{5.72}$$

$$\int \delta(x)e^{-ixk}dx = 1, \tag{5.73}$$

$$\int \frac{e^{-ikx}}{x \pm i0}dx = \mp 2\pi i\theta(\pm k), \tag{5.74}$$

$$\mathcal{P}\int \frac{e^{-ikx}}{x}dx = -\pi i\mathrm{sgn}(k), \tag{5.75}$$

$$\int e^{-i\xi s}(s-\lambda)^{-1}d\ s = \begin{cases} -2\pi i\theta(\xi)e^{-i\lambda\xi} & \mathrm{Im}\lambda < 0, \\ 2\pi i\theta(-\xi)e^{-i\lambda\xi} & \mathrm{Im}\lambda > 0; \end{cases} \tag{5.76}$$

$$\mathcal{P}\int e^{-i\xi s}(s-\lambda)^{-1}d\ s = -\pi i\mathrm{sgn}(\xi)e^{-i\lambda\xi}, \quad \mathrm{Im}\lambda = 0. \tag{5.77}$$

5.2 Homogeneous distributions of integral order

Define for $n = 0, 1, 2, \ldots$

$$\frac{1}{x^{n+1}} := \frac{1}{2}\Big(\frac{1}{(x+i0)^{n+1}} + \frac{1}{(x-i0)^{n+1}}\Big).$$

Clearly

$$\partial_x^n \frac{1}{x} = (-1)^n n! \frac{1}{x^{n+1}}.$$

$$\int x^n e^{-ixk}dx = 2\pi i^n \delta^{(n)}(k), \tag{5.78}$$

$$\int x^n \theta(\pm x)e^{-ixk}dx = \pm\frac{(-i)^{n+1}n!}{(k \mp i0)^{n+1}}, \tag{5.79}$$

$$\int x^n \mathrm{sgn}(x)e^{-ixk}dx = 2(-i)^{n+1}n!\frac{1}{k^{n+1}}, \tag{5.80}$$

$$\int \delta^{(n)}(x)e^{-ixk}dx = i^n k^n, \tag{5.81}$$

$$\int \frac{e^{-ikx}}{(x \pm i0)^{n+1}}dx = \pm\frac{2\pi(-i)^{n+1}}{n!}k^n\theta(\pm k), \tag{5.82}$$

$$\mathcal{P}\int \frac{e^{-ikx}}{x^{n+1}}dx = \frac{\pi(-i)^{n+1}}{n!}k^n\mathrm{sgn}(k). \tag{5.83}$$

5.3 Homogeneous distributions of arbitrary order I

For any $\lambda \in \mathbb{C}$

$$(\pm \mathrm{i}x + 0)^\lambda := \lim_{\epsilon \to 0} (\pm \mathrm{i}x + \epsilon)^\lambda.$$

is a tempered distribution. If $\mathrm{Re}\lambda > -1$, then it is simply the distribution given by the locally integrable function

$$\mathrm{e}^{\pm \mathrm{i}\,\mathrm{sgn}(x) \frac{\pi}{2}\lambda} |x|^\lambda. \tag{5.84}$$

The functions

$$x_\pm^\lambda := (\pm x)^\lambda \theta(\pm x) \tag{5.85}$$

define distributions only for $\mathrm{Re}\lambda > -1$. We can extend them to $\lambda \in \mathbb{C}$ except for $\lambda = -1, -2, \dots$ by putting

$$x_\pm^\lambda := \frac{1}{2\mathrm{i}\sin \pi\lambda}\Big(-\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\lambda}(\mp \mathrm{i}x + 0)^\lambda + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\lambda}(\pm \mathrm{i}x + 0)^\lambda \Big). \tag{5.86}$$

Instead of $x_\pm^\lambda$ it is often more convenient to consider

$$\rho_\pm^\lambda(x) := \frac{x_\pm^\lambda}{\Gamma(\lambda+1)} \tag{5.87}$$

$$= \frac{\Gamma(-\lambda)}{2\pi\mathrm{i}}\Big(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\lambda}(\mp \mathrm{i}x + 0)^\lambda - \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\lambda}(\pm \mathrm{i}x + 0)^\lambda \Big). \tag{5.88}$$

Note that using (5.87) and (5.88) we have defined $\rho_\pm^\lambda$ for all $\lambda \in \mathbb{C}$.

Theorem 5.1 *The distributions $\rho_\pm^\lambda$ satisfy the recurrence relations*

$$\partial_x \rho_\pm^\lambda(x) = \pm \rho_\pm^{\lambda-1}(x).$$

At integers we have

$$\rho_\pm^n = \frac{x_\pm^n}{n!}, \quad n = 0, 1, \dots; \tag{5.89}$$

$$\rho_\pm^{-n-1} = (\pm 1)^n \delta^n(x), \quad n = 0, 1, \dots. \tag{5.90}$$

Their Fourier transforms are below:

$$\int \mathrm{e}^{-\mathrm{i}\xi x} \rho_\pm^\lambda(x) \mathrm{d}x = (\pm \mathrm{i}\xi + 0)^{-\lambda-1},$$

$$\int \mathrm{e}^{-\mathrm{i}\xi x} (\mp \mathrm{i}\xi + 0)^\lambda \mathrm{d}\xi = 2\pi \rho_\pm^{-\lambda-1}(x).$$

Proof. (5.90) follows from

$$\rho_\pm^{-n-1}(x) = \frac{(\mp 1)^{n+1} n!}{2\pi\mathrm{i}}\Big((x \pm \mathrm{i}0)^{-n-1} - (x \mp \mathrm{i}0)^{-n-1} \Big) \tag{5.91}$$

$$= -\frac{(\pm 1)^n}{2\pi\mathrm{i}} \partial_x^n \Big((x \pm \mathrm{i}0)^{-1} - (x \mp \mathrm{i}0)^{-1} \Big). \tag{5.92}$$

□

Theorem 5.2 *Let* $-n-1<\mathrm{Re}\lambda$, $\lambda\not\in\{\dots,-2,-1\}$. *Then for any* $a>0$,

$$\begin{aligned}\mathcal{P}\int x_+^\lambda\phi(x)\mathrm{d}x=&\int_a^\infty x^\lambda\phi(x)\mathrm{d}x\\ &+\int_0^a x^\lambda\Big(\phi(x)-\sum_{j=0}^{n-1}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x\\ &+\sum_{j=0}^{n-1}a^{\lambda+j+1}\phi^{(j)}(0)\sum_{l=0}^{j}\frac{(-1)^l}{(j-l)!(\lambda+1)\cdots(\lambda+1+l)}.\end{aligned}\tag{5.93}$$

If $-n-1<\mathrm{Re}\lambda<-n$, *we can even go with* a *to infinity*

$$\mathcal{P}\int x_+^\lambda\phi(x)\mathrm{d}x=\int_0^\infty x^\lambda\Big(\phi(x)-\sum_{j=0}^{n-1}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x.\tag{5.94}$$

Proof. We use induction. Suppose that the formula is true for λ

$$-\lambda\mathcal{P}\int x_+^{\lambda-1}\phi(x)\mathrm{d}x=\mathcal{P}\int x_+^\lambda\partial_x\phi(x)\mathrm{d}x\tag{5.95}$$

$$\begin{aligned}=&\int_a^\infty x^\lambda\partial_x\phi(x)\mathrm{d}x\\ &+\int_0^a x^\lambda\partial_x\Big(\phi(x)-\sum_{j=0}^{n}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x\\ &+\sum_{j=0}^{n-1}a^{\lambda+j+1}\phi^{(j+1)}(0)\sum_{l=0}^{j}\frac{(-1)^l}{(j-l)!(\lambda+1)\cdots(\lambda+1+l)}.\end{aligned}\tag{5.96}$$

Then we integrate by parts, obtaining the identity for $\lambda-1$. □

5.4 Homogeneous distributions of arbitrary order II

We also can define

$$|x|^\lambda=\frac{1}{2\cos(\frac{\pi}{2}\lambda)}\Big((-\mathrm{i}x+0)^\lambda+(\mathrm{i}x+0)^\lambda\Big),\tag{5.97}$$

$$|x|^\lambda\mathrm{sgn}(x)=\frac{1}{2\mathrm{i}\sin(\frac{\pi}{2}\lambda)}\Big(-(-\mathrm{i}x+0)^\lambda+(\mathrm{i}x+0)^\lambda\Big).\tag{5.98}$$

The Fourier transforms:

$$\int|k|^\lambda\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k=\pi^{\frac12}\frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(-\frac{\lambda}{2})}\Big|\frac{x}{2}\Big|^{-\lambda-1},\tag{5.99}$$

$$\int|k|^\lambda\mathrm{sgn}(k)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k=-\mathrm{i}\pi^{\frac12}\frac{\Gamma(\frac{\lambda+2}{2})}{\Gamma(\frac{1-\lambda}{2})}\Big|\frac{x}{2}\Big|^{-\lambda-1}\mathrm{sgn}(x),\tag{5.100}$$

Especially symmetric expressions for Fourier transforms are obtained if we introduce

$$\eta_{\rm ev}^\lambda(x) := \Gamma\Big(\frac{\lambda}{2}+\frac12\Big)^{-1}\Big(\frac{x^2}{2}\Big)^{\frac{\lambda}{2}} \tag{5.101}$$

$$= (2\pi)^{-1}\Gamma\Big(-\frac{\lambda}{2}+\frac12\Big)2^{-\frac{\lambda}{2}}\Big((\mathrm{i}x+0)^\lambda + (-\mathrm{i}x+0)^\lambda\Big) \tag{5.102}$$

$$= \frac{2^{\frac{\lambda}{2}}}{\sqrt{\pi}}\Gamma\Big(1+\frac{\lambda}{2}\Big)\Big(\rho_+^\lambda(x)+\rho_-^\lambda(x)\Big), \tag{5.103}$$

$$\eta_{\rm odd}^\lambda(x) := \Gamma\Big(\frac{\lambda}{2}+1\Big)^{-1}\Big(\frac{x^2}{2}\Big)^{\frac{\lambda+1}{2}}\frac1x \tag{5.104}$$

$$= \mathrm{i}(2\pi)^{-1}\Gamma\Big(-\frac{\lambda}{2}\Big)2^{-\frac{\lambda}{2}-\frac12}\Big((\mathrm{i}x+0)^\lambda - (-\mathrm{i}x+0)^\lambda\Big) \tag{5.105}$$

$$= \frac{2^{\frac{\lambda}{2}-\frac12}}{\sqrt{\pi}}\Gamma\Big(\frac12+\frac{\lambda}{2}\Big)\Big(\rho_+^\lambda(x)-\rho_-^\lambda(x)\Big). \tag{5.106}$$

We then have the following relations:

$$\partial_x\eta_{\rm ev}^\lambda = \lambda\eta_{\rm odd}^{\lambda-1}, \qquad \partial_x\eta_{\rm odd}^\lambda = \eta_{\rm ev}^{\lambda-1}, \tag{5.107}$$

$$x\eta_{\rm ev}^\lambda(x) = (\lambda+1)\eta_{\rm odd}^{\lambda+1}(x), \qquad x\eta_{\rm odd}^\lambda(x) = \eta_{\rm ev}^{\lambda+1}(x); \tag{5.108}$$

$$\mathcal{F}\eta_{\rm ev}^\lambda = \eta_{\rm ev}^{-\lambda-1}, \qquad \mathcal{F}\eta_{\rm odd}^\lambda = -\mathrm{i}\eta_{\rm odd}^{-\lambda-1}; \tag{5.109}$$

$$\eta_{\rm ev}^{-1-2m}(x) = \frac{(-1)^m\sqrt2}{2^m\big(\frac12\big)_m}\delta^{(2m)}(x), \quad m=0,1,\dots; \tag{5.110}$$

$$\eta_{\rm odd}^{-2m}(x) = \frac{(-1)^m\sqrt2}{2^m\big(\frac12\big)_m}\delta^{(2m-1)}(x), \quad m=1,2,\dots. \tag{5.111}$$

5.5 Anomalous distributions of degree -1

We introduce the distributions $\frac{\theta(\pm k)}{k} = \pm k_\pm^{-1}$:

$$\int k_+^{-1}\phi(k)\mathrm{d}k := -\int_0^\infty \log(k)\phi^{(1)}(k)\mathrm{d}k \tag{5.112}$$

$$= \int_0^a \frac{\phi(k)-\phi(0)}{k}\mathrm{d}k + \log(a)\phi(0) + \int_a^\infty \frac{\phi(k)}{k}\mathrm{d}k \tag{5.113}$$

$$= \lim_{\epsilon\searrow0}\Big(\int_\epsilon^\infty \frac{\phi(k)}{k} + \phi(0)\ln(\epsilon)\Big), \tag{5.114}$$

$$\int k_-^{-1}\phi(k)\mathrm{d}k := -\int_{-\infty}^0 \log(-k)\phi^{(1)}\mathrm{d}k \tag{5.115}$$

$$= -\int_{-a}^0 \frac{\phi(k)-\phi(0)}{k}\mathrm{d}k + \log(a)\phi(0) - \int_{-\infty}^{-a} \frac{\phi(k)}{k}\mathrm{d}k \tag{5.116}$$

$$= \lim_{\epsilon\searrow0}\Big(-\int_{-\infty}^{-\epsilon} \frac{\phi(k)}{k} + \phi(0)\ln(\epsilon)\Big). \tag{5.117}$$

We have
$$\frac{1}{k} = -k_-^{-1} + k_+^{-1},$$
We also define
$$\frac{1}{|k|} = k_-^{-1} + k_+^{-1}.$$

Proposition 5.3 *Here are the Fourier transform of various forms of* $\frac{1}{|k|}$ *and the logarthm:*

$$\int k_\pm^{-1} \mathrm{e}^{-\mathrm{i}xk} \mathrm{d}k = -\log(\pm \mathrm{i}x + 0) - \gamma \tag{5.118}$$
$$= -\log|x| \mp \frac{\mathrm{i}\pi}{2}\mathrm{sgn}(x) - \gamma, \tag{5.119}$$
$$\mathcal{P}\int \frac{1}{|k|} \mathrm{e}^{-\mathrm{i}xk} \mathrm{d}k = -2\log|x| - 2\gamma, \tag{5.120}$$
$$\int \log|x| \mathrm{e}^{-\mathrm{i}xk} \mathrm{d}x = -\pi\mathcal{P}\frac{1}{|k|} - 2\pi\gamma\delta(k), \tag{5.121}$$
$$\int \log(\pm \mathrm{i}x + 0)\mathrm{e}^{-\mathrm{i}xk} \mathrm{d}x = -2\pi\frac{\theta(\mp k)}{|k|} - 2\pi\gamma\delta(k), \tag{5.122}$$
$$\int \log(x \mp \mathrm{i}0)\mathrm{e}^{-\mathrm{i}xk} \mathrm{d}x = -2\pi k_\mp^{-1} + (-2\pi\gamma \mp \mathrm{i}\pi)\delta(k), \tag{5.123}$$
$$\int \log(x - \lambda)\mathrm{e}^{-\mathrm{i}xk} \mathrm{d}x = \mathrm{e}^{-\mathrm{i}\lambda k}\Big(- 2\pi k_\mp^{-1} + (-2\pi\gamma \mp \mathrm{i}\pi)\delta(k)\Big), \quad \pm\mathrm{Im}\lambda > 0. \tag{5.124}$$

Proof. We start from one of the formulas for the Euler constant. We change the variable from k to yk, with $y > 0$:

$$\begin{aligned}
-\gamma &= \int_0^\infty \mathrm{e}^{-k} \log(k)\mathrm{d}k \\
&= \int_0^\infty \mathrm{e}^{-yk} \log(ky)\mathrm{d}(ky) \\
&= y\log(y)\int_0^\infty \mathrm{e}^{-ky}\mathrm{d}k + y\int_0^\infty \mathrm{e}^{-ky}\log(k)\mathrm{d}k \\
&= y\log(y)\frac{1}{y} - \int_0^1 \big(\partial_k(\mathrm{e}^{-ky} - 1)\big)\log(k)\mathrm{d}k - \int_1^\infty \big(\partial_k \mathrm{e}^{-ky}\big)\log(k)\mathrm{d}k \\
&= \log(y) + \int_0^1 \frac{\mathrm{e}^{-ky} - 1}{k}\mathrm{d}k + \int_1^\infty \frac{\mathrm{e}^{-ky}}{k}\mathrm{d}k.
\end{aligned}$$

The rhs is analytic in y on the right halfplane. It is constant on the positive halfline. So it is constant on the whole halfplane. Therefore, we can replace y with $\mathrm{i}x$. This proves (5.118), which implies (5.119) and (5.120).

By inverting the Fourier transform we obtain (5.121) and (5.123). We can also get (5.123) from (5.121):

$$\int \log(x \mp \mathrm{i}0)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = \int \big(\log|x| \mp \mathrm{i}\pi\theta(-x)\big)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x \tag{5.125}$$

$$= -\pi\mathcal{P}\frac{1}{|k|} - 2\pi\gamma\delta(k) \pm \pi\frac{1}{(k+\mathrm{i}0)} \tag{5.126}$$

$$= -2\pi\mathcal{P}\frac{1}{k_\mp} + (-2\pi\gamma \mp \mathrm{i}\pi)\delta(k). \tag{5.127}$$

□

Here is an alternative approach:

$$\frac{1}{k_\pm} = \lim_{\nu\searrow 0}\Big(\frac{1}{k_\pm^{1+\nu}} - \frac{1}{\nu}\delta(k)\Big), \tag{5.128}$$

$$\frac{1}{|k|} = \lim_{\nu\searrow 0}\Big(\frac{1}{|k|^{1+\nu}} - \frac{2}{\nu}\delta(k)\Big) \tag{5.129}$$

Here is the computation of the Fourier transform by this method:

$$\mathcal{P}\int\frac{\mathrm{e}^{-\mathrm{i}kx}}{k_\pm}\mathrm{d}k \approx \int\frac{\mathrm{e}^{-\mathrm{i}kx}}{k_\pm^{1+\nu}}\mathrm{d}k - \frac{1}{\nu} \tag{5.130}$$

$$= \Gamma(\nu)(\pm\mathrm{i}k+0)^{-\nu} - \frac{1}{\nu}, \tag{5.131}$$

$$\approx \Big(\frac{1}{\nu} - \gamma\Big)\big(1 - \nu\log(\pm\mathrm{i}k+0)\big) - \frac{1}{\nu}, \tag{5.132}$$

$$\approx -\log(\pm\mathrm{i}k+0) - \gamma. \tag{5.133}$$

5.6 Anomalous distributions of integral degree

Define

$$k_\pm^{-n-1} := \frac{(\mp 1)^n}{n!}\partial_k^n k_\pm^{-1}, \tag{5.134}$$

$$\frac{\mathrm{sgn}(k)}{k^{n+1}} := k_+^{-n-1} + (-1)^n k_-^{-n-1}. \tag{5.135}$$

Theorem 5.4 *Using H_n defined in (.358), we have*

$$\frac{1}{x_\pm^{1+n}} + (\mp 1)^n H_n \frac{\delta^{(n)}(x)}{n!} \tag{5.136}$$

$$= \lim_{\nu\to 0}\Big(\frac{1}{x_\pm^{1+n-\nu}} - \frac{(\mp 1)^n}{\nu}\frac{\delta^{(n)}(x)}{n!}\Big), \tag{5.137}$$

$$\frac{1}{|x|^{1+n}} + \big((-1)^n+1\big)H_n\frac{\delta^{(n)}(x)}{n!} \tag{5.138}$$

$$= \lim_{\nu\to 0}\Big(\frac{1}{|x|^{1+n-\nu}} - \frac{\big((-1)^n+1\big)}{\nu}\frac{\delta^{(n)}(x)}{n!}\Big), \tag{5.139}$$

$$\frac{\mathrm{sgn}(x)}{|x|^{1+n}} + \big((-1)^n-1\big)H_n\frac{\delta^{(n)}(x)}{n!} \tag{5.140}$$

$$= \lim_{\nu\to 0}\Big(\frac{\mathrm{sgn}(x)}{|x|^{1+n-\nu}} - \frac{\big((-1)^n-1\big)}{\nu}\frac{\delta^{(n)}(x)}{n!}\Big). \tag{5.141}$$

Proof. It is enough to consider only x_+^{-n-1}.

$$\mathcal{P}\int x_+^{-n-1+\nu}\phi(x)\mathrm{d}x = \mathcal{P}\int_0^\infty \frac{\big(\partial_x^{n+1}x^\nu\big)\phi(x)}{\nu(\nu-1)\cdots(\nu-n)}\mathrm{d}x \tag{5.142}$$

$$= \int_0^\infty \frac{x_+^\nu\phi^{(n+1)}(x)}{(-\nu)(1-\nu)\cdots(n-\nu)}\mathrm{d}x \tag{5.143}$$

$$= \int_0^\infty \frac{(x_+^\nu-1)\phi^{(n+1)}(x)}{(-\nu)(1-\nu)\cdots(n-\nu)}\mathrm{d}x \tag{5.144}$$

$$+ \int_0^\infty \frac{\phi^{(n+1)}(x)}{(-\nu)(1-\nu)\cdots(n-\nu)}\mathrm{d}x \tag{5.145}$$

$$= -\int_0^\infty \frac{\log(x)\phi^{(n+1)}(x)}{n!}\mathrm{d}x \tag{5.146}$$

$$+ \frac{1}{\nu}\frac{\phi^{(n)}(0)}{(1-\nu)\cdots(n-\nu)} \tag{5.147}$$

$$= \mathcal{P}\int x_+^{-n-1}\phi(x)\mathrm{d}x \tag{5.148}$$

$$+ \frac{1}{\nu}\frac{\phi^{(n)}(0)}{n!} + H_n\frac{\phi^{(n)}(0)}{n!} + O(\nu). \tag{5.149}$$

□

The above proof is taken from Hörmander, sect. 3.2. Note that Hörmander treats (5.137) as the standard regularization of $x_\pm^{-n-1}$. We prefer the definition (5.134).

Theorem 5.5

$$\mathcal{P}\int x_+^{-n-1}\phi(x)\mathrm{d}x = \int_0^\infty \frac{1}{x^{n+1}}\Big(\phi(x) - \sum_{j=0}^{n-1}\frac{x^j}{j!}\phi^{(j)}(0) - \frac{x^n}{n!}\phi^{(n)}(x)\Big)\mathrm{d}x$$
$$+ \int_0^1 \frac{1}{x}\Big(\frac{\phi^{(n)}(x)}{n!} - \frac{\phi^{(n)}(0)}{n!}\Big)\mathrm{d}x + \int_1^\infty \frac{1}{x}\frac{\phi^{(n)}(x)}{n!}\mathrm{d}x$$
$$- \frac{\phi^{(n)}(0)}{n!}H_n. \tag{5.150}$$

Proof. Let $a > 0$. If we assume that $\mathrm{Re}\nu > -1$, then we can use (5.93) with n replaced with $n+1$:

$$\mathcal{P}\int \frac{1}{x_+^{n+1-\nu}}\phi(x)\mathrm{d}x \tag{5.151}$$
$$= \int_a^\infty \frac{1}{x^{n+1-\nu}}\phi(x)\mathrm{d}x$$
$$+ \int_0^a \frac{1}{x^{n+1-\nu}}\Big(\phi(x) - \sum_{j=0}^{n}\frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x$$
$$- \sum_{j=0}^{n} a^{-n+j+\nu}\phi^{(j)}(0)\sum_{l=0}^{j}\frac{1}{(j-l)!(n-l-\nu)\cdots(n-\nu)}. \tag{5.152}$$

The last term of (5.152) is

$$- a^\nu\phi^{(n)}(0)\sum_{l=0}^{n-1}\frac{1}{(n-l)!(n-l-\nu)\cdots(n-\nu)} \tag{5.153}$$
$$- a^\nu\phi^{(n)}(0)\frac{1}{(-\nu)\cdots(n-\nu)} \tag{5.154}$$
$$= - \phi^{(n)}(0)\frac{H_n}{n!} \tag{5.155}$$
$$+ \frac{1}{\nu}\phi^{(n)}(0)\frac{1}{n!} + \log(a)\phi^{(n)}(0)\frac{1}{n!} \tag{5.156}$$
$$+ \phi^{(n)}(0)\frac{H_n}{n!} + O(\nu) \tag{5.157}$$
$$= \frac{1}{\nu}\phi^{(n)}(0)\frac{1}{n!} + \log(a)\phi^{(n)}(0)\frac{1}{n!} + O(\nu). \tag{5.158}$$

Thus we have proven that

$$\begin{aligned}
\mathcal{P}\int x_+^{-n-1}\phi(x)\mathrm{d}x = &\int_a^\infty \frac{1}{x^{n+1}}\phi(x)\mathrm{d}x \\
&+\int_0^a \frac{1}{x^{n+1}}\Big(\phi(x)-\sum_{j=0}^n \frac{x^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}x \\
&-\sum_{j=0}^{n-1} a^{-n+j}\phi^{(j)}(0)\sum_{l=0}^{j}\frac{1}{(j-l)!(n-l)\cdots n} \qquad (5.159)\\
&-\frac{\phi^{(n)}(0)}{n!}H_n+\log(a)\frac{\phi^{(n)}(0)}{n!} \qquad (5.160)
\end{aligned}$$

Then we take $a\to\infty$, noting that

$$\int_1^a x^{-1}\mathrm{d}x = \log(a).$$

□

Proposition 5.6 *The Fourier transform:*

$$\begin{aligned}
\mathcal{P}\int k_\pm^{-n-1}\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}k &= \frac{(\mp\mathrm{i}x)^n}{n!}\Big(-\log(\pm\mathrm{i}x+0)-\gamma\Big) \qquad (5.161)\\
&= \frac{(\mp\mathrm{i}x)^n}{n!}\Big(-\log|x|\mp\frac{\mathrm{i}\pi}{2}\mathrm{sgn}(x)-\gamma\Big) \qquad (5.162)
\end{aligned}$$

Proof. We use (5.137):

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}xk}}{k^{n+1}}\mathrm{d}k+\frac{(\mp\mathrm{i}x)^n}{n!}H_n &= \lim_{\nu\searrow 0}\Big(\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}xk}}{k^{1+n-\nu}}\mathrm{d}k-\frac{(\mp\mathrm{i}k)^n}{\nu n!}\Big) \qquad (5.163)\\
&= \lim_{\nu\searrow 0}\Big(\Gamma(-n+\nu)(\pm\mathrm{i}x+0)^{n-\nu}-\frac{(\mp\mathrm{i}k)^n}{\nu n!}\Big) \qquad (5.164)\\
= \lim_{\nu\searrow 0}\left(\frac{(-1)^n}{n!}\Big(\frac{1}{\nu}-\gamma+H_n\Big)(\pm\mathrm{i}x)^n\big(1-\nu\log(\pm\mathrm{i}x+0)\big)-\frac{(\mp\mathrm{i}k)^n}{\nu n!}\right)&\\
&= \frac{(\mp\mathrm{i}x)^n}{n!}\big(-\gamma+H_n-\log(\pm\mathrm{i}x+0)\big). \qquad (5.165)
\end{aligned}$$

Note that the terms containg H_n cancel. □

5.7 Infrared regularized distributions

Theorem 5.7 *Let $n+1>2\alpha>n$. Then*

$$\begin{aligned}
\frac{\theta(k)}{k^{2\alpha}} = &\lim_{m\to 0}\Big(\frac{\theta(k)}{(k^2+m^2)^\alpha} \qquad (5.166)\\
&-\sum_{j=0}^{n-1}\frac{\Gamma(\alpha-\frac{j}{2}-\frac{1}{2})\Gamma(\frac{j}{2}+\frac{1}{2})}{2m^{2\alpha-j-1}\Gamma(\alpha)j!}(-1)^j\delta^{(j)}(k)\Big). \qquad (5.167)
\end{aligned}$$

Proof. Clearly,

$$\mathcal{P}\int \frac{\theta(k)\phi(k)}{k^{2\alpha}}\mathrm{d}k = \int_0^\infty \frac{1}{k^{2\alpha}}\Big(\phi(k) - \sum_{j=0}^{n-1}\frac{k^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}k \tag{5.168}$$

is the limit as $m \to 0$ of

$$\int_0^\infty \frac{1}{(k^2+m^2)^\alpha}\Big(\phi(k) - \sum_{j=0}^{n-1}\frac{k^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}k. \tag{5.169}$$

Now

$$\int_0^\infty \frac{k^j}{(k^2+m^2)^\alpha}\mathrm{d}k = \frac{\Gamma(\alpha-\frac{j}{2}-\frac{1}{2})\Gamma(\frac{j}{2}+\frac{1}{2})}{2m^{2\alpha-j-1}\Gamma(\alpha)} \tag{5.170}$$

□

Theorem 5.8 *Let* $2p+1 > 2\alpha > 2p-1$, $p = 1, 2, \ldots$. *Then*

$$\frac{1}{|k|^{2\alpha}} = \lim_{m\to 0}\Big(\frac{1}{(k^2+m^2)^\alpha} \tag{5.171}$$

$$- \sum_{l=0}^{p-1}\frac{\pi^{\frac{3}{2}}m^{-2\alpha+2l+1}(-1)^l}{\Gamma(\alpha)\sin\big(\pi(\alpha-\frac{1}{2})\big)2^{2l}l!\Gamma(\frac{3}{2}-\alpha+l)}\delta^{(2l)}(k)\Big). \tag{5.172}$$

Proof. Clearly,

$$\mathcal{P}\int \frac{\phi(k)}{k^{2\alpha}}\mathrm{d}k = \int \frac{1}{k^{2\alpha}}\Big(\phi(k) - \sum_{l=0}^{p-1}\frac{k^{2l}}{(2l)!}\phi^{(2l)}(0)\Big)\mathrm{d}k \tag{5.173}$$

is the limit as $m \to 0$ of

$$\int \frac{1}{(k^2+m^2)^\alpha}\Big(\phi(k) - \sum_{l=0}^{p-1}\frac{k^{2l}}{(2l)!}\phi^{(2l)}(0)\Big)\mathrm{d}k. \tag{5.174}$$

Now

$$\frac{1}{(2l)!}\int \frac{k^{2l}}{(k^2+m^2)^\alpha}\mathrm{d}k = \frac{m^{-2\alpha+2l+1}\Gamma(\alpha-l-\frac{1}{2})\Gamma(l+\frac{1}{2})}{\Gamma(\alpha)(2l)!} \tag{5.175}$$

$$= \frac{\pi^{\frac{3}{2}}m^{-2\alpha+2l+1}(-1)^l}{\Gamma(\alpha)\sin\big(\pi(\alpha-\frac{1}{2})\big)2^{2l}l!\Gamma(\frac{3}{2}-\alpha+l)}. \tag{5.176}$$

□

Theorem 5.9 *Let $n = 0, 1, \ldots$. Then*

$$\frac{\theta(k)}{k^{n+1}} = \lim_{m\to 0}\Big(\frac{\theta(k)}{(k^2+m^2)^{\frac{n}{2}+\frac{1}{2}}} \tag{5.177}$$

$$-\sum_{j=0}^{n-1}\frac{m^{-n+j}\Gamma(\frac{n}{2}-\frac{j}{2})\Gamma(\frac{j}{2}+\frac{1}{2})}{2\Gamma(\alpha)j!}(-1)^j\delta^{(j)}(k)\Big) \tag{5.178}$$

$$-\begin{cases}\Big(\frac{1}{2}H_{p+1}(\frac{1}{2})+\log(m)\Big)\frac{1}{(2p+1)!}(-1)\delta^{(2p+1)}(k), & n=2p+1;\\ \Big(\frac{1}{2}H_p-\log(2)+\log(m)\Big)\frac{1}{(2p)!}\delta^{(2p)}(k), & n=2p.\end{cases} \tag{5.179}$$

Proof. Clearly,

$$\mathcal{P}\int\frac{\theta(k)\phi(k)}{k^{2\alpha}}\mathrm{d}k \tag{5.180}$$

is the limit as $m \to 0$ of

$$\int_0^\infty\frac{1}{(k^2+m^2)^{n+1}}\Big(\phi(k)-\sum_{j=0}^{n-1}\frac{k^j}{j!}\phi^{(j)}(0)\Big)\mathrm{d}k \tag{5.181}$$

$$-\int_0^1\frac{k^n}{(k^2+m^2)^{\frac{n}{2}+\frac{1}{2}}}\frac{\phi^{(n)}(0)}{n!}-H_n\frac{\phi^{(n)}(0)}{n!}. \tag{5.182}$$

Now for $n = 2p + 1$ we have

$$\int_0^1\frac{k^{2p+1}}{(k^2+m^2)^{p+1}}\mathrm{d}k \tag{5.183}$$

$$=-\sum_{j=1}^{p}\frac{k^{2j}}{2j(k^2+m^2)^j}\Big|_0^1+\int_0^1\frac{k}{k^2+m^2}\mathrm{d}k \tag{5.184}$$

$$=-\sum_{j=1}^{p}\frac{1}{2j(1+m^2)^j}+\frac{1}{2}\arctan\Big(\frac{t}{m}\Big) \tag{5.185}$$

$$=-\frac{1}{2}H_p-\log(m)+o(m^0). \tag{5.186}$$

Then we use

$$H_{2p+1}-\frac{1}{2}H_p=\frac{1}{2}H_{p+1}\Big(\frac{1}{2}\Big). \tag{5.187}$$

For $n = 2p$ we compute

$$\int_0^1 \frac{k^{2p}}{(k^2+m^2)^{p+\frac12}}\mathrm{d}k \tag{5.188}$$

$$= -\sum_{j=0}^{p-1} \frac{k^{2j+1}}{(2j+1)(k^2+m^2)^{j+\frac12}}\Big|_0^1 + \int_0^1 \frac{k}{(k^2+m^2)^{\frac12}}\mathrm{d}k \tag{5.189}$$

$$= -\sum_{j=1}^{p} \frac{1}{(2j+1)(1+m^2)^{j+\frac12}} + \log\left(1+\sqrt{1+m^2}\right) - \log(m) \tag{5.190}$$

$$= -\frac12 H_p\Big(\frac12\Big) + \log(2) - \log(m) + o(m^0). \tag{5.191}$$

Then we use

$$H_{2p} - \frac12 H_p\Big(\frac12\Big) = \frac12 H_p. \tag{5.192}$$

□

Using (2.67) we derive

$$\int \frac{\mathrm{e}^{-\mathrm{i}kx}}{(k^2+m^2)^\alpha}\mathrm{d}k = \frac{2\pi^{\frac12}m^{-2\alpha+1}}{\Gamma(\alpha)}\Big(\frac{m|x|}{2}\Big)^{\alpha-\frac12}K_{\alpha-\frac12}(m|x|). \tag{5.193}$$

Note that (5.193) is bounded if $\alpha > \frac12$, has a logarithmic singularity at zero if $\alpha = \frac12$, and has a singularity $|x|^{2\alpha-1}$ if $\alpha < \frac12$. Therefore, it is no longer a regular distribution if $\alpha < 0$. However, by applying $(-\partial_x^2+m^2)$ sufficiently many times to (5.193) we can interpret it as a distribution for all α. (For $\alpha = -n$, $n = 0, -1, -2, \dots$ we simply obtain $(-\partial_x^2+m^2)^n$).

Suppose now that the assumptions of Thm 5.8 are satisfied. Let us compute the Fourier transform of the linear combination of the deltas:

$$\int \sum_{l=0}^{p-1} \frac{\pi^{\frac32}m^{-2\alpha+2l+1}(-1)^l}{\Gamma(\alpha)\sin\left(\pi(\alpha-\frac12)\right)2^{2l}l!\Gamma(\frac32-\alpha+l)}\delta^{(2l)}(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \tag{5.194}$$

$$= \sum_{l=0}^{p-1} \frac{\pi^{\frac32}m^{-2\alpha+2l+1}}{\Gamma(\alpha)\sin\left(\pi(\alpha-\frac12)\right)2^{2l}l!\Gamma(\frac32-\alpha+l)}x^{2l}. \tag{5.195}$$

Now the rhs of (5.193), using the identity (2.48), can be written as

$$\frac{\pi^{\frac32}m^{-2\alpha+1}}{\Gamma(\alpha)\sin\left(\pi(\alpha-\frac12)\right)}\Big(\frac{m|x|}{2}\Big)^{\alpha-\frac12}I_{-\alpha+\frac12}(m|x|) \tag{5.196}$$

$$-\frac{\pi^{\frac32}m^{-2\alpha+1}}{\Gamma(\alpha)\sin\left(\pi(\alpha-\frac12)\right)}\Big(\frac{m|x|}{2}\Big)^{\alpha-\frac12}I_{\alpha-\frac12}(m|x|) \tag{5.197}$$

Now (5.196) is equal to (5.195) modulo $O(m^{-2\alpha+2p+1})$. (5.196) is equal to

$$\frac{\pi^{\frac12}\Gamma(\frac12-\alpha)}{\Gamma(\alpha)}\Big(\frac{|x|}{2}\Big)^{2\alpha-1} \tag{5.198}$$

modulo $O(m^2)$. This is a confirmation of the correctness of Thm 5.8.

5.8 Distributions on halfline

We will denote by $C^\infty[0,\infty[$ smooth function having all right-sided derivatives at 0. We set

$$C_{\rm N}^\infty[0,\infty[:= \{\phi\in C^\infty[0,\infty[:\ \phi^{(2m+1)}(0)=0,\ m=0,1,\dots\}, \tag{5.199}$$
$$C_{\rm D}^\infty[0,\infty[:= \{\phi\in C^\infty[0,\infty[:\ \phi^{(2m)}(0)=0,\ m=0,1,\dots\}. \tag{5.200}$$

$\mathcal{S}_{\rm N}[0,\infty[$, $\mathcal{S}_{\rm D}[0,\infty[$ have obvious definitions. We set $\mathcal{S}_{\rm N}'[0,\infty[$, $\mathcal{S}_{\rm D}'[0,\infty[$ to be their duals.

Note that ∂_x and the multiplication by x map $\mathcal{S}_{\rm N}[0,\infty[$ into $\mathcal{S}_{\rm D}[0,\infty[$ and vice versa, as well as $\mathcal{S}_{\rm N}'[0,\infty[$ into $\mathcal{S}_{\rm D}'[0,\infty[$ and vice vera.

The cosine transformation with the kernel

$$\mathcal{F}_{\rm N}(x,k) := \sqrt{\frac{2}{\pi}}\cos(xk)$$

maps $\mathcal{S}_{\rm N}'[0,\infty[$ into into itself. We have

Likewise, the sine transformation with the kernel

$$\mathcal{F}_{\rm N}(x,k) := \sqrt{\frac{2}{\pi}}\sin(xk)$$

maps $\mathcal{S}_{\rm D}'[0,\infty[$ into into itself.

Let $I\phi(x) := \phi(-x)$. I maps $\mathcal{S}(\mathbb{R})$, as well as extends to a map of $\mathcal{S}'(\mathbb{R})$ into itself. We will write

$$\mathcal{S}_{\rm ev}(\mathbb{R}) := \{\phi\in\mathcal{S}(\mathbb{R})\ :\ I\phi=\phi\}, \tag{5.201}$$
$$\mathcal{S}_{\rm ev}'(\mathbb{R}) := \{\lambda\in\mathcal{S}'(\mathbb{R})\ :\ I\lambda=\lambda\}, \tag{5.202}$$
$$\mathcal{S}_{\rm odd}(\mathbb{R}) := \{\phi\in\mathcal{S}(\mathbb{R})\ :\ I\phi=-\phi\}, \tag{5.203}$$
$$\mathcal{S}_{\rm odd}'(\mathbb{R}) := \{\lambda\in\mathcal{S}'(\mathbb{R})\ :\ I\lambda=-\lambda\}. \tag{5.204}$$

If $\phi\in\mathcal{S}_{\rm N}[0,\infty[$, we set

$$\phi^{\rm ev}(x) := \begin{cases}\phi(x) & x\geq 0;\\ \phi(-x) & x\leq 0.\end{cases}$$

Note that $\phi^{\rm ev}\in\mathcal{S}_{\rm ev}(\mathbb{R})$.

If $\lambda_{\rm ev}$ is an even distribution in $\mathcal{S}'(\mathbb{R})$, then we can associate with it a distribution in $\mathcal{S}_{\rm N}'[0,\infty[$ by

$$\int_0^\infty \lambda_{\rm N}(x)\phi(x){\rm d}x := \frac12\int\lambda_{\rm ev}(x)\phi^{\rm ev}(x){\rm d}x.$$

Similarly, if $\phi\in\mathcal{S}_{\rm D}[0,\infty[$, we set

$$\phi^{\rm odd}(x) := \begin{cases}\phi(x) & x\geq 0;\\ -\phi(-x) & x\leq 0.\end{cases}$$

Note that $\phi^{\rm odd} \in \mathcal{S}_{\rm odd}(\mathbb{R})$.

If $\lambda_{\rm odd}$ is an odd distribution in $\mathcal{S}'(\mathbb{R})$, then we can associate with it a distribution in $\mathcal{S}'_{\rm D}[0,\infty[$ by We set

$$\int_0^\infty \lambda_{\rm D}(x)\phi(x){\rm d}x := \frac{1}{2}\int \lambda_{\rm odd}(x)\phi^{\rm odd}(x){\rm d}x.$$

The usual Fourier transform $\mathcal{F}$ preserves $\mathcal{S}_{\rm ev}(\mathbb{R})$ and $\mathcal{S}_{\rm odd}(\mathbb{R})$. The Fourier transform on even distributions is closely related to the cosine transform and on odd distributions to the sine transform:

$$\mathcal{F}_{\rm N}\lambda_{\rm N} = (\mathcal{F}\lambda)_{\rm N}, \quad \lambda \in \mathcal{S}'_{\rm ev}(\mathbb{R}), \tag{5.205}$$
$$\mathcal{F}_{\rm D}\lambda_{\rm D} = {\rm i}(\mathcal{F}\lambda)_{\rm D}, \quad \lambda \in \mathcal{S}'_{\rm odd}(\mathbb{R}). \tag{5.206}$$

An example of an even distribution is $\eta_{\rm ev}$. Let $\eta_{\rm N}$ denote the corresponding distributtion in $\mathcal{S}'_{\rm N}[0,\infty[$.

Likewise, an example of an odd distribution is $\eta_{\rm odd}$. Let $\eta_{\rm D}$ denote the corresponding distributtion in $\mathcal{S}'_{\rm D}[0,\infty[$.

We have

$$\mathcal{F}_{\rm N}\eta_{\rm N}^\lambda = \eta_{\rm N}^{-\lambda-1}, \qquad \mathcal{F}_{\rm D}\eta_{\rm D}^\lambda = \eta_{\rm D}^{-\lambda-1}; \tag{5.207}$$

6 Distributions in arbitrary dimension

6.1 Sphere S^{d-1}

Consider the Euclidean space $\mathbb{R}^d$. Introduce two varieties of spherical coordinates on a $d-1$-dimensional sphere

$$(\theta_{d-2},\dots,\theta_1,\phi) \in [0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi[,$$

$$(w_{d-2},\dots,w_1,\phi) \in [0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi[,$$

with $w_j = \cos\theta_j$, The spherical measure on $\mathbb{S}^{d-1}$ is

$$\begin{aligned} & \sin^{d-2}\theta_{d-2}{\rm d}\theta_{d-2}\cdots\sin\theta_1{\rm d}\theta_1{\rm d}\phi \\ = \quad & (1-w_{d-2}^2)^{(d-3)/2}{\rm d}w_{d-2}\cdots{\rm d}w_1{\rm d}\phi. \end{aligned}$$

Theorem 6.1 *The area of the $d-1$-dimensional sphere is*

$$S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})},$$

or, in a more elementary form,

$$S_{2m+1} = \frac{2\pi^{m+1}}{m!}, \quad m = 0,1,\dots; \tag{6.208}$$
$$S_{2m} = \frac{2\pi^m}{(\frac{1}{2})_m}, \quad m = 0,1,\dots. \tag{6.209}$$

Proof. Method I. We compute in two ways the Gaussian integral: in the Cartesian coordinates

$$\int \mathrm{e}^{-x_1^2-\cdots-x_d^2}\mathrm{d}x_1\cdots\mathrm{d}x_d = \pi^{\frac{d}{2}},$$

and in spherical coordinates:

$$S_{d-1}\int_0^\infty \mathrm{e}^{-r^2}r^{d-1}\mathrm{d}r = \frac{1}{2}\Gamma\Big(\frac{d}{2}\Big). \tag{6.210}$$

Method II. We compute the area of the sphere in the spherical coordinates:

$$S_{d-1} = \int_0^\pi \sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1}\cdots\int_0^\pi \sin\phi_2\mathrm{d}\phi_2\int_0^{2\pi}\mathrm{d}\phi_1$$

Then we use

$$\int_0^\pi \sin^{k-1}\phi_k\mathrm{d}\phi_k = \frac{\sqrt{\pi}\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2})},\quad k=2,\dots,d-1;\qquad \int_0^{2\pi}\mathrm{d}\phi_1 = 2\pi.$$

□

6.2 Homogeneous functions in arbitrary dimension

Theorem 6.2 *Let* $-d<\lambda<0$*. Then on* $\mathbb{R}^d$

$$\int |x|^\lambda \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x = \pi^{\frac{d}{2}}\frac{\Gamma(\frac{\lambda+d}{2})}{\Gamma(-\frac{\lambda}{2})}\Big|\frac{\xi}{2}\Big|^{-\lambda-d}. \tag{6.211}$$

Proof. We use the spherical coordinates:

$$\begin{aligned}
&\int |x|^\lambda \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x & (6.212)\\
=\;&\int_0^\infty \mathrm{d}r\int_0^\pi \mathrm{d}\phi_{d-1}r^{\lambda+d-1}\mathrm{e}^{-\mathrm{i}r|\xi|\cos\phi_{d-1}}r^{\lambda+d-1}\sin^{d-2}\phi_{d-1}S_{d-2} & (6.213)\\
=\;&\Gamma(\lambda+d)\int_0^{\frac{\pi}{2}}\Big((\mathrm{i}|\xi|\cos\phi_{d-1}+0)^{-\lambda-d}+(-\mathrm{i}|\xi|\cos\phi_{d-1}+0)^{-\lambda-d}\Big)\sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1}S_{d-2}\\
=\;&\Gamma(\lambda+d)2\cos\Big(\frac{\lambda+d}{2}\pi\Big)|\xi|^{-\lambda-d}\int_0^{\frac{\pi}{2}}\cos^{-\lambda-d}\phi_{d-1}\sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1}S_{d-2}.
\end{aligned}$$

Then we apply

$$\begin{aligned}
S_{d-2} &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})},\\
\int_0^{\frac{\pi}{2}}\cos^{-\lambda-d}\phi_{d-1}\sin^{d-2}\phi_{d-1}\mathrm{d}\phi_{d-1} &= \frac{1}{2}\frac{\Gamma(\frac{-\lambda-d+1}{2})\Gamma(\frac{d-1}{2})}{\Gamma(-\frac{\lambda}{2})},\\
\Gamma(\lambda+d) &= \pi^{-\frac{1}{2}}2^{\lambda+d-1}\Gamma\Big(\frac{\lambda+d}{2}\Big)\Gamma\Big(\frac{\lambda+d+1}{2}\Big)\\
\cos\Big(\frac{\lambda+d}{2}\pi\Big) &= \frac{\pi}{\Gamma(\frac{\lambda+d+1}{2})\Gamma(\frac{-\lambda-d+1}{2})},
\end{aligned}$$

and we obtain (6.211) □

In order to express (6.211) in a more symmetric way, define

$$\eta^\lambda(x) := \frac{1}{\Gamma(\frac{\lambda+d}{2})}\Big(\frac{x^2}{2}\Big)^{\frac{\lambda}{2}}, \quad \lambda > -d.$$

We extend it to $\lambda \leq -d$ by setting

$$\eta^{\lambda-2m}(x) := \frac{(-2)^m}{\big(-\frac{\lambda}{2}\big)_m}\Delta^m \eta^\lambda(x). \tag{6.214}$$

Then

$$\mathcal{F}\eta^\lambda = \eta^{-\lambda-d}, \tag{6.215}$$
$$x^2\eta^\lambda = (\lambda+d)\eta^{\lambda+2}, \tag{6.216}$$
$$\Delta\eta^\lambda = \lambda\eta^{\lambda-2}. \tag{6.217}$$

6.3 Renormalizing the $|k|^{-d}$ function

Define the distribution $|k|^{-d}$ on $\mathbb{R}^d$:

$$\mathcal{P}\int |k|^{-d}\phi(k)\mathrm{d}k := \int_{|k|<1} |k|^{-d}\big(\phi(k)-\phi(0)\big)\mathrm{d}k + \int_{|k|>1} |k|^{-d}\phi(k)\mathrm{d}k..$$

Theorem 6.3 *We have an alternative definition of $|k|^{-d}$:*

$$|k|^{-d} = \lim_{\nu\searrow 0}\Big(|k|^{-d+\nu} - \frac{2\pi^{\frac{d}{2}}}{\nu\Gamma(\frac{d}{2})}\delta(k)\Big). \tag{6.218}$$

Here is its Fourier transform:

$$\begin{aligned}
\frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\mathcal{P}\int |k|^{-d}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k &= -\log\Big(\frac{r}{2}\Big) + \frac{1}{2}\psi\Big(\frac{d}{2}\Big) - \frac{1}{2}\gamma \\
&= -\log r - \gamma, \qquad d=1; \\
&= -\log\Big(\frac{r}{2}\Big) - \gamma, \qquad d=2; \\
&= -\log r - \gamma + \frac{1}{2}H_m\Big(\frac{1}{2}\Big), \qquad d=2m+1; \\
&= -\log\Big(\frac{r}{2}\Big) - \gamma + \frac{1}{2}H_m, \qquad d=2(m+1).
\end{aligned}$$

Proof.

$$\int_{|k|<1}|k|^{-d+\nu}\mathrm{d}k = \int_{|k|<1}|k|^{-1+\nu}\mathrm{d}|k|S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\nu\Gamma(\frac{d}{2})}. \tag{6.219}$$

This proves (6.218).

$$\int |k|^{-d+\nu}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \tag{6.220}$$
$$= \Big(\frac{r}{2}\Big)^{-\nu}\frac{\pi^{\frac{d}{2}}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{d}{2}-\frac{\nu}{2})} \tag{6.221}$$
$$\approx \Big(1-\nu\log\big(\frac{r}{2}\big)\Big)\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\Big(1+\frac{\nu}{2}\psi\big(\frac{d}{2}\big)\Big)\Big(\frac{2}{\nu}-\gamma\Big)$$
$$\approx \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\Big(\frac{1}{\nu}-\log\Big(\frac{r}{2}\Big)+\frac{1}{2}\psi\Big(\frac{d}{2}\Big)-\frac{1}{2}\gamma\Big). \tag{6.222}$$

□

7 Bessel potentials

In this section we analyze some distinguished solutions of

$$(1+\Box)^{\alpha}D(x)=\delta(x), \tag{7.223}$$

where $\Box$ is the Laplacian on the pseudo-Euclidean space $\mathbb{R}^{q,d-q}$ (q minuses and $d-q$ pluses). Formally, D is given by the inverse Fourier transform

$$D(x)=\int\frac{\mathrm{e}^{\mathrm{i}px}}{(1+p^2)^{\alpha}}\frac{\mathrm{d}p}{(2\pi)^d}, \tag{7.224}$$

where p^2 is the square of p wrt the scalar product of $\mathbb{R}^{q,d-q}$. However usually we need to regularize the integrand of (7.224)

Note that if D solves (7.223), and $m>0$, then

$$(m^2+\Box)^{\alpha}m^{d-2\alpha}D(mx)=\delta(x). \tag{7.225}$$

The following identities will be useful:

$$\frac{1}{A^{\alpha}}=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\mathrm{e}^{-sA}s^{\alpha-1}\mathrm{d}s, \tag{7.226}$$
$$\frac{1}{(A\pm\mathrm{i}0)^{\alpha}}=\frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\mathrm{e}^{\pm\mathrm{i}tA}t^{\alpha-1}\mathrm{d}t. \tag{7.227}$$

We will also need the Fourier transform of the Gaussian function on the Euclidean space $\mathbb{R}^d$, and of the Fresnel function on the pseudo-Euclidean space $\mathbb{R}^{q,d-q}$ (with q minuses):

$$\int\mathrm{d}p\mathrm{e}^{-\frac{sp^2}{2}}\mathrm{e}^{\mathrm{i}px}=\Big(\frac{2\pi}{s}\Big)^{\frac{d}{2}}\mathrm{e}^{-\frac{x^2}{2s}}, \tag{7.228}$$
$$\int\mathrm{d}p\mathrm{e}^{\pm\mathrm{i}\frac{tp^2}{2}}\mathrm{e}^{\mathrm{i}px}=(\mp\mathrm{i})^q\Big(\frac{2\pi}{t}\Big)^{\frac{d}{2}}\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}d}\mathrm{e}^{\mp\mathrm{i}\frac{x^2}{2t}}. \tag{7.229}$$

7.1 Euclidean and anti-Euclidean signature

Consider the Euclidean space $\mathbb{R}^d$, with $|x|$ denoting the Euclidean norm.

Theorem 7.1

$$\int \frac{\mathrm{e}^{\mathrm{i}px}}{(p^2+1)^\alpha}\frac{\mathrm{d}p}{(2\pi)^d} = \frac{2}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Big(\frac{|x|}{2}\Big)^{\alpha-\frac{d}{2}}K_{\alpha-\frac{d}{2}}(|x|), \tag{7.230}$$

$$\int \frac{\mathrm{e}^{\mathrm{i}px}}{(-p^2+1\pm\mathrm{i}0)^\alpha}\frac{\mathrm{d}p}{(2\pi)^d} = \frac{\pi(\mp\mathrm{i})^{d-1}}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Big(\frac{|x|}{2}\Big)^{\alpha-\frac{d}{2}}H^{\pm}_{\alpha-\frac{d}{2}}(|x|). \tag{7.231}$$

Proof. By (7.226) and then by (2.45),

$$\begin{aligned}
&\int \frac{\mathrm{e}^{\mathrm{i}px}\mathrm{d}p}{(1+p^2)^\alpha}\\
=\ &\Big(\frac{|x|}{2}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}s\int \mathrm{d}p s^{\alpha-1}\mathrm{e}^{-(1+p^2)\frac{|x|s}{2}}\mathrm{e}^{\mathrm{i}px}\\
=\ &\Big(\frac{|x|}{2}\Big)^{\alpha-\frac{d}{2}}\frac{\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}s s^{\alpha-\frac{d}{2}-1}\mathrm{e}^{-(s+\frac{1}{s})\frac{|x|}{2}}\\
=\ &\frac{2\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\Big(\frac{|x|}{2}\Big)^{\alpha-\frac{d}{2}}K_{\alpha-\frac{d}{2}}(|x|).
\end{aligned}$$

By (7.227), and then by (3.110),

$$\begin{aligned}
&\int \frac{\mathrm{e}^{\mathrm{i}px}\mathrm{d}p}{(1-p^2\pm\mathrm{i}0)^\alpha}\\
=&\Big(\frac{|x|}{2}\Big)^{\alpha}\frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\int \mathrm{d}p\mathrm{e}^{\pm\mathrm{i}t(1-p^2)\frac{|x|}{2}}t^{\alpha-1}\mathrm{e}^{\mathrm{i}px}\\
=&\Big(\frac{|x|}{2}\Big)^{(\alpha-\frac{d}{2})}\frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha+\frac{d}{2})}\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\mathrm{e}^{\pm\mathrm{i}\frac{|x|}{2}(t+\frac{1}{t})}t^{\alpha-\frac{d}{2}-1}\\
=&\frac{\pi(\mp\mathrm{i})^{d-1}\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\Big(\frac{|x|}{2}\Big)^{\alpha-\frac{d}{2}}H^{\pm}_{\alpha-\frac{d}{2}}(|x|).
\end{aligned}$$

7.2 General signature

Consider now a pseudo-Euclidean space of general signature $\mathbb{R}^{q,d-q}$.

Theorem 7.2

$$\int \frac{\mathrm{e}^{\mathrm{i}px}}{(1+p^2\pm\mathrm{i}0)^\alpha}\frac{\mathrm{d}p}{(2\pi)^d} \tag{7.232}$$

$$=\frac{2(\mp\mathrm{i})^q}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Big(\frac{\sqrt{x^2\mp\mathrm{i}0}}{2m}\Big)^{\alpha-\frac{d}{2}}K_{\alpha-\frac{d}{2}}\big(\sqrt{x^2\mp\mathrm{i}0}\big) \tag{7.233}$$

$$=\pm\frac{\pi\mathrm{i}(\mp\mathrm{i})^q}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Big(\frac{\sqrt{-x^2\pm\mathrm{i}0}}{2}\Big)^{\alpha-\frac{d}{2}}H^{\pm}_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2\pm\mathrm{i}0}\big). \tag{7.234}$$

Remark 7.3 *In (7.233) and (7.234) we use the notation explained in (3.118a) and (3.118b). Note that (7.233) works best for* $x^2 > 0$*, because then we can ignore* $\mp$i0*. Likewise, (7.234) is best suited for* $x^2 < 0$*, because then we can ignore* $\pm$i0*.*

Proof of Thm 7.2. using (7.227) and (2.49) we obtain

$$\begin{aligned}&\int \frac{\mathrm{e}^{\mathrm{i}px}\mathrm{d}p}{(1+p^2\pm\mathrm{i}0)^\alpha}\\ =&\frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}}{2^\alpha\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\int \mathrm{d}p\mathrm{e}^{\pm\frac{\mathrm{i}}{2}t(1+p^2)}t^{\alpha-1}\mathrm{e}^{\mathrm{i}px}\\ =&\frac{(\mp\mathrm{i})^q\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha-\frac{d}{2})}\pi^{\frac{d}{2}}}{2^{\alpha-\frac{d}{2}}\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\mathrm{e}^{\pm\frac{\mathrm{i}}{2}(t-\frac{x^2}{t})}t^{\alpha-\frac{d}{2}-1}. \qquad (7.235)\end{aligned}$$

For $x^2 \geq 0$, we change the variable $t = \pm\mathrm{i}s\sqrt{x^2}$, so that (7.235) becomes

$$\begin{aligned}&\Big(\frac{\sqrt{x^2}}{2}\Big)^{(\alpha-\frac{d}{2})}\frac{(\mp\mathrm{i})^q\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}s\mathrm{e}^{-\frac{\sqrt{x^2}}{2}(s+\frac{1}{s})}s^{\alpha-\frac{d}{2}-1}\\ =&\frac{2(\mp\mathrm{i})^q\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\Big(\frac{\sqrt{x^2}}{2}\Big)^{\alpha-\frac{d}{2}}K_{\alpha-\frac{d}{2}}\big(\sqrt{x^2}\big),\end{aligned}$$

For $x^2 \leq 0$, we change the variable $t = s\sqrt{-x^2}$. By (7.227) and (3.110) we transform (7.235) into

$$\begin{aligned}&\Big(\frac{\sqrt{-x^2}}{2}\Big)^{(\alpha-\frac{d}{2})}\frac{(\mp\mathrm{i})^q\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha-\frac{d}{2})}\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}s\mathrm{e}^{\pm\mathrm{i}\frac{\sqrt{-x^2}}{2}(s+\frac{1}{s})}s^{\alpha-\frac{d}{2}-1}\\ =&\frac{2(\mp\mathrm{i})^q\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\Big(\mp\mathrm{i}\frac{\sqrt{-x^2}}{2}\Big)^{\alpha-\frac{d}{2}}K_{\alpha-\frac{d}{2}}\big(\mp\mathrm{i}\sqrt{-x^2}\big)\\ =&\pm\frac{\pi\mathrm{i}(\mp\mathrm{i})^q\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\Big(\frac{\sqrt{-x^2}}{2}\Big)^{\alpha-\frac{d}{2}}H^{\pm}_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2}\big).\end{aligned}$$

Then we notice that $\sqrt{x^2}$ and $\mp\mathrm{i}\sqrt{-x^2}$ can be joined in $\sqrt{x^2\mp\mathrm{i}0}$. □

7.3 The Minkowski signature

$\mathbb{R}^{1,d-1}$ is called the Minkowski space. (We use the signature "mostly pluses"). This case is especially important and rich.

Let us state the special case of Thm 7.2 for the Minkowski signature as a separate theorem. We also introduce special notation.

Theorem 7.4

$$D_\alpha^{\overline{\mathrm{F}}/\mathrm{F}}(x) := \int \frac{\mathrm{e}^{\mathrm{i}px}}{(1+p^2 \pm \mathrm{i}0)^\alpha} \frac{\mathrm{d}p}{(2\pi)^d} \tag{7.236}$$

$$= \mp \frac{2\mathrm{i}}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}} \Big(\frac{\sqrt{x^2 \mp \mathrm{i}0}}{2}\Big)^{\alpha-\frac{d}{2}} K_{\alpha-\frac{d}{2}}\big(\sqrt{x^2\mp\mathrm{i}0}\big) \tag{7.237}$$

$$= \frac{\pi}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}} \Big(\frac{\sqrt{-x^2 \pm \mathrm{i}0}}{2}\Big)^{\alpha-\frac{d}{2}} H^{\pm}_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2\pm\mathrm{i}0}\big). \tag{7.238}$$

Suppose that x^0 denotes the first coordinate of $\mathbb{R}^{1,d-1}$, which we assume to be timelike (having a negative coefficient in the scalar product). The remaining, spacelike coordinates will be denoted $\vec{x}$. Then we set

$$\begin{aligned} J^\vee &:= \{x \in \mathbb{R}^{1,d-1} \ : \ x^2 \le 0, \quad x^0 \ge 0\}, \\ J^\wedge &:= \{x \in \mathbb{R}^{1,d-1} \ : \ x^2 \le 0, \quad x^0 \le 0\}. \end{aligned}$$

Theorem 7.5

$$D_\alpha^{\vee/\wedge}(x) := \int \frac{\mathrm{e}^{\mathrm{i}px}}{(1+p^2 \mp \mathrm{i}0\mathrm{sgn}p^0)^\alpha} \frac{\mathrm{d}p}{(2\pi)^d} \tag{7.239}$$

$$\begin{aligned} =&\theta(\pm x^0)\frac{\pi}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Bigg(\Big(\frac{\sqrt{-x^2+\mathrm{i}0}}{2m}\Big)^{\alpha-\frac{d}{2}} H^{+}_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2+\mathrm{i}0}\big) \\ &+\Big(\frac{\sqrt{-x^2-\mathrm{i}0}}{2m}\Big)^{\alpha-\frac{d}{2}} H^{-}_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2-\mathrm{i}0}\big)\Bigg) \end{aligned} \tag{7.240}$$

is a distribution whose support is contained in $J^{\vee/\wedge}$. Inside $J^{\vee/\wedge}$ we have the identity

$$D_\alpha^{\vee/\wedge}(x) = \frac{2\pi}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}} \Big(\frac{\sqrt{-x^2}}{2}\Big)^{\alpha-\frac{d}{2}} J_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2}\big). \tag{7.241}$$

We also have

$$D_\alpha^{\mathrm{F}}(x) + D_\alpha^{\overline{\mathrm{F}}}(x) = D_\alpha^{\vee}(x) + D_\alpha^{\wedge}(x). \tag{7.242}$$

Proof. Let us now prove that the support of (7.239) with the minus signed is contained in $J^\vee$. By the Lorentz invariance it suffices to prove that it is zero for $x^0 < 0$. We write

$$\int \frac{\mathrm{e}^{\mathrm{i}px}\mathrm{d}p}{(p^2+1-\mathrm{i}0\mathrm{sgn}p^0)^\alpha} = \int \frac{\mathrm{e}^{-\mathrm{i}p^0x^0+\mathrm{i}\vec{p}\vec{x}}\mathrm{d}p^0\mathrm{d}\vec{p}}{\big(\vec{p}^2+1-(p^0+\mathrm{i}0)^2\big)^\alpha}$$

Next we continuously deform the contour of integration, replacing $p^0 + \mathrm{i}0$ by $p^0 + \mathrm{i}R$, where $R \in [0, \infty[$. We do not cross any singularities of the integrand and note that $\mathrm{e}^{-\mathrm{i}x^0(p^0+\mathrm{i}R)}$ goes to zero (remember that $x^0 < 0$).

Next we note that

$$\int\Big(\frac{1}{(1+p^2+\mathrm{i}0)^\alpha}+\frac{1}{(1+p^2-\mathrm{i}0)^\alpha}\Big)\mathrm{e}^{\mathrm{i}px}\mathrm{d}p \tag{7.243}$$

$$=\frac{\pi\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\Bigg(\Big(\frac{\sqrt{-x^2+\mathrm{i}0}}{2}\Big)^{\alpha-\frac{d}{2}}H^+_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2+\mathrm{i}0}\big)$$

$$+\Big(\frac{\sqrt{-x^2-\mathrm{i}0}}{2}\Big)^{\alpha-\frac{d}{2}}H^-_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2-\mathrm{i}0}\big)\Bigg). \tag{7.244}$$

Taking into account the support properties, we obtain (7.240). Finally, using (3.106) we obtain (7.241). □

We also introduce special notation for some solutions of

$$(1+\Box)^\alpha D(x)=0. \tag{7.245}$$

$$D^{\mathrm{PJ}}_\alpha(x):=D^\vee_\alpha(x)-D^\wedge_\alpha(x), \tag{7.246}$$

$$D^{(+)/(-)}_\alpha(x):=-\mathrm{i}D^{\mathrm{F}}_\alpha(x)+\mathrm{i}D^{\vee/\wedge}_\alpha(x) \tag{7.247}$$

$$=\mathrm{i}D^{\overline{\mathrm{F}}}_\alpha(x)-\mathrm{i}D^{\wedge/\vee}_\alpha(x). \tag{7.248}$$

Here are expicit formulas for these solutions:

$$D^{\mathrm{PJ}}_\alpha(x)=\mathrm{sgn}(x^0)\frac{\pi}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Bigg(\Big(\frac{\sqrt{-x^2+\mathrm{i}0}}{2m}\Big)^{\alpha-\frac{d}{2}}H^+_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2+\mathrm{i}0}\big)$$

$$+\Big(\frac{\sqrt{-x^2-\mathrm{i}0}}{2m}\Big)^{\alpha-\frac{d}{2}}H^-_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2-\mathrm{i}0}\big)\Bigg) \tag{7.249}$$

$$D^{(\pm)}_\alpha=\frac{2}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Big(\frac{\sqrt{x^2\pm\mathrm{i}\mathrm{sgn}x^0 0}}{2}\Big)^{\alpha-\frac{d}{2}}K_{\alpha-\frac{d}{2}}\big(\sqrt{x^2\pm\mathrm{i}\mathrm{sgn}x^0 0}\big). \tag{7.250}$$

Note the identities

$$D^{\mathrm{PJ}}_\alpha(x)=-\mathrm{i}D^{(+)}_\alpha(x)+\mathrm{i}D^{(-)}_\alpha(x), \tag{7.251}$$

$$D^{\mathrm{F}}_\alpha(x)-D^{\overline{\mathrm{F}}}_\alpha(x)=\mathrm{i}D^{(+)}_\alpha(x)+\mathrm{i}D^{(-)}_\alpha(x). \tag{7.252}$$

The support of D^{PJ}_α is contained in $J^\vee\cup J^\wedge$. In the interior of $J^\vee\cup J^\wedge$ we have the identity

$$D^{\mathrm{PJ}}_\alpha(x)=\mathrm{sgn}(x^0)\frac{2\pi}{\Gamma(\alpha)(4\pi)^{\frac{d}{2}}}\Big(\frac{\sqrt{-x^2}}{2}\Big)^{\alpha-\frac{d}{2}}J_{\alpha-\frac{d}{2}}\big(\sqrt{-x^2}\big). \tag{7.253}$$

7.4 Fourier transforms and Bessel type functions

We will have two generic notations for elements of $\mathbb{R}^d$: for $k \in \mathbb{R}^d$, $p := |k|$, and for $x \in \mathbb{R}^d$, $r := |x|$.

Theorem 7.6 *Let* $\mathrm{Re}\alpha > \frac{d}{2}$. *Then*

$$\int (k^2 + m^2)^{-\alpha} \mathrm{d}k = \pi^{\frac{d}{2}} m^{d-2\alpha} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)}, \tag{7.254}$$

$$\int \mathrm{e}^{-\mathrm{i}kx} (k^2 + m^2)^{-\alpha} \mathrm{d}k = \pi^{\frac{d}{2}} m^{d-2\alpha} \frac{2}{\Gamma(\alpha)} \Big(\frac{mr}{2}\Big)^{\alpha - \frac{d}{2}} K_{\alpha - \frac{d}{2}}(mr). \tag{7.255}$$

Proof. (7.254) is

$$S_{d-1} \int_0^\infty (p^2 + m^2)^{-\alpha} p^{d-1} \mathrm{d}p = S_{d-1} 2^{-1} m^{d-2\alpha} \frac{\Gamma(\frac{d}{2})\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)}.$$

To prove (7.255) we use

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{e}^{-tA} t^{\alpha-1} \mathrm{d}t.$$

We obtain

$$\begin{aligned} & \int (1 + k^2)^{-\alpha} \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k \\ = & \Big(\frac{r}{2}\Big)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}t \int \mathrm{d}k t^{\alpha-1} \mathrm{e}^{-(1+k^2)\frac{rt}{2}} \mathrm{e}^{-\mathrm{i}kx} \\ = & \Big(\frac{r}{2}\Big)^{\alpha - \frac{d}{2}} \frac{\pi^{\frac{d}{2}}}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}t t^{\alpha - \frac{d}{2} - 1} \mathrm{e}^{-(t+t^{-1})\frac{r}{2}}. \end{aligned}$$

□

Note that the rhs of (7.255) is locally integrable for $\alpha > 0$. Therefore, (7.255) is true also for $\mathrm{Re}\alpha > 0$, if the Fourier transform is interpreted appropriately. We can actually extend (7.255) to all $\alpha \in \mathbb{C}$, in the sense of distributions.

In the range $0 < \mathrm{Re}\alpha < \frac{d}{2}$ both (7.255) and (7.256) are regular distributions Using the asymptotics of the MacDonald function we easily see that as $m \searrow 0$, the rhs of (7.255) converges to

$$\pi^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2} - \alpha)}{\Gamma(\alpha)} \Big(\frac{r}{2}\Big)^{2\alpha - d}. \tag{7.256}$$

In the distributional sense this convergence is true for all $\alpha \in \mathbb{C}$ except for $\alpha = \frac{d}{2} + n$, $n = 0, 1, \ldots$, because then $\frac{1}{k^{2\alpha}} = \frac{1}{k^{d+2n}}$, which is the anomalous case and an additional renormalization is needed.

7.5 Averages of plane waves on sphere

Consider the Euclidean space $\mathbb{R}^d$. Let us take the average of plane waves over the unit sphere $\mathbb{S}^{d-1}$. Let $\mathrm{d}\Omega$ denote the standard measure on the sphere.

$$\int \mathrm{e}^{-\mathrm{i}kx}\mathrm{d}\Omega(k) \tag{7.257}$$

$$= \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}pr\cos\theta}\sin^{d-2}\theta\mathrm{d}\theta S_{d-1} \tag{7.258}$$

$$= \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}prw}(1-w^2)^{\frac{d-3}{2}}\mathrm{d}w S_{d-1} \tag{7.259}$$

$$= (2\pi)^{\frac{d}{2}} J_{\frac{d}{2}-1}(pr)(pr)^{-\frac{d}{2}-1}. \tag{7.260}$$

The Fourier transform of a radial function is radial and we have the identity

$$\int f(|k|)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k = (2\pi)^{\frac{d}{2}} \int f(p) J_{\frac{d}{2}-1}(rp)(rp)^{-\frac{d}{2}+1}p^{d-1}\mathrm{d}p.$$

Here are the low dimensional cases:

$$\begin{aligned} &\int f(|k|)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \\ = \; & 2\int_0^\infty f(p)\cos(pr)\mathrm{d}p, \qquad d=1; \\ = \; & 2\pi\int_0^\infty f(p)pJ_0(pr)\mathrm{d}p, \qquad d=2; \\ = \; & 4\pi\int_0^\infty f(p)p^2\frac{\sin(pr)}{pr}\mathrm{d}p, \qquad d=3. \end{aligned}$$

In particular, in dimension 1 we obtain

$$2\int_0^\infty (1+p^2)^{-\alpha}\cos(pr)\mathrm{d}p = \frac{2\pi^{1/2}}{\Gamma(\alpha)}\Big(\frac{r}{2}\Big)^{\alpha-\frac12}K_{\alpha-\frac12}(r). \tag{7.261}$$

Setting $m=\alpha-\frac12$, we obtain the Poisson representation (2.65).

In dimension $d=2$ we obtain

$$2\pi\int_0^\infty (1+p^2)^{-\alpha}J_0(pr)p\mathrm{d}p = \frac{2\pi}{\Gamma(\alpha)}\Big(\frac{r}{2}\Big)^{\alpha-1}K_{\alpha-1}(r).$$

In dimension $d=3$ we obtain

$$4\pi\int_0^\infty (1+p^2)^{-\alpha}\frac{\sin(pr)}{pr}p\mathrm{d}p = \frac{2\pi^{\frac32}}{\Gamma(\alpha)}\Big(\frac{r}{2}\Big)^{\alpha-\frac32}K_{\alpha-\frac32}(r),$$

which could be also deduced from (7.261) by differentiating wrt r.

7.6 General signature

Suppose that the scalar product on $\mathbb{R}^d$ has a signature with q minuses.

Theorem 7.7

$$\int \mathrm{e}^{-\mathrm{i}kx}(m^2+k^2\pm \mathrm{i}0)^{-\alpha}\mathrm{d}k \tag{7.262}$$

$$= \begin{cases} \pi^{\frac{d}{2}} m^{d-2\alpha} \frac{2(\mp\mathrm{i})^q}{\Gamma(\alpha)} \left(\frac{m\sqrt{x^2}}{2}\right)^{\alpha-\frac{d}{2}} K_{\alpha-\frac{d}{2}}\big(m\sqrt{x^2}\big) & x^2\geq 0; \\ \pm\pi^{\frac{d}{2}} m^{d-2\alpha} \frac{\pi\mathrm{i}(\mp\mathrm{i})^q}{\Gamma(\alpha)} \left(\frac{m\sqrt{-x^2}}{2}\right)^{\alpha-\frac{d}{2}} H^{\pm}_{\alpha-\frac{d}{2}}\big(m\sqrt{-x^2}\big) & x^2\leq 0. \end{cases} \tag{7.263}$$

Proof. We use

$$(A\pm\mathrm{i}0)^{-\alpha} = \frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}}{\Gamma(\alpha)}\int_0^\infty \mathrm{e}^{\pm\mathrm{i}tA}t^{\alpha-1}\mathrm{d}t. \tag{7.264}$$

(7.262) for $x^2\geq 0$ is

$$\left(\frac{\sqrt{x^2}}{2}\right)^{\alpha}\frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\int \mathrm{d}k\mathrm{e}^{\pm\mathrm{i}t(1+k^2)\frac{\sqrt{x^2}}{2}}t^{\alpha-1}\mathrm{e}^{-\mathrm{i}kx} \tag{7.265}$$

$$=\left(\frac{\sqrt{x^2}}{2}\right)^{(\alpha-\frac{d}{2})}\frac{(\mp\mathrm{i})^q\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha-\frac{d}{2})}\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\mathrm{e}^{\pm\mathrm{i}\frac{\sqrt{x^2}}{2}(t-\frac{1}{t})}t^{\alpha-\frac{d}{2}-1} \tag{7.266}$$

$$=\left(\frac{\sqrt{x^2}}{2}\right)^{(\alpha-\frac{d}{2})}\frac{(\mp\mathrm{i})^q\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\mathrm{e}^{-\frac{\sqrt{x^2}}{2}(s+\frac{1}{s})}s^{\alpha-\frac{d}{2}-1}, \tag{7.267}$$

which is the first case of (7.263).

(7.262) for $x^2\leq 0$ is

$$\left(\frac{\sqrt{-x^2}}{2}\right)^{\alpha}\frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\alpha}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\int \mathrm{d}k\mathrm{e}^{\pm\mathrm{i}t(1+k^2)\frac{\sqrt{-x^2}}{2}}t^{\alpha-1}\mathrm{e}^{-\mathrm{i}kx} \tag{7.268}$$

$$=\left(\frac{\sqrt{-x^2}}{2}\right)^{(\alpha-\frac{d}{2})}\frac{(\mp\mathrm{i})^q\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha-\frac{d}{2})}\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}t\mathrm{e}^{\pm\mathrm{i}\frac{\sqrt{-x^2}}{2}(t+\frac{1}{t})}t^{\alpha-\frac{d}{2}-1}, \tag{7.269}$$

which is the second case of (7.263). □

7.7 Averages of plane waves on hyperboloid

Consider the Minkowski space $\mathbb{R}^{1,d-1}$. Let us take the average of plane waves over the unit future hyperboloid $\mathbb{H}_+^{d-1}$. Let $\mathrm{d}\Omega$ denote the standard measure on $\mathbb{H}_+^{d-1}$. Let x be a future oriented vector.

$$
\begin{aligned}
& \int \mathrm{e}^{-\mathrm{i}kx}\mathrm{d}\Omega(k) && (7.270)\\
= & \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}pr\cosh\theta}\sinh^{d-2}\theta\mathrm{d}\theta S_{d-1} && (7.271)\\
= & \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}prw}(w^2-1)^{\frac{d-3}{2}}\mathrm{d}w S_{d-1} && (7.272)\\
= & \mathrm{e}^{-\mathrm{i}\pi(m+\frac{1}{2})}(2\pi)^{\frac{d}{2}}H^{-}_{\frac{d}{2}-1}(pr)(pr)^{-\frac{d}{2}-1}. && (7.273)
\end{aligned}
$$

8 Integrals of Bessel functions

8.1 Scalar products

Theorem 8.1 *We have the following indefinite integrals:*

$$
\begin{aligned}
\int_y^\infty xK_m(ax)K_m(bx)\mathrm{d}x &= \frac{y}{a^2-b^2}\Big(aK_{m-1}(ay)K_m(by)-bK_m(ay)K_{m-1}(by)\Big),\\
&\quad \mathrm{Re}(a+b)>0,\\
\int_y^\infty xK_m(ax)^2\mathrm{d}x &= \frac{y^2}{2}K_m(ay)^2+\frac{my}{a}K_m(ay)K_{m-1}(ay)-\frac{y^2}{2}K_{m-1}(ay)^2,\\
&\quad \mathrm{Re}a>0.
\end{aligned}
$$

Proof. Let us prove the first identity. Using $K_m = K_{-m}$, we write

$$
\begin{aligned}
& y\Big(aK_{m-1}(ay)K_m(by)-bK_m(ay)K_{m-1}(by)\Big)\\
= & ay^{-m+1}K_{-m+1}(ay)y^mK_m(by)-by^mK_m(ay)y^{-m+1}K_{-m+1}(by).
\end{aligned}
$$

We differentiate using the recurrence relations. We obtain

$$
\begin{aligned}
& a^2y^{-m+1}K_{-m}(ay)y^mK_m(by)+aby^{-m+1}K_{-m+1}(ay)y^mK_{m-1}(by)\\
& -aby^mK_{m-1}(ay)y^{-m+1}K_{-m+1}(by)-b^2y^mK_m(ay)y^{-m+1}K_{-m}(by)\\
= & (a^2-b^2)yK_m(ay)K_m(by).
\end{aligned}
$$

□

Theorem 8.2 *We have the following definite integrals:*

$$\begin{aligned}
\int_0^\infty xK_m(ax)K_m(bx)\mathrm{d}x &= \frac{\pi(a^m b^{-m} - a^{-m}b^m)}{2\sin m\pi(a^2-b^2)}, \\
&\quad m \neq 0,\ |\mathrm{Re}m| < 1,\ \mathrm{Re}(a+b) > 0; \\
\int_0^\infty xK_0(ax)K_0(bx)\mathrm{d}x &= \frac{\ln a - \ln b}{a^2-b^2}, \quad \mathrm{Re}(a+b) > 0; \\
\int_0^\infty xK_m(ax)^2\mathrm{d}x &= \frac{\pi m}{2\sin m\pi a^2}, \\
&\quad m \neq 0,\ |\mathrm{Re}m| < 1,\ \mathrm{Re}a > 0; \\
\int_0^\infty xK_0(ax)^2\mathrm{d}x &= \frac{1}{2a^2}, \quad \mathrm{Re}a > 0.
\end{aligned}$$

Proof. Assume that $0 < \mathrm{Re}m < 1$. Then for small z

$$\begin{aligned}
K_m(z) &\approx \frac{\pi}{2\sin\pi m}I_{-m}(z) \approx \frac{\pi}{2\sin\pi m\Gamma(1-m)}\Big(\frac{z}{2}\Big)^{-m}, \\
K_{m-1}(z) &\approx -\frac{\pi}{2\sin\pi(m-1)}I_{m-1}(z) \approx \frac{\pi}{2\sin\pi m\Gamma(m)}\Big(\frac{z}{2}\Big)^{m-1}.
\end{aligned}$$

Therefore, for small y,

$$\begin{aligned}
&\frac{y}{a^2-b^2}\Big(aK_{m-1}(ay)K_m(by) - bK_m(ay)K_{m-1}(by)\Big) \\
\approx\ &\frac{\pi y}{(a^2-b^2)(2\sin\pi m)^2\Gamma(m)\Gamma(1-m)}\Big(a\Big(\frac{ay}{2}\Big)^{m-1}\Big(\frac{by}{2}\Big)^{-m} - b\Big(\frac{ay}{2}\Big)^{-m}\Big(\frac{by}{2}\Big)^{m-1}\Big) \\
=\ &\frac{\pi(a^m b^{-m} - a^{-m}b^m)}{2\sin m\pi(a^2-b^2)}.
\end{aligned}$$

8.2 Barnes integrals

Theorem 8.3 *For* $\mathrm{Re}z > 0$, $c > 0$, $c + \mathrm{Re}m > 0$,

$$K_m(z) = \frac{1}{8\pi}\int_{-\infty}^{\infty}\Gamma\Big(c+\frac{\mathrm{i}s}{2}\Big)\Gamma\Big(c+\frac{\mathrm{i}s}{2}+m\Big)\Big(\frac{x}{2}\Big)^{-2c-\mathrm{i}s-m}\mathrm{d}s. \qquad (8.274)$$

Proof.

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_\gamma \Gamma\Big(c+\frac{\mathrm{i}s}{2}\Big)\Gamma\Big(c+\frac{\mathrm{i}s}{2}+m\Big)\Big(\frac{z}{2}\Big)^{-2c-\mathrm{i}s-m}\frac{\mathrm{i}\mathrm{d}s}{2} \\
=\ &\sum_{n=0}^\infty\frac{(-1)^n}{n!}\Gamma(m-n)\Big(\frac{z}{2}\Big)^{2n-m} + \sum_{n=0}^\infty\frac{(-1)^n}{n!}\Gamma(-m-n)\Big(\frac{z}{2}\Big)^{2n+m} \\
=\ &\sum_{n=0}^\infty\frac{\pi}{n!\Gamma(1-m+n)\sin\pi m}\Big(\frac{z}{2}\Big)^{2n-m} + \sum_{n=0}^\infty\frac{\pi}{n!\Gamma(1+m+n)\sin\pi m}\Big(\frac{z}{2}\Big)^{2n+m} \\
=\ &\frac{\pi}{\sin\pi m}\Big(I_{-m}(z) - I_m(z)\Big) = 2K_m(z).
\end{aligned}$$

The integral is convergent because of the estimates

$$\Big|\Gamma\Big(c+\frac{\mathrm{i}s}{2}\Big)\Gamma\Big(c+\frac{\mathrm{i}s}{2}+m\Big)\Big| \leq c\langle s\rangle^{2c+\mathrm{Re}m}\mathrm{e}^{-\frac{\pi}{2}s}. \tag{8.275}$$

$$\Big|\Big(\frac{z}{2}\Big)^{-2c-\mathrm{i}s-m}\Big| \leq |z|^{-2c-\mathrm{Re}m}\mathrm{e}^{s\arg z}. \tag{8.276}$$

Of course, we have a version for the Hankel functions.

The following representation holds only for real x

Theorem 8.4 *For* $0 < c < \frac{1}{2}\mathrm{Re}m$*,*

$$J_m(x) = \frac{1}{4\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\Gamma(c+\frac{\mathrm{i}t}{2})}{\Gamma(m+1-c-\frac{\mathrm{i}t}{2})}\Big(\frac{x}{2}\Big)^{m+\frac{1}{2}-2c-\mathrm{i}t}\mathrm{d}t. \tag{8.277}$$

8.3 Integral of Sonine and Schafheitlin

([1] Exercise 4.14, p. 236):

$$\begin{aligned}
&\int_0^\infty \frac{J_m(x\xi)J_k(y\xi)\mathrm{d}\xi}{\xi^\lambda} \\
= \quad & \frac{y^k\Gamma\left(\frac{1+m+k-\lambda}{2}\right)}{2^\lambda x^{1+k-\lambda}\Gamma(k+1)\Gamma\left(\frac{1+m-k-\lambda}{2}\right)}F\left(\frac{1-m+k-\lambda}{2},\frac{1+m+k-\lambda}{2};k+1;\frac{y^2}{x^2}\right),\ y<x; \\
= \quad & \frac{x^m\Gamma\left(\frac{1+m+k-\lambda}{2}\right)}{2^\lambda y^{1+m-\lambda}\Gamma(m+1)\Gamma\left(\frac{1-m+k-\lambda}{2}\right)}F\left(\frac{1+m-k-\lambda}{2},\frac{1+m+k-\lambda}{2};m+1;\frac{x^2}{y^2}\right),\ x<y.
\end{aligned}$$

8.4 Another integral

The following integrals essentially Watson, 13.31 (1):

$$\int_0^\infty \exp(-qy^2)J_m(ay)J_m(by)y\mathrm{d}y \tag{8.278}$$

$$=\frac{1}{2q}\exp\Big(-\frac{a^2+b^2}{4q}\Big)I_m\Big(\frac{ab}{2q}\Big), \quad \mathrm{Re}m > -1, \quad |\arg q| < \frac{\pi}{2}. \tag{8.279}$$

Here is its another version:

$$\int_0^\infty \exp(-qy^2)I_m(\alpha y)I_m(\beta y)y\mathrm{d}y \tag{8.280}$$

$$=\frac{1}{2q}\exp\Big(\frac{\alpha^2+\beta^2}{4q}\Big)I_m\Big(\frac{\alpha\beta}{2q}\Big), \quad \mathrm{Re}m > -1, \quad |\arg q| < \frac{\pi}{2}. \tag{8.281}$$

9 Klein-Gordon equation in $1+1$ dimensions

9.1 Hyperbolic coordinates

We have the coordinates

$$x_+ := \frac{1}{2}(t+y), \quad x_- := \frac{1}{2}(t-y). \tag{9.282}$$

The space $\mathbb{R}^{1,1}$ is divided into 4 sectors:

$$\begin{aligned}
J_{++} &= \{(t,y)\ :\ t>|y|\} &&= \{x_+>0,\ x_->0\},\\
J_{--} &= \{(t,y)\ :\ t<-|y|\} &&= \{x_+<0,\ x_-<0\},\\
J_{+-} &= \{(t,y)\ :\ y>|t|\} &&= \{x_+>0,\ x_-<0\},\\
J_{-+} &= \{(t,y)\ :\ y<-|t|\} &&= \{x_+<0,\ x_->0\}.
\end{aligned}$$

There are 4 hyperbolic coordinate systems:

$$\begin{aligned}
J_{++}: \quad & t = r\cosh\phi, \quad y = r\sinh\phi,\\
& r=\sqrt{t^2-y^2}, \quad \phi = \frac{1}{2}\log(t+y) - \frac{1}{2}\log(t-y),\\
& x_+ = \frac{1}{2}r\mathrm{e}^{\phi}, \quad x_- = \frac{1}{2}r\mathrm{e}^{-\phi};\\
J_{--}: \quad & t = -r\cosh\phi, \quad y = -r\sinh\phi,\\
& r=\sqrt{t^2-y^2}, \quad \phi = \frac{1}{2}\log(-t-y) - \frac{1}{2}\log(-t+y),\\
& x_+ = -\frac{1}{2}r\mathrm{e}^{\phi}, \quad x_- = -\frac{1}{2}r\mathrm{e}^{-\phi};\\
J_{+-}: \quad & t = r\sinh\phi, \quad y = r\cosh\phi,\\
& r=\sqrt{y^2-t^2}, \quad \phi = \frac{1}{2}\log(y-t) - \frac{1}{2}\log(y+t),\\
& x_+ = \frac{1}{2}r\mathrm{e}^{\phi}, \quad x_- = -\frac{1}{2}r\mathrm{e}^{-\phi};\\
J_{-+}: \quad & t = -r\sinh\phi, \quad y = -r\cosh\phi,\\
& r=\sqrt{y^2-t^2}, \quad \phi = \frac{1}{2}\log(-y+t) - \frac{1}{2}\log(-y-t),\\
& x_+ = -\frac{1}{2}r\mathrm{e}^{\phi}, \quad x_- = \frac{1}{2}r\mathrm{e}^{-\phi};
\end{aligned}$$

The d'Alembertian in all sectors is

$$\Box = -\partial_t^2 + \partial_y^2 = -\partial_{x_+}\partial_{x_-} = \partial_r^2 + r^{-1}\partial_r - r^{-2}\partial_\phi^2. \tag{9.283}$$

9.2 Plane waves

We look for solutions of the Klein-Gordon equation

$$(-\Box+1)F = 0$$

of the form

$$F(t,y)=\frac{1}{2\pi}\int\int F(\xi,\eta)\mathrm{e}^{\mathrm{i}(-t\tau+y\eta)}\mathrm{d}\xi\mathrm{d}\eta.$$

We obtain

$$(-\tau^2+\eta^2+1)F(\xi,\eta)=0.$$

Positive frequency plane waves are parametrized by $\psi\in\mathbb{R}$ and given by

$$f_\psi(x,y):=\mathrm{e}^{\mathrm{i}(-x\cosh\psi+y\sinh\psi)}.$$

They are solutions of the Klein-Gordon equation. Here is the plane wave in hyperbolic coordinates:

$$\begin{aligned}
J_{++}:\quad f_\psi(r,\phi)\quad &=\mathrm{e}^{-\mathrm{i}r\cosh(\phi-\psi)},\\
J_{--}:\quad f_\psi(r,\phi)\quad &=\mathrm{e}^{\mathrm{i}r\cosh(\phi-\psi)},\\
J_{+-}:\quad f_\psi(r,\phi)\quad &=\mathrm{e}^{-\mathrm{i}r\sinh(\phi-\psi)},\\
J_{-+}:\quad f_\psi(r,\phi)\quad &=\mathrm{e}^{\mathrm{i}r\sinh(\phi-\psi)}.
\end{aligned}$$

Positive frequency solutions are given by

$$g(x,y)=\int f_\psi(x,y)g(\psi)\mathrm{d}\psi, \tag{9.284}$$

where g is a distribution on $\mathbb{R}$. We will denote by $\mathcal{H}$ the Hilbert space of functions on $\mathbb{R}^2$ of the form (9.284) with $g\in L^2(\mathbb{R})$. We will treat them as "nice" solutions of the Klein-Gordon equation.

9.3 Hyperbolic waves

Let us introduce for $\mu\in\mathbb{R}$

$$f_\mu:=\frac{1}{2\pi}\int f_\psi\mathrm{e}^{\mathrm{i}\mu\psi}\mathrm{d}\psi. \tag{9.285}$$

Proposition 9.1 *Here are the expressions of hyperbolic waves in hyperbolic coordinates:*

$$J_{++}:\quad f_\mu(r,\phi)\quad =\frac{1}{\pi}K_{\mathrm{i}\mu}(\mathrm{i}r)\mathrm{e}^{\mathrm{i}\mu\phi}, \tag{9.286}$$

$$J_{--}:\quad f_\mu(r,\phi)\quad =\frac{1}{\pi}K_{\mathrm{i}\mu}(-\mathrm{i}r)\mathrm{e}^{\mathrm{i}\mu\phi}, \tag{9.287}$$

$$J_{+-}:\quad f_\mu(r,\phi)\quad =\frac{\mathrm{e}^{\pm\frac{\pi}{2}m}}{\pi}K_{\mathrm{i}\mu}(\pm r)\mathrm{e}^{\mathrm{i}\mu\phi}, \tag{9.288}$$

$$J_{-+}:\quad f_\mu(r,\phi)\quad =\frac{\mathrm{e}^{\mp\frac{\pi}{2}m}}{\pi}K_{\mathrm{i}\mu}(\pm r)\mathrm{e}^{\mathrm{i}\mu\phi}. \tag{9.289}$$

Proposition 9.2 *We can expand plane waves in hyperbolic waves:*

$$f_\psi=\int f_\mu\mathrm{e}^{-\mathrm{i}\mu\psi}\mathrm{d}\mu. \tag{9.290}$$

9.4 Wave equation in $1+1$ dimension

We will use polar coordinates. Using the expressions (**??**) we obtain

$$\begin{aligned}\Box := \partial_x^2 - \partial_t^2 &= \cos(2\phi)\partial_r^2 - 2\sin(2\phi)r^{-1}\partial_r\partial_\phi - \cos(2\phi)r^{-2}\partial_\phi^2 \\ &\quad +2r^{-2}\sin(2\phi)\partial_\phi - r^{-1}\cos(2\phi)\partial_r. \end{aligned} \tag{9.291}$$

Thus

$$r^2\Box = \cos(2\phi)\big(r^2\partial_r^2 - r\partial_r\big) - 2\sin(2\phi)\big(r\partial_r - 1\big)\partial_\phi - \cos(2\phi)\partial_\phi^2$$

On functions of the form $r^\lambda f(\phi)$ we obtain an operator

$$\Lambda_\lambda = \cos(2\phi)(\lambda^2 - 2\lambda) - 2\sin(2\phi)(\lambda-1)\partial_\phi - \cos(2\phi)\partial_\phi^2.$$

Let us substitute $w = \sin(2\phi)$. One can distinguish two regions:

$$\begin{aligned}\partial_\phi = 2\sqrt{1-w^2}\partial_w, &\quad \cos(2\phi) = \sqrt{1-w^2}, \quad \cos(2\phi) > 0, \\ \partial_\phi = -2\sqrt{1-w^2}\partial_w, &\quad \cos(2\phi) = -\sqrt{1-w^2}, \quad \cos(2\phi) < 0.\end{aligned}$$

We obtain

$$\Lambda_\lambda = -4\sqrt{1-w^2}\begin{cases}\Big(\frac{\lambda}{2}\Big(1-\frac{\lambda}{2}\Big) + 2\Big(\frac{\lambda}{2}-1\Big)w\partial_w + (1-w^2)\partial_w^2\Big), \\ \Big(\frac{\lambda}{2}\Big(\frac{\lambda}{2}-1\Big) - \lambda w\partial_w + (1-w^2)\partial_w^2\Big).\end{cases}$$

This corresponds to the Jacobi equation

$$(1-w^2)\partial_w^2 - 2(m+1)w\partial_w + n(n+2m+1) \tag{9.292}$$

with

$$n = -m = \frac{\lambda}{2}, \tag{9.293}$$

$$-n = m = \frac{\lambda}{2}. \tag{9.294}$$

10 Elements of partial differential equations

10.1 General formalism

Let

$$P(k) = \sum_\alpha P_\alpha k^\alpha \tag{10.295}$$

be a polynomial in d variables $k = (k_1, \dots, k_d)$. We set

$$D_i := \frac{1}{\mathrm{i}}\partial_i. \tag{10.296}$$

We consider the differential operator

$$P(D) = \sum_\alpha P_\alpha D^\alpha. \tag{10.297}$$

One can consider two problems: find solutions of the *homogeneous problem*

$$P(D)\zeta = 0, \tag{10.298}$$

and, given f, find solutions of the *inhomogeneous problem*.

$$P(D)\zeta = f. \tag{10.299}$$

To solve the inhomogeneous problem, it is useful to introduce a *Green's function* or a *fundamental solution* of $P(D)$, which is a distribution G satisfying

$$PG(x) = \delta(x). \tag{10.300}$$

Note that if we know Green's function, then

$$Gf(x) = \int G(x-y)f(y)\mathrm{d}y \tag{10.301}$$

solves the inhomogeneous equation.

Green's function is not uniquely defined. In fact, if G is Green's function and ζ solves the homogeneous problem, then $G + \zeta$ is also Green's function. We will see, however, that often we will have distinguished Green's functions. Sometimes we will also have distinguished solutions.

We can look for Green's functions using the Fourier transformation. In fact, suppose that $G \in \mathcal{S}(\mathbb{R}^d)$. We can write

$$G(x) = \int G(k)\mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{d}k}{(2\pi)^d}, \tag{10.302}$$

$$\delta(x) = \int \mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{d}k}{(2\pi)^d}. \tag{10.303}$$

Equation (10.300) becomes

$$\int P(k)G(k)\mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{d}k}{(2\pi)^d} = \int \mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{d}k}{(2\pi)^d}. \tag{10.304}$$

Thus formally,

$$G(x) = \int \frac{1}{P(k)}\mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{d}k}{(2\pi)^d}. \tag{10.305}$$

If (10.305) is well defined, then it provides a distinguished Green's function for $P(D)$. Unfortunately, often, especially if P has zeros, $\frac{1}{P(k)}$ is not a well defined distribution and (10.305) is problematic.

10.2 Laplace equation in $d = 1$

Consider $P(D) = -\partial_x^2$, so that $P(k) = k^2$.

Space of solutions is

$$a + bx. \tag{10.306}$$

Examples of Green's functions:

$$\begin{aligned} G^+(x) &= -\theta(x)x, & (10.307)\\ G^-(x) &= -\theta(-x)|x|, & (10.308)\\ G^0(x) &= -\frac{1}{2}|x|. & (10.309) \end{aligned}$$

$\frac{1}{k^2}$ is not a distribution, but it can be regularized.

10.3 Helmholtz equation in $d = 1$

Consider $P(D) = -\partial_x^2 + m^2$, so that $P(k) = k^2 + m^2$.

Space of solutions is

$$a_+ \mathrm{e}^{mx} + a_- \mathrm{e}^{-mx} \tag{10.310}$$

Examples of Green's functions:

$$\begin{aligned} G^+(x) &= -\theta(x)\frac{\sinh(mx)}{m}, & (10.311)\\ G^-(x) &= -\theta(-x)\frac{|\sinh(mx)|}{m}, & (10.312)\\ G(x) &= \frac{\mathrm{e}^{-m|x|}}{2m}. & (10.313) \end{aligned}$$

$\frac{1}{k^2+m^2}$ is a distribution. By the method of residues we can compute:

$$\frac{1}{2\pi}\int \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2+m^2}\mathrm{d}k = \frac{\mathrm{e}^{-m|x|}}{2m}, \tag{10.314}$$

which reproduces (10.313).

10.4 Laplace equation in $d = 2$

Consider $P(D) = -\partial_x^2 - \partial_y^2$, so that $P(k) = k_x^2 + k_y^2$. Introduce complex coordinates

$$z = \frac{1}{2}(x + \mathrm{i}y), \quad \overline{z} = \frac{1}{2}(x - \mathrm{i}y). \tag{10.315}$$

Then

$$\partial_z = \partial_x - \mathrm{i}\partial_y, \quad \partial_{\overline{z}} = \partial_x + \mathrm{i}\partial_y. \tag{10.316}$$

We have

$$-\partial_x^2 - \partial_y^2 = \partial_{\overline{z}}\partial_z. \tag{10.317}$$

Therefore, solutions are sums of a holomorphic and antiholomorphic function:

$$\zeta_1(z) + \zeta_2(\overline{z}). \tag{10.318}$$

In polar coordinates

$$x = r\cos\phi, \quad y = r\sin\phi,$$

the Laplacian is

$$-\partial_x^2 - \partial_y^2 = -\frac{1}{r}\partial_r r\partial_r - \frac{1}{r^2}\partial_\phi^2. \tag{10.319}$$

We claim that rotationally symmetric Green's functions have the form

$$a - \frac{1}{2\pi}\log r = a - \frac{1}{4\pi}\log(z) - \frac{1}{4\pi}\log(\overline{z}) + \log 4. \tag{10.320}$$

We easily check that $a + b\log(r)$ solves the Laplace equation outside of the origin, either in the polar coordinates, or noticing the decomposition into a holomorphic and an antiholomorphic function. It is more difficult to determine the coefficient in front of $\log r$.

Note that $PG = \delta$ means that for any test function $\phi \in C_{\rm c}^\infty(\mathbb{R}^2)$

$$\int P(D)\phi(x,y)G(x,y)\mathrm{d}x\mathrm{d}y = \int \phi(x,y)P(D)G(x,y)\mathrm{d}x\mathrm{d}y = \phi(0,0). \tag{10.321}$$

Assume that ϕ is rotationally symmetric and $G(r) = a + b\log(r)$. We have

$$\int\limits_{x^2+y^2>\epsilon^2} P(D)\phi(x,y)G(x,y)\mathrm{d}x\mathrm{d}y \tag{10.322}$$

$$= 2\pi\int_\epsilon^\infty (-\partial_r r\partial_r)\phi(r)G(r)\mathrm{d}r \tag{10.323}$$

$$= 2\pi\int_\epsilon^\infty \phi(r)(-\partial_r r\partial_r)G(r)\mathrm{d}r \tag{10.324}$$

$$+2\pi(\phi'(\epsilon)G(\epsilon)\epsilon - \phi(\epsilon)G'(\epsilon)\epsilon) \tag{10.325}$$

$$\to -2\pi b\phi(0). \tag{10.326}$$

Hence, $b = -\frac{1}{2\pi}$.

$\frac{1}{k}\theta(k)$ is not a distribution. We can regularize it as in (**??**). Then we obtain a Green's function with a rather strange looking constant:

$$G(x,y) = \iint\limits_{k_x^2+k_y^2<1} \frac{\left(\mathrm{e}^{\mathrm{i}xk_x+\mathrm{i}yk_y} - 1\right)}{(k_x^2+k_y^2)}\frac{\mathrm{d}k_x\mathrm{d}k_y}{(2\pi)^2} \tag{10.327}$$

$$+ \iint\limits_{k_x^2+k_y^2>1} \frac{\mathrm{e}^{\mathrm{i}xk_x+\mathrm{i}yk_y}}{(k_x^2+k_y^2)}\frac{\mathrm{d}k_x\mathrm{d}k_y}{(2\pi)^2} \tag{10.328}$$

$$= -\frac{1}{2\pi}\log\left(\frac{r}{2}\right) - \frac{\gamma}{2\pi}. \tag{10.329}$$

10.5 Helmholtz equation in $d = 2$

Consider $P(D) = -\partial_x^2 - \partial_y^2 + m^2$, so that $P(k) = k_x^2 + k_y^2 + m^2$. The method of Fourier transformation gives a distinguished Green's function

$$\begin{aligned} G(x,y) &= \int\int \frac{e^{ixk_x + iyk_y}}{(m^2 + k_x^2 + k_y^2)} \frac{dk_x dk_y}{(2\pi)^2} && (10.330) \\ &= \frac{1}{(2\pi)^2} \int\int \frac{e^{i|k|r\cos\phi}}{(m^2 + |k|^2)} |k| d|k| d\phi && (10.331) \\ &= \frac{1}{2\pi} K_0(mr), && (10.332) \end{aligned}$$

where $K_0(z)$ is the 0th MacDonald function. Note the asymptotics around zero:

$$K_0(z) \simeq -\log\frac{z}{2} - \gamma. \tag{10.333}$$

Thus, in order to obtain a zero-mass Green's function, we need to renormalize. Writing $G_m(r)$ for the Green's function with mass m, as defined in (10.329) and (10.332), we obtain the massless Green's functions by the following limit:

$$\lim_{m \to 0} \big(G_m + \frac{1}{2\pi}\log m\big) = G_0. \tag{10.334}$$

10.6 Wave equation in $d = 1 + 1$

Consider $\Box = \partial_t^2 - \partial_y^2$. Introducing coordinates

$$u_+ := \frac{1}{2}(t+y), \qquad u_- := \frac{1}{2}(t-y), \tag{10.335}$$

we have

$$\Box = \partial_{u_+}\partial_{u_-}. \tag{10.336}$$

Therefore, the general solution is

$$\chi_+(t+y) + \chi_-(t-y). \tag{10.337}$$

We have the retarded Green's function, the advanced Green's function, and the Pauli-Jordan solution:

$$\begin{aligned} D^+(t,y) &= \theta(t-|x|) = \theta(t+x)\theta(t-x), && (10.338) \\ D^-(t,y) &= \theta(-t-|x|) = \theta(-t-x)\theta(-t+x), && (10.339) \\ D^{PJ}(t,y) &= \theta(t-|x|) - \theta(-t-|x|) && (10.340) \\ &= \theta(t-x) - \theta(-t-x) = \theta(t+x) - \theta(-t+x). && (10.341) \end{aligned}$$

Let us compute the retarded Green's function by the Fourier transform method. We introduce E, p, the dual variables to t, y. Besides, $p_+ := E + p$ and

$p_- := E - p$. We have $\mathrm{d}E\mathrm{d}p = \frac{1}{2}\mathrm{d}p_+\mathrm{d}p_-$.

$$\begin{aligned} D^+(t,y) &= \int \frac{\mathrm{e}^{\mathrm{i}(-Et+px)}\mathrm{d}E\mathrm{d}p}{(-E^2+p^2-\mathrm{i}0\mathrm{sgn}E)(2\pi)^2} && (10.342)\\ &= \frac{1}{2}\int \frac{\mathrm{e}^{-\mathrm{i}(u_+p_-+u_-p_+)}\mathrm{d}p_-\mathrm{d}p_+}{\big(-p_+p_- - \mathrm{i}0\mathrm{sgn}(p_++p_-)\big)(2\pi)^2} && (10.343)\\ &= -\frac{1}{2}\int \frac{\mathrm{e}^{-\mathrm{i}u_-p_+}\mathrm{d}p_+}{(p_++\mathrm{i}0)2\pi}\int \frac{\mathrm{e}^{-\mathrm{i}u_+p_-}\mathrm{d}p_-}{(p_-+\mathrm{i}0)2\pi} && (10.344)\\ &= \theta(u_+)\theta(u_-). && (10.345)\end{aligned}$$

The Feynman propagator is obtained by the Wick rotation from the Euclidean propagator (Green's function of the Laplacian). More precisely, we set $E = \mathrm{i}k_x$, $t = \mathrm{i}x$:

$$\begin{aligned} D^{\mathrm{F}}(t,y) &= \frac{1}{(2\pi)^2}\int\int \frac{\mathrm{e}^{-\mathrm{i}Et+\mathrm{i}py}\mathrm{d}E\mathrm{d}p}{-E^2+p^2-\mathrm{i}0} && (10.346)\\ &= \frac{\mathrm{i}}{(2\pi)^2}\int\int \frac{\mathrm{e}^{-\mathrm{i}k_xx+\mathrm{i}py}\mathrm{d}E\mathrm{d}p}{-E^2+p^2-\mathrm{i}0} = \mathrm{i}D^{\mathrm{E}}(\mathrm{i}^{-1}t,y). && (10.347)\end{aligned}$$

Setting

$$D^{\mathrm{E}}(x,y) = -\frac{1}{4\pi}\log\Big(\frac{x^2+y^2}{4}\Big) - \frac{\gamma}{2\pi}, \qquad (10.348)$$

we obtain

$$\begin{aligned} D^{\mathrm{F}}(t,y) &= -\frac{\mathrm{i}}{4\pi}\log\Big(\frac{-t^2+y^2+\mathrm{i}0}{4}\Big) - \frac{\mathrm{i}\gamma}{2\pi}\\ &= -\frac{\mathrm{i}}{4\pi}\log\Big(\frac{|-t^2+y^2|}{4}\Big) + \frac{1}{2}\theta\big(|t|-|y|\big) - \frac{\mathrm{i}\gamma}{2\pi},\\ D^{(\pm)}(t,x) &= \mp\frac{\mathrm{i}}{4\pi}\log\Big(\frac{|-t^2+y^2|}{4}\Big) + \frac{1}{2}\theta\big(t-|y|\big) - \frac{1}{2}\theta\big(-t-|y|\big) \mp \frac{\mathrm{i}\gamma}{2\pi}.\end{aligned}$$

11 Miscellanea

11.1 ...

Identity

$$\begin{aligned}(2m+\kappa)\frac{(1-w^2)}{r}\partial_w + (2m+\kappa)w\partial_r &= \Big(\frac{m+\kappa}{r}+\partial_r\Big)\big(mw+(1-w^2)\partial_w\big)\\ &\quad +\Big(\frac{m}{r}-\partial_r\Big)\big(-(m+\kappa)w+(1-w^2)\partial_w\big).\end{aligned}$$

11.2

We set $z = \cos\phi$:

$$\begin{aligned}
& \partial_x^2 + \partial_y^2 + (\frac{1}{4} - m^2)\frac{1}{y^2} \\
= \quad & \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\Big(\partial_\phi^2 + (\frac{1}{4} - m^2)\frac{1}{\sin^2\phi}\Big) \\
= \quad & \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\Big((1-z^2)\partial_z^2 - z\partial_z + (\frac{1}{4} - m^2)\frac{1}{1-z^2}\Big)
\end{aligned}$$

We use

$$(1-z^2)^{-\frac{1}{2}(m+\frac{1}{2})}\partial_z(1-z^2)^{\frac{1}{2}(m+\frac{1}{2})} = \partial_z - (m+\frac{1}{2})\frac{z}{1-z^2},$$

obtaining

$$\begin{aligned}
& (1-z^2)^{-\frac{1}{2}(m+\frac{1}{2})}\Big((1-z^2)\partial_z^2 - z\partial_z + (\frac{1}{4} - m^2)\frac{1}{1-z^2}\Big)(1-z^2)^{\frac{1}{2}(m+\frac{1}{2})} \\
= \quad & (1-z^2)\partial_z^2 - (2m+2)z\partial_z - \big(m + \frac{1}{2}\big)^2.
\end{aligned}$$

11.3

We set $t = \cos 2\phi$, so that $\sin^2\phi = \frac{1-t}{2}$, $\cos^2\phi = \frac{1+t}{2}$, $\partial_\phi = -2\sqrt{1-t^2}\partial_t$.

$$\begin{aligned}
& \partial_x^2 + (\frac{1}{4} - \alpha^2)\frac{1}{x^2} + \partial_y^2 + (\frac{1}{4} - \beta^2)\frac{1}{y^2} + \partial_z^2 \\
= \quad & \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Bigg((1-w^2)\partial_w^2 - 2w\partial_w + \\
& \frac{1}{1-w^2}\Big(4(1-t^2)\partial_t^2 - 4t\partial_t + (\frac{1}{4} - \alpha^2)\frac{2}{1+t} + (\frac{1}{4} - \beta^2)\frac{2}{1-t}\Big)\Bigg).
\end{aligned}$$

We use

$$\begin{aligned}
& (1+t)^{-\frac{1}{2}(\alpha+\frac{1}{2})}(1-t)^{-\frac{1}{2}(\beta+\frac{1}{2})}\partial_t(1+t)^{\frac{1}{2}(\alpha+\frac{1}{2})}(1-t)^{\frac{1}{2}(\beta+\frac{1}{2})} \\
= \quad & \partial_t + \frac{1}{2}(\alpha + \frac{1}{2})\frac{1}{1+t} - \frac{1}{2}(\beta + \frac{1}{2})\frac{1}{1-t}.
\end{aligned}$$

We obtain

$$\begin{aligned}
& (1+t)^{-\frac{1}{2}(\alpha+\frac{1}{2})}(1-t)^{-\frac{1}{2}(\beta+\frac{1}{2})} \\
& \times\Big(4(1-t^2)\partial_t^2 - 4t\partial_t + (\frac{1}{4} - \alpha^2)\frac{2}{1+t} + (\frac{1}{4} - \beta^2)\frac{2}{1-t}\Big) \\
& \times(1+t)^{\frac{1}{2}(\alpha+\frac{1}{2})}(1-t)^{\frac{1}{2}(\beta+\frac{1}{2})} \\
= \quad & 4\Bigg((1-t^2)\partial_t^2 + \Big(\frac{\alpha-\beta}{2} - \frac{\alpha+\beta+2}{2}t\Big)\partial_t + \frac{(\alpha+\beta+2)(\alpha+\beta)}{4}\Bigg).
\end{aligned}$$

.1 The digamma function

In our paper we use the digamma function:

$$\psi(z) := \frac{\partial_z \Gamma(z)}{\Gamma(z)}. \tag{.349}$$

Here are its properties:

$$\psi(1+z) = \psi(z) + \frac{1}{z}, \tag{.350}$$
$$\psi(z) - \psi(1-z) = -\pi \cot(\pi z), \tag{.351}$$
$$\psi\Big(\frac{1}{2}+z\Big) - \psi\Big(\frac{1}{2}-z\Big) = \pi \tan(\pi z), \tag{.352}$$
$$2\log 2 + \psi(z) + \psi\Big(z+\frac{1}{2}\Big) = 2\psi(2z), \tag{.353}$$
$$\psi(1) = -\gamma, \tag{.354}$$
$$\psi\Big(\frac{1}{2}\Big) = -\gamma - 2\log 2. \tag{.355}$$

The inverse of the Gamma function is an analytic function with the derivative

$$\partial_z \frac{1}{\Gamma(z)} = -\frac{\psi(z)}{\Gamma(z)}, \tag{.356}$$
$$\partial_z \frac{1}{\Gamma(z)}\Big|_{z=-n} = (-1)^n n!, \quad n = 0, 1, 2, ... \tag{.357}$$

It is also useful to introduce

the shifted kth harmonic number $$H_k(z) := \frac{1}{z} + \cdots + \frac{1}{z+k-1}, \tag{.358}$$

the kth harmonic number $$H_k := \frac{1}{1} + \cdots + \frac{1}{k} = H_k(1), \tag{.359}$$

the Pochhammer symbol $$(z)_k := \frac{\Gamma(z+k)}{\Gamma(z)} = \begin{cases} (z)(z+1)\cdots(z+k-1), & k \geq 0, \\ \frac{1}{(z+k)(z+k+1)\cdots(z-1)}, & k \leq 0. \end{cases} \tag{.360}$$

Some of their properties are collected below:

$$H_{k+n}(z) = H_n(z) + H_k(z+n), \tag{.361}$$
$$H_k(z) = -H_k(1-z-k), \tag{.362}$$
$$\psi(z+k) = \psi(z) + H_k(z), \tag{.363}$$
$$\psi(1+k) = -\gamma + H_k, \tag{.364}$$
$$(z)_k = (-1)^k (1-k-z)_k, \tag{.365}$$
$$\partial_z (z)_n = H_n(z)(z)_n. \tag{.366}$$

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