BOGOLIUBOV TRANSFORMATIONS

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Based on joint work with

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Bogoliubov transformations – linear transformations of field operators preserving the CCR/CAR.

Sometimes they can be implemented by unitary Bogoliubov implementers.

Sometimes they can be implemented by natural Bogoliubov implementers.

Implementers are the exponentials of quadratic Hamiltonians.

1. BOSONIC CASE

Let $(\mathcal{Y}, \omega)$ be a real vector space equipped with an antisymmetric form. A regular representations of the canonical commutation relations or a regular CCR representation over $(\mathcal{Y}, \omega)$ on a Hilbert space $\mathcal{H}$ is a map

$$\mathcal{Y} \ni y \mapsto \phi(y) \tag{1}$$

with values in self-adjoint operators on $\mathcal{H}$ such that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\phi(y)}\mathrm{e}^{\mathrm{i}\phi(y')} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}y\cdot\omega y'}\mathrm{e}^{\mathrm{i}\phi(y+y')}, \\ \phi(ty) &= t\phi(y), \quad t \in \mathbb{R} \end{aligned}$$

$\phi(y)$ are called field operators. They satisfy

$$\begin{aligned} \phi(y+y') &= \phi(y)+\phi(y'), \\ [\phi(y),\phi(y')] &= \mathrm{i}y{\cdot}\omega y' \end{aligned}$$

on appropriate domains.

Consider the Hilbert space $L^2(\mathbb{R}^d)$. Let ϕ_i denote the ith coordinate of $\mathbb{R}^d$. Let ϕ_i denote the operator of multiplication by the variable ϕ_i on and π_i the momentum operator $\frac{1}{\mathrm{i}}\partial_{\phi_i}$. Then,

$$\mathbb{R}^d \oplus \mathbb{R}^d \ni (\eta, q) \mapsto \eta \cdot \phi + q \cdot \pi \tag{2}$$

is an irreducible regular CCR representation on $L^2(\mathbb{R}^d)$. (2) is called the Schrödinger representation over the symplectic space $\mathbb{R}^d \oplus \mathbb{R}^d$.

Let $(\mathcal{Y}, \omega)$ be a symplectic space. We will write $Sp(\mathcal{Y})$ for the symplectic group, that is, the group of invertible symplectic transformations.

Assume in addition that $\mathcal{Y}$ is finite dimensional. Clearly, $\mathcal{Y}$ is always equivalent to $\mathbb{R}^d \oplus \mathbb{R}^d$ with a natural symplectic form. The Stone–von Neumann Theorem says that all irreducible regular CCR representations over $\mathcal{Y}$ are unitarily equivalent to the Schrödinger representation.

Let $\mathcal{Y} \ni y \mapsto \phi(y)$ be a regular CCR representation on $\mathcal{H}$. We define two groups

The c-metaplectic group $Mp^c(\mathcal{Y})$ consists $U \in U(\mathcal{H})$ such that

$$\{U\phi(y)U^* \ : \ y \in \mathcal{Y}\} = \{\phi(y) \ : \ y \in \mathcal{Y}\}.$$

The metaplectic group $Mp(\mathcal{Y})$ is the subgroup of $Mp^c(\mathcal{Y})$ generated by $\mathrm{e}^{\mathrm{i}\phi(y)^2}$, $y \in \mathcal{Y}$.

We have a homomorphism $Mp^c(\mathcal{Y}) \ni U \mapsto r \in Sp(\mathcal{Y})$ given by

$$U\phi(y)U^* = \phi(ry).$$

Various homomorphism related to the metaplectic group can be described by the following diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & U(1) & \rightarrow & U(1) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Mp(\mathcal{Y}) & \rightarrow & Mp^{\mathrm{c}}(\mathcal{Y}) & \rightarrow & U(1) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Sp(\mathcal{Y}) & \rightarrow & Sp(\mathcal{Y}) & \rightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \qquad (3)$$

Let $\mathcal{Z}$ be a complex Hilbert space. Consider the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$. We use the standard notation for creation/annihilation operators $a^*(z)$, $a(z)$, $z \in \mathcal{Z}$.

$$\mathcal{Y} := \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) = \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\}$$

is equipped with a symplectic form

$$(z, \overline{z}) \cdot \omega(z', \overline{z}') := 2\mathrm{Im}(z|z').$$

The following regular CCR representation is called the Fock representation.

$$\mathcal{Y} \ni (z, \overline{z}) \mapsto \phi(z, \overline{z}) := a^*(z) + a(z)$$

Note that $(\mathcal{Y}, \omega)$ besides the symplectic form has more structure. In particular, it has a symplectic map

$$\mathrm{j}(z, \overline{z}) := (\mathrm{i}z, -\mathrm{i}\overline{z})$$

which satisfies

$$\mathrm{j}^2 = -\mathbb{1}.$$

j is thus a complex structure (anti-involution). Using the terminology from differential geometry, one can say that the space $\mathcal{Y}$ is equipped with a Kähler structure. With help of j we can recover $\mathcal{Z}$ from $\mathcal{Y}$:

$$\mathcal{Z} = \frac{\mathbb{1} - \mathrm{i}\mathrm{j}}{2}\mathbb{C}\mathcal{Y}.$$

Let $r \in Sp(\mathcal{Y})$. Its complexification is a complex linear map on $\mathbb{C}\mathcal{Y} \simeq \mathcal{Z} \oplus \overline{\mathcal{Z}}$ that can be written as

$$r_{\mathbb{C}} = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, \tag{4}$$

with $p \in B(\mathcal{Z})$, $q \in B(\overline{\mathcal{Z}}, \mathcal{Z})$. It satisfies the following conditions:

$$p^*p - q^\#\overline{q} = \mathbb{1}, \quad p^*q - q^\#\overline{p} = 0,$$

$$pp^* - qq^* = \mathbb{1}, \quad pq^\# - qp^\# = 0.$$

Note that

$$pp^* \geq \mathbb{1}, \qquad p^*p \geq \mathbb{1}.$$

Hence p^{-1} and p^{*-1} are bounded operators, and we can set

$$\begin{aligned} c &:= q^{\#}(p^{\#})^{-1}, \\ d &:= q\overline{p}^{-1}. \end{aligned}$$

Note that $c^{\#} = c$, $d = d^{\#}$. One has the following factorization:

$$r_{\mathbb{C}} = \begin{bmatrix} \mathbb{1} & d \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ \overline{c} & \mathbb{1} \end{bmatrix}. \tag{5}$$

Define the restricted symplectic group

$$\begin{aligned} Sp_{\mathrm{j}}(\mathcal{Y}) &:= \{r \in Sp(\mathcal{Y}) \ : \ r\mathrm{j} - \mathrm{j}r \ \text{ is H.S.}\}, \\ &= \{r \in Sp(\mathcal{Y}) \ : \ q \ \text{ is H.S.}\}, \\ &= \{r \in Sp(\mathcal{Y}) \ : \ c \ \text{ is H.S.}\}. \end{aligned}$$

Shale Theorem. *$r \in Sp(\mathcal{Y})$ is unitarily implementable iff $r \in Sp_{\mathrm{j}}(\mathcal{Y})$. Its implementer, up to a phase factor, is*

$$|\det pp^*|^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}a(c)}.$$

Above we use 2-particle creation/annihilation operators. This means the following. Let $\mathcal{Z} = L^2(\Xi, \mathrm{d}\xi)$. Then for 1-particle creation/annihilation operators of $z \in \mathcal{Z}$ we write

$$\begin{aligned} a^*(z) &= \int z(\xi) a^*(\xi) \mathrm{d}\xi, \\ a(z) &= \int \overline{z(\xi)} a(\xi) \mathrm{d}\xi. \end{aligned}$$

Let $d(\xi, \xi')$ be the integral kernel of $d = d^{\#}$. Then

$$\begin{aligned} a^*(d) &= \int d(\xi, \xi') a^*(\xi) a^*(\xi') \mathrm{d}\xi \mathrm{d}\xi', \\ a(d) &= \int \overline{d(\xi, \xi')} a(\xi') a(\xi) \mathrm{d}\xi \mathrm{d}\xi'. \end{aligned}$$

Let $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ be the group of unitaries on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ such that

$$\{U\phi(y)U^* \;:\; y \in \mathcal{Y}\} = \{\phi(y) \;:\; y \in \mathcal{Y}\}.$$

Then we have the short exact sequence

$$\mathbb{1} \to U(1) \to Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) \to Sp_{\mathrm{j}}(\mathcal{Y}) \to \mathbb{1}.$$

Define the anomaly-free symplectic group $Sp_{\mathrm{j,af}}(\mathcal{Y})$ as the set of elements of $Sp_{\mathrm{j}}(\mathcal{Y})$ with $\mathbb{1} - p$ trace class. Let $Mp_{\mathrm{j,af}}(\mathcal{Y})$ consist of operators

$$(\det p^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}a(c)}.$$

We have the short exact sequence

$$\mathbb{1} \to \mathbb{Z}_2 \to Mp_{\mathrm{j,af}}(\mathcal{Y}) \to Sp_{\mathrm{j,af}}(\mathcal{Y}) \to \mathbb{1}.$$

$\mathcal{Y}$ possesses also a natural scalar product

$$(z, \overline{z}) \cdot \nu(z', \overline{z}') := 2\mathrm{Re}(z|z').$$

We can use it to distinguish special cases of $r \in Sp(\mathcal{Y})$:

1. r is orthogonal or equivalently, complex (preserves the complex structure j). Then $q = 0$ and p is unitary. The Bogoliubov implementer is $\Gamma(p)$, or $(\det p)^{1/2}\Gamma(p)$ in the metaplectic group.

2. r is positive. Then $p > 0$ and $c = d$. We can parametrize r by c:

$$p = (\mathbb{1} - cc^*)^{-\frac{1}{2}}, \quad q = (\mathbb{1} - cc^*)^{-\frac{1}{2}}c.$$

The Bogoliubov implementer is

$$\det(\mathbb{1} - cc^*)^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}a^*(c)} \Gamma(\mathbb{1} - cc^*)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a(c)}.$$

Every $r \in Sp(\mathcal{Y})$ has a unique polar decomposition $r = u|r|$. Both u and $|r|$ are symplectic. Besides, u is orthogonal, r is positive.

Consider now bosonic Bogoliubov transformations from the infinitesimal point of view.

We consider first the case of a finite number of degrees of freedom.

Let $sp(\mathcal{Y})$ denote the symplectic Lie algebra. Every element of $sp(\mathcal{Y})$ can be written as $\zeta\omega$, where $\zeta^{\#} = \zeta$. The polynomial given by the symmetric operator ζ will be called the classical Hamiltonian of $\zeta\omega$. We can quantize it using the Weyl quantization:

$$\mathrm{Op}(\zeta) := \sum_{i,j} \phi_i \zeta_{ij} \phi_j .$$

Note that $\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\zeta)} \in Mp(\mathcal{Y})$.

Consider in particular $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ and the Fock representation. The symplectic form and the symmetric form defining the Hamiltonian can be extended by complex linearity:

$$\omega : \mathcal{Z} \oplus \overline{\mathcal{Z}} \to \overline{\mathcal{Z}} \oplus \mathcal{Z}, \quad \omega = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix},$$

$$\zeta : \overline{\mathcal{Z}} \oplus \mathcal{Z} \to \mathcal{Z} \oplus \overline{\mathcal{Z}}, \quad \zeta = \begin{bmatrix} g & h \\ h^{\#} & \overline{g} \end{bmatrix},$$

where $h^* = h$, $g^{\#} = g$.

The complexification of the corresponding element of $sp(\mathcal{Y})$ is

$$\zeta\omega = \begin{bmatrix} -h & g \\ -\overline{g} & h^{\#} \end{bmatrix}.$$

The quantum Hamiltonian is

$$\mathrm{Op}(\zeta) \ = \ \sum_{ij} \Big(h_{ij}(a_i^* a_j + a_j a_i^*) + g_{ij} a_i^* a_j^* + \overline{g}_{ij} a_j a_i \Big).$$

Assume now that $\zeta > 0$. There is a nice formula for the infimum of the quantum quadratic Hamiltonian:

$$\inf \mathrm{Op}(\zeta) = \frac{1}{2}\mathrm{Tr}\sqrt{-(\zeta\omega)^2}. \tag{6}$$

The operator $\zeta\omega$ is anti-self-adjoint on $(\mathcal{Y}, \zeta^{-1})$. Besides, its kernel is $\{0\}$. Therefore if we set

$$\mathrm{j}_1 := (-\zeta\omega\zeta\omega)^{-1/2}\zeta\omega,$$

then $\mathrm{j}_1 \in Sp(\mathcal{Y})$ is an anti-involution and we have the polar decomposition

$$\zeta\omega = (-\zeta\omega\zeta\omega)^{1/2}\mathrm{j}_1 = \mathrm{j}_1(-\zeta\omega\zeta\omega)^{1/2}$$

Double creation/annihilation terms disappear iff $\mathrm{j} = \mathrm{j}_1$.

Set $k := -\mathrm{j}\mathrm{j}_1$. One can check that k is a positive symplectic transformation.

Define $r := k^{1/2}$. Clearly, r is also positive and symplectic. Now

$$r\mathrm{j}_1 r^{-1} = r^2 \mathrm{j}_1 = k\mathrm{j}_1 = \mathrm{j}.$$

Therefore, if U denotes the quantization of r, then the quadratic Hamiltonian $U\mathrm{Op}(\zeta)U^*$ has no double creation/annihilation terms.

If we Wick order the Hamiltonian we obtain the following formula for the ground state energy

$$\inf \sum_{ij} \left(2h_{ij} a_i^* a_j + g_{ij} a_i^* a_j^* + \overline{g}_{ij} a_j a_i \right)$$

$$= \frac{1}{2} \mathrm{Tr} \left(\begin{bmatrix} h^2 - gg^* & -hg + gh^\# \\ g^* h - h^\# g^* & h^{\#2} - g^* g \end{bmatrix}^{\frac{1}{2}} - \begin{bmatrix} h & 0 \\ 0 & h^\# \end{bmatrix} \right).$$

2. FERMIONIC CASE

Let $(\mathcal{Y}, \nu)$ be a real Hilbert space. A representations of the canonical anticommutation relations or a CAR representation over $(\mathcal{Y}, \nu)$ on a Hilbert space $\mathcal{H}$ is a linear map

$$\mathcal{Y} \ni y \mapsto \phi(y) \tag{7}$$

with values in bounded self-adjoint operators on $\mathcal{H}$ such that

$$[\phi(y), \phi(y')]_+ \;=\; 2y \cdot \nu y'$$

For any real Hilbert space $\mathcal{Y}$, its orthogonal group will be denoted by $O(\mathcal{Y})$.

Assume in addition that $\dim \mathcal{Y}$ be finite. Elementary arguments show that if $\dim \mathcal{Y}$ is even, then all irreducible representations are equivalent. If $\dim \mathcal{Y}$ is odd, there are exactly two inequivalent irreducible representations.

It is convenient to introduce the (complex) C^*-algebra $\mathrm{CAR}(\mathcal{Y})$ generated by $\phi(y)$, $y \in \mathcal{Y}$. We have

$$\begin{aligned} \mathrm{CAR}(\mathbb{R}^{2m}) &\simeq B(\mathbb{C}^{2^m}), \\ \mathrm{CAR}(\mathbb{R}^{2m+1}) &\simeq B(\mathbb{C}^{2^m}) \oplus B(\mathbb{C}^{2^m}). \end{aligned}$$

$\mathrm{CAR}(\mathcal{Y})$ contains a real subalgebra generated by $\phi(y)$, $y \in \mathcal{Y}$, which will be denoted $\mathrm{Cliff}(\mathcal{Y})$.

The even subalgebra of $\mathrm{CAR}(\mathcal{Y})$ will be denoted $\mathrm{CAR}_0(\mathcal{Y})$.

We define 4 groups: $Pin^{\mathrm{c}}(\mathcal{Y})$ is the set of unitary U in $\mathrm{CAR}(\mathcal{Y})$ such that

$$\{U\phi(y)U^* \ : \ y \in \mathcal{Y}\} = \{\phi(y) \ : \ y \in \mathcal{Y}\}.$$

$$\begin{aligned} Pin(\mathcal{Y}) &:= Pin^c(\mathcal{Y}) \cap \mathrm{Cliff}(\mathcal{Y}), \\ Spin^{\mathrm{c}}(\mathcal{Y}) &:= Pin^c(\mathcal{Y}) \cap \mathrm{CAR}_0(\mathcal{Y}), \\ Spin(\mathcal{Y}) &:= Pin^c(\mathcal{Y}) \cap \mathrm{Cliff}_0(\mathcal{Y}). \end{aligned}$$

The most useful homomorphism $Pin^c(\mathcal{Y}) \ni U \mapsto r \in O(\mathcal{Y})$ is given not by the implementation but by the det-implementation:

$$U\phi(y)U^* = \det(r)\,\phi(ry).$$

Various homomorphism related to the Pin group can be described by the following diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & U(1) & \rightarrow & U(1) & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Pin(\mathcal{Y}) & \rightarrow & Pin^{\mathrm{c}}(\mathcal{Y}) & \rightarrow & U(1) & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & O(\mathcal{Y}) & \rightarrow & O(\mathcal{Y}) & \rightarrow & 1 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{8}$$

Let $\mathcal{Z}$ be a complex Hilbert space. Consider the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. We use the standard notation for creation/annihilation operators $a^*(z)$, $a(z)$, $z \in \mathcal{Z}$.

$$\mathcal{Y} := \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) = \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\}$$

is equipped with the scalar product

$$(z, \overline{z}) \cdot \nu (z', \overline{z}') := 2\mathrm{Re}(z|z').$$

The following CAR representation is called the Fock representation.

$$\mathcal{Y} \ni (z, \overline{z}) \mapsto \phi(z, \overline{z}) := a^*(z) + a(z)$$

Again, $(\mathcal{Y}, \nu)$ besides the scalar product has more structure. In particular, it has an orthogonal anti-involution j

$$\mathrm{j}(z, \overline{z}) := (\mathrm{i}z, -\mathrm{i}\overline{z})$$

which satisfies

$$\mathrm{j}^2 = -\mathbb{1}.$$

j is thus a complex structure (anti-involution). Using the terminology from differential geometry, one can say that the space $\mathcal{Y}$ has a Kähler structure.

Let $r \in O(\mathcal{Y})$. Its complexification is a complex linear map on $\mathbb{C}\mathcal{Y} \simeq \mathcal{Z} \oplus \overline{\mathcal{Z}}$ that can be written as

$$r_{\mathbb{C}} = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, \tag{9}$$

with $p \in B(\mathcal{Z})$, $q \in B(\overline{\mathcal{Z}}, \mathcal{Z})$. It satisfies the following conditions:

$$p^*p + q^{\#}\overline{q} = \mathbb{1}, \quad p^*q + q^{\#}\overline{p} = 0,$$

$$pp^* + qq^* = \mathbb{1}, \quad pq^{\#} + qp^{\#} = 0.$$

We will say that $r \in O(\mathcal{Y})$ is j-nondegenerate iff $\mathrm{Ker}(r\mathrm{j} + \mathrm{j}r) = \{0\}$, or equivalently, if $\mathrm{Ker} p = \{0\}$. For such r, define

$$\begin{aligned} c &:= q^{\#}(p^{\#})^{-1}, \\ d &:= -q\overline{p}^{-1}. \end{aligned}$$

Note that $c^{\#} = -c$, $d = -d^{\#}$. One has the following factorization:

$$r_{\mathbb{C}} = \begin{bmatrix} \mathbb{1} & d \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ \overline{c} & \mathbb{1} \end{bmatrix}. \quad (10)$$

Define the restricted orthogonal group

$$\begin{aligned} O_{\mathrm{j}}(\mathcal{Y}) &:= \{r \in O(\mathcal{Y}) \ : \ r\mathrm{j} - \mathrm{j}r \ \text{ is H.S.}\}, \\ &= \{r \in O(\mathcal{Y}) \ : \ q \ \text{ is H.S.}\}. \end{aligned}$$

Note that j-nondegenerate elements of $O_{\mathrm{j}}(\mathcal{Y})$ automatically belong to $SO_{\mathrm{j}}(\mathcal{Y})$.

Shale-Stinespring Theorem. *$r \in O(\mathcal{Y})$ is unitarily det-implementable iff $r \in O_{\mathrm{j}}(\mathcal{Y})$. In particular, if r is* j*-nondegenerate, its det-implementer, up to a phase factor, is*

$$|\det pp^*|^{\frac{1}{4}} \mathrm{e}^{\frac{1}{2}a^*(d)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}a(c)}.$$

Above we use 2-particle creation/annihilation operators. This means the following. Let $\mathcal{Z} = L^2(\Xi, \mathrm{d}\xi)$. Then for 1-particle creation/annihilation operators of $z \in \mathcal{Z}$ we write

$$\begin{aligned} a^*(z) &= \int z(\xi) a^*(\xi) \mathrm{d}\xi, \\ a(z) &= \int \overline{z(\xi)} a(\xi) \mathrm{d}\xi. \end{aligned}$$

Let $d(\xi, \xi')$ be the integral kernel of $d = -d^{\#}$. Then

$$\begin{aligned} a^*(d) &= \int d(\xi, \xi') a^*(\xi) a^*(\xi') \mathrm{d}\xi \mathrm{d}\xi', \\ a(d) &= \int \overline{d(\xi, \xi')} a(\xi') a(\xi) \mathrm{d}\xi \mathrm{d}\xi'. \end{aligned}$$

Let $Pin^{\mathrm{c}}_{\mathrm{j}}(\mathcal{Y})$ be the group of unitaries on $\Gamma_{\mathrm{a}}(\mathcal{Z})$ such that

$$\{U\phi(y)U^* \ : \ y \in \mathcal{Y}\} = \{\phi(y) \ : \ y \in \mathcal{Y}\}.$$

Then we have the short exact sequence

$$\mathbb{1} \to U(1) \to Pin^{\mathrm{c}}_{\mathrm{j}}(\mathcal{Y}) \to O_{\mathrm{j}}(\mathcal{Y}) \to \mathbb{1}.$$

Define the anomaly-free orthogonal group $O_{\mathrm{j,af}}(\mathcal{Y})$ as the set of elements of $O_{\mathrm{j}}(\mathcal{Y})$ with $\mathbb{1}-p$ trace class. Let $Pin_{\mathrm{j,af}}(\mathcal{Y})$ be the group generated by operators of the form

$$(\det p)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}a^*(d)}\Gamma\big((p^*)^{-1}\big)\mathrm{e}^{\frac{1}{2}a(c)}.$$

We have the short exact sequence

$$\mathbb{1}\to\mathbb{Z}_2\to Pin_{\mathrm{j,af}}(\mathcal{Y})\to O_{\mathrm{j,af}}(\mathcal{Y})\to\mathbb{1}.$$

Let $\kappa \in O(\mathcal{Y})$ be an involution, that is $\kappa^2 = \mathbb{1}$. It is easy to see that κ is det-implementable iff $\mathrm{Ker}(\kappa + \mathbb{1})$ is finite dimensional and then it is det-implemented by

$$\phi(e_1) \cdots \phi(e_n) \tag{11}$$

where $e_1, \ldots, e_n$ is an o.n. basis of $\mathrm{Ker}(\kappa + \mathbb{1})$. Note that (11) belongs to $Pin_{\mathrm{af,j}}(\mathcal{Y})$. It belongs to $Spin_{\mathrm{af,j}}(\mathcal{Y})$ iff n is even.

Note special cases of $r \in O(\mathcal{Y})$, defined using the complex structure j:

1. If r is complex (commutes with j), then $q = 0$ and p is unitary. The Bogoliubov implementer is $\Gamma(p)$, or $(\det p)^{-1/2}\Gamma(p)$ in the Spin group.

2. We say that r is j-positive if $p > 0$. Then $c = d$ and we can express r with c:

$$p = (\mathbb{1} + cc^*)^{-\frac{1}{2}}, \quad q = (\mathbb{1} + cc^*)^{-\frac{1}{2}}c.$$

The Bogoliubov implementer is

$$\det(\mathbb{1} + cc^*)^{-1/4} \mathrm{e}^{\frac{1}{2}a^*(c)} \Gamma(\mathbb{1} + cc^*)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a(c)}.$$

Every j-non-degenerate $r \in SO(\mathcal{Y})$ has a unique j-polar decomposition $r = u|r|$. Both u and $|r|$ are orthogonal. Besides, u is complex and r is j-positive.

Consider now fermionic Bogoliubov transformations from the infinitesimal point of view. For simplicity, we consider only the case of a finite number of degrees of freedom.

Let $so(\mathcal{Y})$ denote the orthogonal Lie algebra. Every element of $so(\mathcal{Y})$ can be written as $\mathrm{i}\zeta\nu$, where $\zeta^{\#} = -\zeta$, $\overline{\zeta} = -\zeta$. We quantize it using the so-called antisymmetric quantization, which is the fermionic analog of the Weyl quantization:

$$\mathrm{Op}(\zeta) := \sum_{i,j} \phi_i \zeta_{ij} \phi_j.$$

We can compute the infimum and supremum of the quantum Hamiltonian:

$$\inf \mathrm{Op}(\zeta) = -\mathrm{Tr}|\zeta\nu|, \quad \sup \mathrm{Op}(\zeta) = \mathrm{Tr}|\zeta\nu|.$$

Consider in particular $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ and the Fock representation. The scalar product form and the antisymmetric form ζ can be extended by complex linearity:

$$\nu : \mathcal{Z} \oplus \overline{\mathcal{Z}} \to \overline{\mathcal{Z}} \oplus \mathcal{Z}, \quad \nu = \frac{1}{2} \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix},$$

$$\zeta : \overline{\mathcal{Z}} \oplus \mathcal{Z} \to \mathcal{Z} \oplus \overline{\mathcal{Z}}, \quad \zeta = \begin{bmatrix} g & h \\ -h^{\#} & -\overline{g} \end{bmatrix},$$

where $h^* = h$, $g^{\#} = -g$.

The complexification of the corresponding element of $o(\mathcal{Y})$ is

$$\zeta\nu = \frac{1}{2}\begin{bmatrix} h & g \\ -\overline{g} & -h^{\#} \end{bmatrix}.$$

The quantum Hamiltonian is

$$\mathrm{Op}(\zeta) \;=\; \sum_{ij} \big(h_{ij}(a_i^* a_j - a_j a_i^*) + g_{ij} a_i^* a_j^* + \overline{g}_{ij} a_j a_i\big).$$

Assume that ζ is non-degenerate. The operator $\zeta\nu$ is anti-self-adjoint on $(\mathcal{Y}, \nu)$. Besides, its kernel is $\{0\}$. Therefore if we set

$$\mathrm{j}_1 := (-\zeta\nu\zeta\nu)^{-1/2}\zeta\nu,$$

then $\mathrm{j}_1 \in O(\mathcal{Y})$ is an anti-involution and we have the polar decomposition

$$\zeta\nu = (-\zeta\nu\zeta\nu)^{1/2}\mathrm{j}_1 = \mathrm{j}_1(-\zeta\nu\zeta\nu)^{1/2}$$

Double creation/annihilation terms disappear iff $\mathrm{j} = \mathrm{j}_1$.

Assume now that $\mathrm{Ker}(\mathrm{j}+\mathrm{j}_1)=\{0\}$. Set $k:=-\mathrm{j}\mathrm{j}_1$. Then k is an orthogonal transformation satisfying $\mathrm{Ker}(k+1\!\!1)=\{0\}$. Therefore, $r:=k^{\frac{1}{2}}$ is well defined. Then r is orthogonal and

$$r\mathrm{j}_1 r^{-1}=\mathrm{j}.$$

Therefore, if U denotes the quantization of r, then the quadratic quantum Hamiltonian $U\mathrm{Op}(\zeta)U^*$ has no double creation/annihilation terms.

If we Wick order the Hamiltonian we obtain the following formula for the ground state energy

$$\inf \sum_{ij} \left(2h_{ij}a_i^* a_j + g_{ij}a_i^* a_j^* + \overline{g}_{ij}a_j a_i\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left(- \begin{bmatrix} h^2 + gg^* & hg - gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#2} + g^*g \end{bmatrix}^{\frac{1}{2}} + \begin{bmatrix} h & 0 \\ 0 & h^{\#} \end{bmatrix} \right).$$

3. QUADRATIC HAMILTONIANS WITH AN INFINITE NUMBER OF DEGREES OF FREEDOM

Consider a Fock bosonic or fermionic representation with an arbitrary number of degrees of freedom. We say that a self-adjoint operator H is a quadratic or Bogoliubov Hamiltonian on $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ if there exists a strongly continuous 1-parameter unitary group $\mathbb{R} \ni t \mapsto U(t) \in Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})/Spin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ such that

$$U(t) = \mathrm{e}^{\mathrm{i}tH}.$$

Clearly, for a finite number of degrees of freedom quadratic Hamiltonians have the form

$$\sum_{ij} \Big(2h_{ij} a_i^* a_j + g_{ij} a_i^* a_j^* + \overline{g}_{ij} a_j a_i \Big) + c.$$

The Bogoliubov transformation generated by H does not depend on c, so its choice is to some extent arbitrary. One of its possible choices is $c = 0$, corresponding to the Wick ordering. This choice can be often generalized to an infinite number of degrees of freedom. Indeed, if $\Omega \in \mathrm{Dom}|H|^{\frac{1}{2}}$, then $c := (\Omega|H\Omega)$ is well defined and $H - c$ is the Wick-ordered H.

However, many quadratic Hamiltonians cannot be Wick-ordered. In this case the representation of H of the above form is true only formally with $c = \infty$. In QFT this corresponds to the so-called infinite vacuum polarization. The infinity needs to be removed by renormalization.

Suppose that $\mathbb{R} \ni t \mapsto \zeta(t)$ is a family of time-dependent classical quadratic Hamiltonians. We can quantize it by the Weyl/antisymmetric quantization. Then the quantum dynamics

$$\mathrm{T}\exp\Big(-\int_{t_-}^{t_+} \mathrm{i}\mathrm{Op}(\zeta_t)\mathrm{d}t\Big)$$

belongs the the anomaly-free group $Mp_{\mathrm{j,af}}(\mathcal{Y})/Spin_{\mathrm{j,af}}(\mathcal{Y})$. So the classical dynamics fixes the quantum dynamics up to a sign.

For an infinite number of degrees of freedom, we can seldom use the Weyl/antisymmetric quantization. We can only say that the dynamics belongs to $Mp_{\mathrm{j}}(\mathcal{Y})/Spin_{\mathrm{j,af}}(\mathcal{Y})$. In particular, the phase is not fixed by the classical transformation and depends on the exact history of the dynamics.

The following example illustrates the above point: We consider a bosonic system with 1 degree of freedom whose dynamics given by a time dependent Hamiltonian. Its classical scattering operator is identity. With the Wick quantization we obtain

$$\begin{aligned} \mathrm{e}^{\mathrm{i}s\sinh^2 t} &= \exp\Big(\mathrm{i}t(a^{*2}+a^2)\Big)\exp\left(\mathrm{i}sa^*a\right)\exp\Big(-\mathrm{i}t(a^{*2}+a^2)\Big) \\ &\quad\times\exp\Big(\mathrm{i}s\big(\cosh t\sinh t(a^{*2}+a^2)-(\cosh^2 t+\sinh^2 t)a^*a\big)\Big). \end{aligned}$$

The Weyl quantization yields

$$\begin{aligned} 1 &= \exp\Big(\mathrm{i}t(a^{*2}+a^2)\Big)\exp\left(\mathrm{i}s(a^*a+1/2)\right)\exp\Big(-\mathrm{i}t(a^{*2}+a^2)\Big) \\ &\quad\times\exp\Big(\mathrm{i}s\big(\cosh t\sinh t(a^{*2}+a^2)-(\cosh^2 t+\sinh^2 t)(a^*a+1/2)\big)\Big). \end{aligned}$$

In this computation we used

$$\exp\left(\mathrm{i}t(a^{*2}+a^2)\right) a \exp\left(-\mathrm{i}t(a^{*2}+a^2)\right) = \cosh t a - \sinh t a^*.$$