RETURN TO EQUILIBRIUM

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Conventional wisdom

(1) In a generic situation, a small system interacting with a large reservoir at temperature T goes to equilibrium at the same temperature.
(2) The behavior of a small system interacting with reservoirs at distinct temperatures is much more difficult to desribe than in the case of a reservoir at a fixed temperature.

Rigorous expression
of conventional wisdom

Rigorous theorems proven for nontrivial, explicit and realistic models:

(1) Return to equilibrium for a generic small system interacting with a thermal reservoir.

(2) Absence of normal stationary states for a generic small system interacting with a non-equilibrium reservoir.

Mathematical techniques involved in the study of return to equilibrium.

- **Operator algebras:**
  - KMS states,
  - Standard forms, Liouvilleans;
- **Quantum field theory:**
  - Quasi-free (Araki-Woods) representations of the CCR;
- **Spectral theory:**
  - Fermi Golden Rule, the Feshbach method,
  - The positive commutator (Mourre) method.

Plan of the lecture

(1) **Small system interacting with a bosonic reservoir**

(2) W^***-algebraic background.**

(3) **Fermi golden rule**

SMALL SYSTEM INTERACTING WITH A BOSONIC RESERVOIR

Bosonic Fock space

$$\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)) := \mathop{\oplus}_{n=0}^{\infty} \otimes_{\mathrm{s}}^n L^2(\mathbb{R}^d).$$

Creation/annihilation operators:

$$[a^*(\xi_1), a^*(\xi_2)] = 0, \quad [a(\xi_1), a(\xi_2)] = 0,$$

$$[a^*(\xi_1), a(\xi_2)] = \delta(\xi_1 - \xi_2).$$

$$\int a^*(\xi) f(\xi) \mathrm{d}\xi \Phi := \sqrt{n+1} f \otimes_{\mathrm{s}} \Phi, \qquad \Phi \in \otimes_{\mathrm{s}}^n L^2(\mathbb{R}^d).$$

Vacuum: $\Omega = 1 \in \otimes_{\mathrm{s}}^0 L^2(\mathbb{R}^d) = \mathbb{C}$.

Free Hamiltonian of fotons or phonons:

$$H = \int |\xi| a^*(\xi) a(\xi) \mathrm{d}\xi.$$

Quasi-free representations of the CCR

Radiation density: $\mathbb{R}^d \ni \xi \mapsto \rho(\xi) \in [0,\infty[$. **We look for creation/annihillation operators** $a_{\rho,\mathrm{l}}^*(\xi)/a_{\rho,\mathrm{l}}(\xi)$**, with a quasi-free state of density** ρ **given by a cyclic vector** Ω**:**

$$[a_{\rho,\mathrm{l}}^*(\xi_1), a_{\rho,\mathrm{l}}^*(\xi_2)] = 0, \quad [a_{\rho,\mathrm{l}}(\xi_1), a_{\rho,\mathrm{l}}(\xi_2)] = 0,$$

$$[a_{\rho,\mathrm{l}}(\xi_1), a_{\rho,\mathrm{l}}^*(\xi_2)] = \delta(\xi_1 - \xi_2).$$

$$W_{\rho,\mathrm{l}}(f) := \exp\Big(\frac{\mathrm{i}}{\sqrt{2}}\int (f(\xi)a_{\rho,\mathrm{l}}^*(\xi) + \overline{f}(\xi)a_{\rho,\mathrm{l}}(\xi))\mathrm{d}\xi\Big).$$

$$(\Omega|W_{\rho,\mathrm{l}}(f)\Omega) = \exp\Big(-\frac{1}{4}\int |f(\xi)|^2(1+2\rho(\xi))\mathrm{d}\xi\Big).$$

Araki-Woods representation of CCR

We will write $a_{\mathrm{l}}^*(\xi)/a_{\mathrm{l}}(\xi)$, $a_{\mathrm{r}}(\xi)/a_{\mathrm{r}}^*(\xi)$ for the creation/ annihilation operators corresponding to the left and right $L^2(\mathbb{R}^d)$ resp. acting on the Fock space

$$\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)).$$

Left Araki-Woods creation/annihillation operators are defined as

$$a_{\rho,\mathrm{l}}^*(\xi) := \sqrt{1+\rho(\xi)}a_{\mathrm{l}}^*(\xi) + \sqrt{\rho(\xi)}a_{\mathrm{r}}(\xi),$$
$$a_{\rho,\mathrm{l}}(\xi) := \sqrt{1+\rho(\xi)}a_{\mathrm{l}}(\xi) + \sqrt{\rho(\xi)}a_{\mathrm{r}}^*(\xi).$$

Left Araki-Woods algebra is denoted by $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$ and defined as the W^*-algebra generated by the operators $W_{\rho,\mathrm{l}}(f)$. The vacuum Ω defines a quasi-free state of density ρ.

Commutant of the Araki-Woods algebra

Define an involution ϵ **on** $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ **by** $\epsilon(f_1, \overline{f}_2) := (f_1, \overline{f}_2)$**. Set** $J := \Gamma(\epsilon)$**. Then** J **is the modular involution for the state** $(\Omega| \cdot \Omega)$**.**

Right Araki-Woods creation/annihillation operators:

$$a^*_{\rho,\mathrm{r}}(\xi) := \sqrt{\rho(\xi)}a_{\mathrm{l}}(\xi) + \sqrt{1+\rho(\xi)}a^*_{\mathrm{r}}(\xi),$$
$$a_{\rho,\mathrm{r}}(\xi) := \sqrt{\rho(\xi)}a^*_{\mathrm{l}}(\xi) + \sqrt{1+\rho(\xi)}a_{\mathrm{r}}(\xi).$$

generate the right Araki-Woods algebra denoted by $\mathfrak{M}^{\mathrm{AW}}_{\rho,\mathrm{r}}$**. Note that** $J\mathfrak{M}^{\mathrm{AW}}_{\rho,\mathrm{l}}J = \mathfrak{M}^{\mathrm{AW}}_{\rho,\mathrm{r}}$ **is the commutant of** $\mathfrak{M}^{\mathrm{AW}}_{\rho,\mathrm{l}}$**.**

Dynamics of the quasifree bosons

The Liouvillean of free bosons:

$$L = \int |\xi| a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) \mathrm{d}\xi - \int |\xi| a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi) \mathrm{d}\xi.$$

Note that $JLJ = -L$**.**

$\mathrm{e}^{\mathrm{i}tL} \cdot \mathrm{e}^{-\mathrm{i}tL}$ **defines a dynamics on** $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$**. The state** $(\Omega| \cdot \Omega)$ **is** β**-KMS iff the density is given by the Planck law:**

$$\rho(\xi) = (\mathrm{e}^{\beta|\xi|} - 1)^{-1}.$$

Small quantum system in contact with Bose gas at zero density

Hilbert space of the small quantum system: $\mathcal{K} = \mathbb{C}^n$.
The Hamiltonian of the free system: K.
The free Pauli-Fierz Hamiltonian:

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \int |\xi| a^*(\xi) a(\xi) \mathrm{d}\xi.$$

$$\mathbb{R}^d \ni \xi \mapsto v(\xi) \in B(\mathcal{K})$$

describes the interaction:

$$V := \int v(\xi) \otimes a^*(\xi) \mathrm{d}\xi + \mathrm{hc}$$

The full Pauli-Fierz Hamiltonian: $H := H_{\mathrm{fr}} + \lambda V$. The Pauli-Fierz system at zero density:

$$\left(B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)), \mathrm{e}^{\mathrm{i}tH} \cdot \mathrm{e}^{-\mathrm{i}tH} \right).$$

Small quantum system in contact with Bose gas at density ρ.

The algebra of observables of the composite system:

$$\mathfrak{M}_\rho := B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}} \subset B\left(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))\right).$$

The free Pauli-Fierz semi-Liouvillean at density ρ:

$$L_{\mathrm{fr}}^{\mathrm{semi}} := K \otimes 1 + 1 \otimes \left(\int |\xi| a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) \mathrm{d}\xi - \int |\xi| a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi) \mathrm{d}\xi\right).$$

The interaction:

$$V_\rho := \int v(\xi) \otimes a_{\rho,\mathrm{l}}^*(\xi) \mathrm{d}\xi + \mathrm{hc.}$$

The full Pauli-Fierz semi-Liouvillean at density ρ:

$$L_\rho^{\mathrm{semi}} := L_{\mathrm{fr}}^{\mathrm{semi}} + \lambda V_\rho.$$

The Pauli-Fierz W^*-dynamical system at density ρ:

$$(\mathfrak{M}_\rho, \sigma_\rho), \quad \textbf{where } \sigma_{\rho,t}(A) := \mathrm{e}^{\mathrm{i}tL_\rho^{\mathrm{semi}}} A\, \mathrm{e}^{-\mathrm{i}tL_\rho^{\mathrm{semi}}}.$$

Relationship between the dynamics at zero density and at density ρ.

Set $\rho = 0$.

$$\mathfrak{M}_0 \simeq B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)) \otimes 1.$$

$$L_0^{\mathrm{semi}} \simeq H \otimes 1 - 1 \otimes \int |\xi| a_r(^*(\xi) a_r(\xi) \mathrm{d}\xi.$$

$$\sigma_{0,t}(A \otimes 1) = \mathrm{e}^{\mathrm{i}tH} A \, \mathrm{e}^{-\mathrm{i}tH} \otimes 1.$$

If we formally replace $a_{\mathrm{l}}(\xi)$, $a_{\mathrm{r}}(\xi)$ with $a_{\rho,\mathrm{l}}(\xi)$, $a_{\rho,\mathrm{r}}(\xi)$ (the CCR do not change!) then $\mathfrak{M}_0$, L_0^{semi}, σ_0 transform into $\mathfrak{M}_\rho$, L_ρ^{semi}, σ_ρ. In the case of a finite number of degrees of freedom this can be implemented by a unitary Bogoliubov transformation. $(\mathfrak{M}_\rho, \sigma_\rho)$ can be viewed as a thermodynamical limit of $(\mathfrak{M}_0, \sigma_0)$.

Theorem I: Return equilibrium in the thermal case. Let the reservoir have inverse temperature β. Assume some conditions about the regularity and effectiveness of $v(\xi)$. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$, $(\mathfrak{M}_\rho, \sigma_\rho)$ has a single normal stationary state ω. This state is β-KMS and for any normal state ϕ and $A \in \mathfrak{M}_\rho$, we have $\lim_{|t|\to\infty} \phi(\sigma_{\rho,t}(A)) = \omega(A)$. Jakšić-Pillet, Jakšić-D., Bach-Fröhlich-Sigal, Fröhlich-Merkli

Theorem II: Absence of normal stationary states in the non-equilibrium case. Suppose that the reservoir has parts at distinct temperatures. Assume some conditions about the regularity and effectiveness of $v(\xi)$. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$, $(\mathfrak{M}_\rho, \sigma_\rho)$ has no normal stationary states. Jakšić-D.

Standard representation of $\mathfrak{M}_\rho$.

In order to prove the above theorems we need to go to the standard representation:

$$\pi : \mathfrak{M}_\rho \to B(\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)),$$

$$\pi(A \otimes B) = A \otimes 1 \otimes B,$$

$$J\Phi_1 \otimes \overline{\Phi}_2 \otimes \Psi = \Phi_2 \otimes \overline{\Phi}_1 \otimes \Gamma(\epsilon)\Psi.$$

The free Pauli-Fierz Liouvillean:

$$L_{\mathrm{fr}} := K \otimes 1 \otimes 1 - 1 \otimes \overline{K} \otimes 1$$

$$+1 \otimes 1 \otimes \int \left(|\xi| \left(a_{\mathrm{l}}^*(\xi)a_{\mathrm{l}}(\xi) - a_{\mathrm{l}}^*(\xi)a_{\mathrm{l}}(\xi)\right)\right) \mathrm{d}\xi,$$

$$\pi(V_\rho) = \int v(\xi) \otimes 1 \otimes a_{\rho,\mathrm{l}}^*(\xi)\mathrm{d}\xi + \mathrm{hc},$$

$$J\pi(V_\rho)J = \int 1 \otimes \overline{v}(\xi) \otimes 1 \otimes a_{\rho,\mathrm{r}}^*(\xi)\mathrm{d}\xi + \mathrm{hc}.$$

The full Pauli-Fierz Liouvillean at density ρ:

$$L\rho = L_{\mathrm{fr}} + \lambda\pi(V_\rho) - \lambda J\pi(V_\rho)J.$$

Theorem I': **Let the reservoir have inverse temperature β. Assume some conditions about the regularity and effectiveness of $v(\xi)$. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$, $\dim \mathrm{Ker} L_\rho = 1$ and L_ρ has absolutely continuous spectrum away from 0.**

Theorem II': **Absence of normal stationary states in the non-equilibrium case.** **Suppose that the reservoir has parts at distinct temperatures. Assume some conditions about the regularity and effectiveness of $v(\xi)$. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$, $\dim \mathrm{Ker} L_\rho = 0$.**

Spectrum of Pauli-Fierz Liouvillean

Spectrum of L_{fr} **is** $\mathbb{R}$**.**
Point spectrum of L_{fr} **is** $\mathrm{sp}K - \mathrm{sp}K$**.**
$\Phi_{\mathrm{fr}} := \mathrm{e}^{-\beta K/2} \otimes \Omega$ **is a** β**-KMS vector of** L_{fr}**.**
By Araki-Jakšić-Pillet-D,

$$\mathrm{e}^{-(L_{\mathrm{fr}}+\lambda\pi(V_\rho))\beta/2}\,\Phi_{\mathrm{fr}}$$

is a β**-KMS vector of** L_ρ**. Therefore,**

$$\mathrm{Ker} L_\rho \geq 1.$$

By a rigorous version of the Fermi Golden Rule, if the interaction is sufficiently regular and effective, then there exists $\lambda_0 > 0$ **such that for** $0 < |\lambda| \leq \lambda_0$

$$\mathrm{Ker} L_\rho \leq 1.$$

W^*-ALGEBRAIC BACKGROUND

2 approaches to quantum systems

(1) **C^*-dynamical system** $(\mathfrak{A}, \alpha_t)$**:**
$\mathfrak{A}$ – C^***-algebra,** $t \mapsto \alpha_t \in \mathrm{Aut}(\mathfrak{A})$ – **strongly continuous 1-parameter group.**

(2) **W^*-dynamical system** $(\mathfrak{M}, \sigma_t)$**:**
$\mathfrak{M}$ – W^***-algebra,** $t \mapsto \sigma_t \in \mathrm{Aut}(\mathfrak{M})$ – σ**-weakly continuous 1-parameter group.**

We use the W^*-dynamical approach

The GNS representation

Suppose that ω is a state on $\mathfrak{M}$. Then we have the GNS representation $\pi : \mathfrak{M} \to B(\mathcal{H})$ with $\Omega \in \mathcal{H}$ – a cyclic vector for $\pi(\mathfrak{M})$ such that

$$\omega(A) = (\Omega|\pi(A)\Omega), \quad A \in \mathfrak{M}.$$

If ω is normal, then so is π.

If in addition ω is stationary wrt a W^*-dynamics σ, then we have a distinguished unitary implementation of σ:

$$\pi(\sigma_t(A)) = \mathrm{e}^{\mathrm{i}tL}\,\pi(A)\,\mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathfrak{M},$$

$$L\Omega = 0.$$

Theorem Return to equilibrium in mean.
Suppose that ω is faithful. Then the following statements are equivalent:

(1) ω **is a unique invariant normal state.**

(2) Ω **is a unique eigenvector of L.**

(3) **For any normal state ϕ and $A \in \mathfrak{M}$,**

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \phi(\sigma_s(A))\mathrm{d}s = \omega(A).$$

Theorem Return to equilibrium.
Suppose that ω is faithful and L has absolutely continuous spectrum away from 0. Then for any normal state ϕ and $A \in \mathfrak{M}$,

$$\lim_{t\to\infty} \phi(\sigma_t(A)) = \omega(A).$$

Standard form of a W^*-algebra
Connes, Araki, Haagerup

A W^*-algebra in a standard form is a quadruple $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$, where $\mathcal{H}$ is a Hilbert space, $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is a W^*-algebra, J is an antiunitary involution on $\mathcal{H}$ (that is, J is anti-linear, $J^2 = 1$, $J^* = J$) and $\mathcal{H}^+$ is a self-dual cone in $\mathcal{H}$ such that:

(1) $J\mathfrak{M}J = \mathfrak{M}'$**;**

(2) $JAJ = A^*$ **for** A **in the center of** $\mathfrak{M}$**;**

(3) $J\Psi = \Psi$ **for** $\Psi \in \mathcal{H}^+$**;**

(4) $AJA\mathcal{H}^+ \subset \mathcal{H}^+$ **for** $A \in \mathfrak{M}$**.**

Standard form in a GNS representation

If ω **is a faithful state,**
$(\pi, \mathcal{H}, \Omega)$ – **the corresponding GNS representation,**
J – **the modular conjugation given by the Tomita-Takesaki theory,**
$\mathcal{H}^+ := \{\pi(A)J\pi(A)\Omega \ : \ A \in \mathfrak{M}\}^{\mathrm{cl}}$, **then**

$$(\pi(\mathfrak{M}), \mathcal{H}, J_\Omega, \mathcal{H}^+),$$

is a standard form.

Standard Liouvillean

For every W^*-dynamics σ there exists a unique self-adjoint operator L called the Liouvillean of σ such that

$$\pi(\sigma_t(A)) = \mathrm{e}^{\mathrm{i}tL}\,\pi(A)\,\mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathfrak{M},$$

$$\mathrm{e}^{\mathrm{i}tL}\,\mathcal{H}_+ \subset \mathcal{H}_+.$$

If the W^*-dynamics σ has a faithful invariant normal state ω, then its Liouvilean L coincides with the operator L introduced in the GNS representation.

Normal states
and vectors in the positive cone

Theorem Every normal state ω has a unique standard vector representative, that is a vector $\Omega \in \mathcal{H}^+$ such that

$$\omega(A) = (\Omega|\pi(A)\Omega), \quad A \in \mathfrak{M}.$$

Theorem

(1) $\dim \mathrm{Ker} L = 0$ **iff the W^*-dynamics σ_t has no normal invariant states.**

(2) $\dim \mathrm{Ker} L = 1$ **iff the W^*-dynamics σ_t has a single normal invariant state.**

KMS states

Let σ be a W^*-dynamics and L the corresponding Liouvillean.

A normal state ω is called a β-KMS state iff

$$\omega(AB) = \omega(B\sigma_{\mathrm{i}\beta}(A)),\ \ A, B \in \mathfrak{M},\ \ B \quad \sigma - \textbf{analytic}.$$

β-KMS states are stationary.

A vector Ω is called a β-KMS vector iff $\Omega \in \mathcal{H}^+$ and

$$\mathrm{e}^{-\beta L/2} A\Omega = JA^*\Omega, \quad A \in \mathfrak{M}.$$

β-KMS vectors belong to $\mathrm{Ker}L$. They are standard vector representatives of β-KMS states.

Theorem **Let** $(\mathfrak{M}, \sigma_{\mathrm{fr}})$ **be a** W^***-dynamical system with the Liouvillean** L_{fr}**. Let** Ω_{fr} **be a** β**-KMS vector for** σ_{fr}**. Let** V **be a self-adjoint operator affiliated to** $\mathfrak{M}$ **satisfying some technical assumptions.**

Then (1) There exists a perturbed dynamics σ **such that**

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_t(A) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\mathrm{fr},t}(A) + \mathrm{i}[V, \sigma_{\mathrm{fr},t}(A)].$$

(2) The Liouvillean of σ **equals**

$$L = L_{\mathrm{fr}} + \pi(V) - J\pi(V)J.$$

(3) $\mathrm{e}^{-\beta\pi(V)/2}\Omega$ **is a** β**-KMS vector for** σ**.**

Araki–bounded V**; Jakšić, Pillet and D.–unbounded** V**.**

Example: type I factor

W^***-algebra:** $B(\mathcal{K})$;

Standard Hilbert space: $\mathcal{K}\otimes\overline{\mathcal{K}} = B^2(\mathcal{K})$;

Standard representation: $\pi(A) = A \otimes 1_{\mathcal{K}} \simeq A\,\cdot$;

Standard positive cone: $B^2_+(\mathcal{K})$;

State: $\omega(A) = \mathrm{Tr}\rho A$, $\rho \in B^1_+(\mathcal{K}),\ \mathrm{Tr}\rho = 1$;

Its vector representative: $\Omega = \rho^{1/2} \in B^2_+(\mathcal{K})$;

W^***-dynamics:** $\sigma_t(A) = \mathrm{e}^{\mathrm{i}tK}\,A\,\mathrm{e}^{-\mathrm{i}tK}$;

Its Liouvillean: $L = K \otimes 1 - 1 \otimes K \simeq [K, \cdot]$;

β**-KMS state:** $\omega_\beta(A) = (\mathrm{Tr}\,\mathrm{e}^{-\beta K})^{-1}\mathrm{Tr}\,\mathrm{e}^{\beta K}\,A$;

β**-KMS vector:** $(\mathrm{Tr}\,\mathrm{e}^{-\beta K})^{-1/2}\,\mathrm{e}^{\beta K/2}$.

$B^2(\mathcal{K})$–**Hilbert-Schmidt operators,** $B^1(\mathcal{K})$–**trace class operators**

RIGOROUS FERMI GOLDEN RULE

2nd order perturbation theory for isolated eigenvalues

Unperturbed operator: L_0. The spectral projection onto an isolated part of spectrum of L_0 consisting of a finite number of eigenvalues is denoted P. We define

$$L_0 P =: E = \sum_{e \in \mathrm{sp} E} e 1_e(E).$$

Perturbation: Q. We assume that there is no 1st order shift of eigenvalues:

$$PQP = 0.$$

Perturbed operator: $L_\lambda := L_0 + \lambda Q$.

Theorem. **For small λ, in a neighborhood of $\mathrm{sp}E$ we have**

$$\mathrm{sp}L_\lambda = \mathrm{sp}(E + \lambda^2\Gamma) + o(\lambda^2),$$

where Γ is the Level Shift Operator

$$\Gamma = \sum_{e\in\mathrm{sp}E} 1_e(E)Q(e - L_0)^{-1}Q1_e(E).$$

Multiplicities of eigenvalues of $E + \lambda^2\Gamma$ coincide with multiplicities of corresponding clusters of eigenvalues of L_λ.

2nd order perturbation theory for isolated eigenvalues

Let L_0, P, Q and L_λ be as above, except that the spectrum of E can be embedded in the rest of spectrum of L_0. Introduce the (upper) Level Shift Operator:

$$\Gamma = \sum_{e \in \mathrm{sp} E} \lim_{\epsilon \downarrow 0} 1_e(E) Q (e + \mathrm{i}\epsilon - L_0)^{-1} Q 1_e(E).$$

Clearly, Γ satisfies

$$\Gamma E = E \Gamma, \quad \frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*) \leq 0.$$

Fermi Golden Rule:
$\frac{1}{2}(\Gamma + \Gamma^*)$ describes energy shift,
$\frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*)$ describes the decay rates.

Theorem. **There exists** $\lambda_0 > 0$ **such that for** $0 < |\lambda| < \lambda_0$

$$\dim 1_{\mathrm{p}}(L_\lambda) \leq \dim \mathrm{Ker} \frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*).$$

Proofs (for Pauli-Fierz Liouvilleans) involve

1) analytic deformation method, Jakšić-Pillet;

2) positive commutator method, Jakšić-D; Merkli;

3) "renormalization group" Bach-Fröhlich-Sigal.

2nd order perturbation theory applied to Pauli-Fierz Liouvilleans

Unperturbed operator: L_0.
Projection: $1_{\mathrm{p}}(L_{\mathrm{fr}})$, **which coincides with the projection onto** $\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Omega$.
Perturbation: $\pi(V_\rho) - J\pi(V_\rho)J$.
Perturbed operator: L_ρ.

Theorem. If the interaction is sufficiently regular and effective, then

$\dim \mathrm{Ker} \frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*) \leq 1$ **in thermal case,**

$\dim \mathrm{Ker} \frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*) = 0$ **in nonthermal case.**