Clifford algebras and fermions

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Contents

These lecture notes are devoted to various algebraic constructions, especially those useful in Quantum Physics.

1 Introduction—examples of algebras

Let $\mathbb{K}$ be a field. In practice, we will restrict ourselves to $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

Let us describe several algebras that appear in quantum physics. All of this section will contain the unit and will be over $\mathbb{K}$. The unit will be denoted $\mathbb{1}$, and for any element A of the algebra will satisfy

$$\mathbb{1}A = A\mathbb{1} = A. \tag{1.1}$$

Example 1.1 *Algebra generated by $x_1, \dots, x_n$ statisfying*

$$x_i x_j = x_j x_i. \tag{1.2}$$

It is the algebra of polynomials in variables $x_1, \dots, x_n$. The standard notation: $\mathbb{K}[x_1, \dots, x_n]$. It has the basis

$$x_1^{m_1} \cdots x_n^{m_n}, \quad m_1, \dots, m_n \in \mathbb{N}_0. \tag{1.3}$$

The differentiation wrt x_i, denoted ∂_{x_i} is the unique linear operator satisfying

$$\partial_{x_i} FG = (\partial_{x_i} F)G + F\partial_{x_i} G, \tag{1.4}$$

$$\partial_{x_i} x_j = \delta_{ij}. \tag{1.5}$$

We have

$$\partial_{x_i} x_1^{m_1} \cdots x_i^{m_i} \cdots x_n^{m_n} = m_i x_1^{m_1} \cdots x_i^{m_i - 1} \cdots x_n^{m_n}. \tag{1.6}$$

Example 1.2 *Algebra over $\mathbb{C}$ generated by $x^1, \dots, x^n, p_1, \dots, p_n$ satisfying*

$$x^i x_j - x^j x^i = p_i p_j - p_j p_i = 0, \tag{1.7}$$

$$x^i p_j - p_j x^i = \mathrm{i}\delta^i_j \mathbb{1} \tag{1.8}$$

It goes sometimes under the name of the Weyl algebra (but this is ambiguous). Other possible names are the Heisenberg algebra or the CCR algebra.

Its basis is

$$(x^1)^{m_1} \cdots (x^n)^{m_n} p_1^{k_1} \cdots p_n^{k_n}, \quad m_i, k_j \in \mathbb{N}_0. \tag{1.9}$$

Standard representation on $\mathbb{C}[x^1, \dots, x^n]$:

$$\hat{x}^i := \text{ multiplication by } x^i, \tag{1.10}$$

$$\hat{p}_i := \frac{1}{\mathrm{i}} \partial_{x^i}. \tag{1.11}$$

Example 1.3 *Algebra generated by $\theta_1, \dots \theta_n$ satisfying*

$$\theta_i\theta_j + \theta_j\theta_i = 0. \tag{1.12}$$

It is called the Grassman algebra or the algebra of polynomials in anticommuting variables. Another name is the exterior algebra. Sometimes it is denoted $\mathbb{K}[\theta_1, \dots, \theta_n]$. Its basis is

$$\theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n}, \qquad \epsilon_i \in \{0, 1\}. \tag{1.13}$$

Elements which are linear compbinations of (1.13) with $\epsilon_1 + \cdots + \epsilon_n$ even/odd are called even/odd. Elements which are either even or odd are called pure. If F is a pure element, then $\operatorname{sgn}(F) := 1$ if F is even and $\operatorname{sgn}(F) := -1$ if F is odd.

We have two kinds of differentiation: the left differentiation $\overleftarrow{\partial}_{\theta_i} = \overleftarrow{\partial}^i$ and the right differentiation $\overrightarrow{\partial}_{\theta_i} = \overrightarrow{\partial}^i$ They are defined by

$$\overleftarrow{\partial}^j \theta_i = \overleftarrow{\partial}^j \theta_i = \delta_i^j, \tag{1.14}$$

$$\overrightarrow{\partial}^j FG = (\overrightarrow{\partial}^j F)G + \operatorname{sgn}(F)F(\overrightarrow{\partial}^j G), \tag{1.15}$$

$$\overleftarrow{\partial}^j FG = \operatorname{sgn}(G)(\overleftarrow{\partial}^j F)G + F(\overleftarrow{\partial}^j G). \tag{1.16}$$

Thus after acting with $\overrightarrow{\partial}^j$, resp. $\overleftarrow{\partial}^j$ on $\theta_1^{\epsilon_1} \cdots \theta_n^{\epsilon_n}$ we obtain 0 if θ_j is not present and the same expression with θ_j omitted multiplied by $(-1)^{\epsilon_1 + \cdots + \epsilon_{j-1}}$, resp. $(-1)^{\epsilon_{j+1} + \cdots + \epsilon_n}$.

We will treat $\overrightarrow{\partial}$ as the standard differentiation, denoting it often by ∂.

Example 1.4 *The algebra generated by $\alpha_1 \dots \alpha_n$ satisfying*

$$\alpha_i\alpha_j + \alpha_i\alpha_j = 2\delta_{ij}. \tag{1.17}$$

It is called the Clifford algebra. For $\mathbb{K} = \mathbb{R}$ it will be denoted $\mathrm{Cl}^+(n\mathrm{Cl}^+(\mathbb{R}^n)$. For $\mathbb{K} = \mathbb{C}$ it is sometimes called the CAR algebra.

Here is a basis of $\mathrm{Cl}^+(n)$

$$\alpha_1^{\epsilon_1} \cdots \alpha_n^{\epsilon_n}, \qquad \epsilon_i \in \{0, 1\}. \tag{1.18}$$

We will discuss this algebra further in more detail.

Example 1.5 *Algebra generated by $x_1, \dots, x_n$ (with no relations).*

It is called the free algebra with the generators $x_1, \dots, x_n$. Its basis are the expressions

$$x_{i_1} \cdots x_{i_k}, \quad k = 0, 1, \dots, \quad i_1, \dots i_k \in \{1, \dots, k\}. \tag{1.19}$$

The product is just the concatenation of these expressions.

2 Quaternions

2.1 Definitions

The algebra over $\mathbb{R}$ with a basis $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ satisfying the relations

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \quad \mathrm{ij} = \mathrm{k}, \quad \mathrm{jk} = \mathrm{i}, \quad \mathrm{ki} = \mathrm{j}, \tag{2.1}$$

is called the algebra *quaternions* and denoted $\mathbb{H}$. Note that the following identities follow from (2.1):

$$\mathrm{ji} = -\mathrm{k}, \quad \mathrm{kj} = -\mathrm{i}, \quad \mathrm{ik} = -\mathrm{j}, \tag{2.2}$$

$\mathbb{H}$ is endowed with $*$ acting as

$$1^* = 1, \quad \mathrm{i}^* = -\mathrm{i}, \quad \mathrm{j}^* = -\mathrm{j}, \quad \mathrm{k}^* = -\mathrm{k}.$$

$*$ is an *involution*, that is $x^{**} = x$, $(xy)^* = y^*x^*$, $x, y \in \mathbb{H}$. $x \in \mathbb{H}$ is called Hermitian, resp. anti-Hermitian if $x = x^*$, resp. $x = -x^*$.

For $x \in \mathbb{H}$ we set

$$\mathrm{Re} x := \frac{1}{2}(x + x^*), \quad |x| := \sqrt{x^* x}.$$

(Note that, że $x^*x = xx^*$ is always positive real).

If $x = x_1 + x_\mathrm{i}\mathrm{i} + x_\mathrm{j}\mathrm{j} + x_\mathrm{k}\mathrm{k}$, where $x_1, x_\mathrm{i}, x_\mathrm{j}, x_\mathrm{k} \in \mathbb{R}$, then

$$\mathrm{Re} x = x_1, \qquad |x| = \sqrt{x_1^2 + x_\mathrm{i}^2 + x_\mathrm{j}^2 + x_\mathrm{k}^2}.$$

Note that $|\cdot|$ is a norm on $\mathbb{H}$. If $x, y \in \mathbb{H}$, then $|xy| = |x||y|$.

$\mathbb{H}$ is equipped with the *quaternionic scalar product* x^*y and the *real scalar product*

$$\langle x|y\rangle := \mathrm{Re} x^* y = x_1 y_1 + x_\mathrm{i} y_\mathrm{i} + x_\mathrm{j} y_j + x_\mathrm{k} y_\mathrm{k}, \quad x, y \in \mathbb{H}.$$

All non-zero elements of $\mathbb{H}$ are invertible (just as in a field). Such algebras are called *division algebras.*

An element $x \in \mathbb{H}$ is called *unitary* if $x^*x = 1$. Equivalently, x is unitary if $|x| = 1$. Every non-zero quaternion can be uniquely written as $x = |x|u$, where u is unitary. Every unitary u can be written as

$$u = \cos\theta + y \sin\theta = \exp(\theta y), \tag{2.3}$$

where $y^2 = -1$. From this it is easy to show that unitary quaternions form a group isomorphic to $SU(2)$, see also (2.9).

Isomorphisms of $\mathbb{H}$ preserve the scalar product $\langle\cdot|\cdot\rangle$. They also preserve the 3-dimensional subspace of anti-Hermitian quaternions. This group is isomorphic to $SO(3)$. Every isomorphism of $\mathbb{H}$ has the form

$$\mathbb{H} \ni x \mapsto uxu^{-1} \in \mathbb{H}, \tag{2.4}$$

where u is a unitary anti-Hermitian quaternion.

2.2 Embedding complex numbers in quaternions

It is easy to see that there exists a unique continuous injective homomorphism $\mathbb{R} \to \mathbb{H}$. Its image is the center of the algebra $\mathbb{H}$, which can be identified with $\mathbb{R}$.

There exist many continuous injective homomorphisms $\mathbb{C} \to \mathbb{H}$. To fix it one has to fix an element of $\mathbb{H}$ whose square is -1. Let us call it i.

Quaternions can be defined as an algebra over $\mathbb{C}$ spanned by 1, j, satisfying the relations

$$z\mathrm{j} = \mathrm{j}\overline{z}. \tag{2.5}$$

This fixes a homomorphism $\mathbb{C} \to \mathbb{H}$. $\mathbb{H}$ is then a vector space over $\mathbb{C}$ of dimension 2. We have

$$x = x_1 + x_\mathrm{i}\mathrm{i} + x_\mathrm{j}\mathrm{j} + x_\mathrm{k}\mathrm{k} = (x_1 + x_\mathrm{i}\mathrm{i})1 + (x_\mathrm{j} + x_\mathrm{k}\mathrm{i})\mathrm{j}. \tag{2.6}$$

The map

$$\mathbb{H} \ni x \mapsto \frac{1}{2}(x - \mathrm{i}x\mathrm{i}) \in \mathbb{C} \tag{2.7}$$

is a projection.

Set

$$(x|y) := \frac{1}{2}(yx^* - \mathrm{i}yx^*\mathrm{i}) \tag{2.8}$$

By (2.7), the values of this scalar product are in $\mathbb{C}$. (2.8) is sesquilinear, because

$$\begin{aligned} (x|zy) &= \frac{1}{2}(zyx^* - \mathrm{i}zyx^*\mathrm{i}) = z(x|y), \\ (zx|y) &= \frac{1}{2}(yx^*\overline{z} - \mathrm{i}yx^*\overline{z}\mathrm{i}) = (x|y)\overline{z}, \quad z \in \mathbb{C}, \end{aligned}$$

Thus (2.8) is a complex sesquilinear scalar product on $\mathbb{H}$, so that $\mathbb{H}$ becomes a 2-dimensional complex Hilbert space. $1, \mathrm{j}$ is an example of an orthonormal basis in $\mathbb{H}$ wrt (2.8).

2.3 Matrix representation of quaternions

Quaternions can be represented by Pauli matrices multiplied by i:

$$\pi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi(\mathrm{i}) = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \pi(\mathrm{j}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \pi(\mathrm{k}) = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}. \tag{2.9}$$

This yields a representation of quaternions acting on the Hilbert space $\mathbb{C}^2$

$$\pi : \mathbb{H} \to B(\mathbb{C}^2). \tag{2.10}$$

In this representation

$$\pi(x^*) = \pi(x)^*, \quad |x|^2 = \det \pi(x). \tag{2.11}$$

$$\pi(\mathbb{H}) = \{\lambda U \ : \ U \in U(2), \quad \lambda \in [0, \infty[\}.$$

Another useful relation within this representation is

$$\pi(\mathbb{H}) = \{A \in B(\mathbb{C}^2) \ : \ A = \pi(\mathrm{j})\overline{A}\pi(\mathrm{j})^{-1}\}, \tag{2.12}$$

where $\overline{A}$ denotes the usual complex conjugation of the matrix A. Indeed,

$$\overline{\pi(1)} = \pi(1), \quad \overline{\pi(\mathrm{i})} = -\pi(\mathrm{i}), \quad \overline{\pi(\mathrm{j})} = \pi(\mathrm{j}), \quad \overline{\pi(\mathrm{k})} = -\pi(\mathrm{k}). \tag{2.13}$$

Replacing (2.10) by $W\pi(\cdot)W^{-1}$ for some invertible W, we replace $\pi(\mathrm{j})$ by $R := W\pi(\mathrm{j})\overline{W}^{-1}$. Note that

$$R\overline{R} = -1\!\!1. \tag{2.14}$$

3 Algebras

3.1 Definitions

Let $\mathbb{K}$ be a field. Let $\mathfrak{A}$ be a vector space over $\mathbb{K}$. We say that $\mathfrak{A}$ is an *algebra* over $\mathbb{K}$ if it is equipped with an operation

$$\mathfrak{A} \times \mathfrak{A} \ni (A, B) \mapsto AB \in \mathfrak{A}$$

satisfying

$$\begin{aligned} A(B + C) = AB + AC, &\quad (B + C)A = BA + CA, \\ (\alpha\beta)(AB) = (\alpha A)(\beta B), &\quad A, B, C \in \mathfrak{A}, \quad \alpha, \beta \in \mathbb{K}. \end{aligned} \tag{3.1}$$

If in addition

$$A(BC) = (AB)C,$$

we say that it is an *associative algebra*. (In practice by an algebra we will usually mean an associative algebra).

We say that $\mathfrak{A}$ is *commutative* if $A, B \in \mathfrak{A}$ implies $AB = BA$.

The *center* of an algebra $\mathfrak{A}$ equals

$$\mathfrak{Z}(\mathfrak{A}) = \{A \in \mathfrak{A} \ : \ AB = BA, \ B \in \mathfrak{A}\}.$$

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras. A map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called a *homomorphism* if it is linear and preserves the multiplication, ie.

(1) $\phi(\lambda A) = \lambda\phi(A)$;

(2) $\phi(A + B) = \phi(A) + \phi(B)$;

(3) $\phi(AB) = \phi(A)\phi(B)$.

The set of all automorphisms of $\mathfrak{A}$ is denoted $\mathrm{Aut}(\mathfrak{A})$.

We say that $1\!\!1 \in \mathfrak{A}$ is a unit if

$$1\!\!1 A = A 1\!\!1 = A, \qquad A \in \mathfrak{A}. \tag{3.2}$$

An algebra is called *unital* if it possesses a unit.

3.2 Subalgebras

Fix an algebra $\mathfrak{A}$. $\mathfrak{B} \subset \mathfrak{A}$ is called a *subalgebra* of $\mathfrak{A}$ if it is a vector subspace of $\mathfrak{A}$ and $A, B \in \mathfrak{B} \Rightarrow AB \in \mathfrak{B}$. Obviously, a subalgebra is an algebra.

If a family $\mathfrak{B}_\alpha \subset \mathfrak{A}$ consists of subalgebras, then $\cap_\alpha \mathfrak{B}_\alpha$ is also a subalgebra. Therefore, for any subset $\mathfrak{B} \subset \mathfrak{A}$ there exists the smallest subalgebra containing $\mathfrak{B}$. It is denoted $\mathrm{Alg}(\mathfrak{B})$ and called the *subalgebra generated by* $\mathfrak{B}$.

Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. Clearly, the set of linear transformations in $\mathcal{V}$, denoted $L(\mathcal{V})$, is an (associative) algebra.

Subalgebras of $L(\mathcal{V})$ are called *concrete algebras.*

A homomorphism of an algebra $\mathfrak{A}$ into $L(\mathcal{V})$ is called a representation of $\mathfrak{A}$ on $\mathcal{V}$.

3.3 $*$-algebras

We say that an algebra $\mathfrak{A}$ over $\mathbb{C}$ (more rarely over $\mathbb{R}$) is a *$*$-algebra* if it is equipped with an antilinear map $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$ such that $(AB)^* = B^*A^*$, $A^{**} = A$ and $A \neq 0$ implies $A^*A \neq 0$.

If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ equipped with the Hermitian conjugation is a $*$-algebra

If $\mathfrak{A}$, $\mathfrak{B}$ are $*$-algebras, then a homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ satisfying $\pi(A^*) = \pi(A)^*$ is called a *$*$-homomorphism.*

3.4 Ideals

$\mathfrak{B}$ is an *ideal* of an algebra $\mathfrak{A}$, if it is a linear subspace of $\mathfrak{A}$ and $A \in \mathfrak{A}$, $B \in \mathfrak{B} \Rightarrow AB, BA \in \mathfrak{B}$.

We say that an ideal $\mathfrak{B}$ is *proper* if $\mathfrak{B} \neq \mathfrak{A}$. We say that an ideal $\mathfrak{B}$ is *nontrivial* if $\mathfrak{B} \neq \mathfrak{A}$ and $\mathfrak{B} \neq \{0\}$.

Theorem 3.1 *The kernel of a homomorphism is an ideal. If $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{B}$ has a natural structure of an algebra. The map*

$$\mathfrak{A} \ni A \mapsto A + \mathfrak{B} \in \mathfrak{A}/\mathfrak{B}$$

is a surjective homomorphism whose kernel is $\mathfrak{B}$. If $\mathfrak{A} \to \mathfrak{C}$ is a different surjective homorphism whose kernel is also equal $\mathfrak{B}$, then $\mathfrak{C} \simeq \mathfrak{A}/\mathfrak{B}$.

Saying that

$$\mathfrak{B} \xrightarrow{\phi} \mathfrak{A} \xrightarrow{\psi} \mathfrak{H}$$

is an exact sequence we mean that $\mathrm{Ker}\psi = \mathrm{Ran}\phi$.

In particular,

$$0 \to \mathfrak{B} \xrightarrow{\phi} \mathfrak{A} \xrightarrow{\psi} \mathfrak{H} \to 0 \tag{3.3}$$

means that ϕ is injective, ψ is surjective and $\mathrm{Ker}\psi = \mathrm{Ran}\phi$. Then ψ generates an isomorphism of $\mathfrak{A}/\phi(\mathfrak{B})$ with $\mathfrak{H}$. (3.3) is called then *a short exact sequence* We say that $\mathfrak{A}$ is an *extension of $\mathfrak{B}$ by $\mathfrak{H}$.*

Theorem 3.2 (1) *If $\mathfrak{H}$, $\mathfrak{B}$ are ideals, then so is $\mathfrak{H} + \mathfrak{B}$ and $\mathfrak{H} \cap \mathfrak{B} = \mathfrak{H} \cdot \mathfrak{B}$.*

(2) *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a surjective homorphism between algebras, then $\mathfrak{C} \mapsto \phi(\mathfrak{C})$ defines a bijection between ideals of $\mathfrak{A}$ containing* $\mathrm{Ker}\phi$ *and ideals of $\mathfrak{B}$.*

3.5 Quaternionic vector spaces

We say that $(\mathcal{V}, +, 0)$ is a *quaternionic vector space* if it is an abelian group equipped with the operations

$$\mathbb{H} \times \mathcal{V} \ni (x, v) \mapsto xv \in \mathcal{V}, \quad \mathcal{V} \times \mathbb{H} \ni (v, x) \mapsto vx \in \mathcal{V},$$

such that

$$(x + y)v = xv + yv, \quad (xy)v = x(yv), \quad x, y \in \mathbb{H}, \quad v \in \mathcal{V}.$$

$$v(x + y) = vx + vy, \quad v(xy) = (vx)y, \quad x, y \in \mathbb{H}, \quad v \in \mathcal{V}.$$

For example, $\mathbb{H}^n$ are quaternionic vector spaces. Quaternionic vector spaces isomorphic to $\mathbb{H}^n$ are said to be of *quaternionic dimension n*.

Transformations $\mathbb{H}$-linear from the left/right on a quaternionic vector space have an obvious definition. The set of $\mathbb{H}$-linear transformations from the right from $\mathcal{V}$ to $\mathcal{W}$ is denoted $L(\mathcal{V}, \mathcal{W})$. As usual, $L(\mathcal{V}) := L(\mathcal{V}, \mathcal{V})$.

Elements of $L(\mathbb{H}^n, \mathbb{H}^m)$ can be obviously represented with matrices $m \times n$ of quaternionic elements.

If we fix the embedding (2.7), then quaternionic vector spaces can be reinterpreted as complex vector spaces, and quaternionic Hilbert spaces as complex Hilbert spaces. If $\mathcal{V}$ is a quaternionic vector space, then $\mathcal{V}_{\mathbb{C}}$ denotes $\mathcal{V}$ understood as a complex space. It is called a *complex form of the space $\mathcal{V}$*.

3.6 Real and complex simple algebras

An algebra over $\mathbb{K}$ that does not contain nontrivial ideals and is different from $\mathbb{K}$ with the zero product is called a *simple algebra*.

It is well known by the Wederburn Theorem that one can classify all finite dimensional algebras over $\mathbb{C}$ and $\mathbb{R}$. The complex case is especially easy:

Theorem 3.3 *Let $\mathfrak{A}$ be a complex finite dimensional simple algebra. Then there exists $n \in \mathbb{N}$ such that $\mathfrak{A}$ is isomorphic to $L(\mathbb{C}^n)$.*

The corresponding real classification is slightly more complicated:

Theorem 3.4 *Let $\mathfrak{A}$ be a real finite dimensional simple algebra. Then thgere exists $n \in \mathbb{N}$ such that $\mathfrak{A}$ is isomorphic to $L(\mathbb{C}^n)$, $L(\mathbb{R}^n)$ or $L(\mathbb{H}^n)$.*

Note that $L(\mathbb{R}^n)$ can be embedded in $L(\mathbb{C}^n)$:

$$L(\mathbb{R}^n) = \{A \in L(\mathbb{C}^n) \ : \ A = \overline{A}\}.$$

$L(\mathbb{H}^n)$ can be embedded in $L^2(\mathbb{C}^2 \otimes \mathbb{C}^n)$:

$$L(\mathbb{H}^n) = \{A \in L(\mathbb{C}^2 \otimes \mathbb{C}^n) \ : \ RA = \overline{A}R\},$$

where $R = \pi(\mathrm{j}) \otimes 1\!\!1$.

3.7 Algebras generated by relations

Suppose that $\{e_i \;:\; i \in I\}$ is a set. It is obvious what is the unital algebra generated by $\{e_i\}_{i\in I}$ is a set. Let us denote it $\mathrm{Free}\{e_i \;:\; i \in I\}$. Suppose that $R \subset \mathrm{Free}\{e_i \;:\; i \in I\}$. Let $\mathrm{Ideal}(R)$ be the ideal generated by R. Then $\mathrm{Free}\{e_i \;:\; i \in I\}/\mathrm{Ideal}(R)$ is called the *algebra generated by* $\{e_i \;:\; i \in I\}$ *with relations* R. We had a few examples oof this construction in the introduction.

4 Second quantization

In this chapter we describe the terminology and notation of multilinear algebra. We will concentrate on the infinite dimensional case, where it is often natural to use the structure of Hilbert spaces. We will introduce Fock spaces and various classes of operators acting on them. In quantum physics the passage from a dynamics on one-particle spaces to a dynamics on Fock spaces is often called *second quantization* – hence the name of the chapter.

We will consider two setups: that of vector spaces and that of Hilbert spaces. If $\mathcal{X}, \mathcal{Y}$ are vector spaces, then $L(\mathcal{X}, \mathcal{Y})$ will denote the set of linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces, then $B(\mathcal{X}, \mathcal{Y})$ will denote the set of bounded operators fro $\mathcal{X}$ to $\mathcal{Y}$.

4.1 Vector and Hilbert spaces

Let $\mathcal{V}$ be a vector space. A set $\{e_i \;:\; i \in I\} \subset \mathcal{V}$ is called linearly independent if for any finite subset $\{e_{i_1}, \dots, e_{i_n}\} \subset \{e_i \;:\; i \in I\}$

$$c_1 e_{i_1} + \cdots + c_n e_{i_n} = 0 \quad \Rightarrow \quad c_1 = \cdots = c_n = 0. \tag{4.1}$$

$\{e_i \;:\; i \in I\}$ is a Hamel basis (or simply a basis) of $\mathcal{V}$ if it is a maximal linearly independent set. It means that it is linearly independent and if we add any $v \in \mathcal{V}$ to $\{e_i \;:\; i \in I\} \subset \mathcal{V}$ then it is not linearly independent any more. Note that every $v \in \mathcal{V}$ can be written as a finite linear combination $v = \sum_{i \in I} \lambda_i e_i$ in a unique way.

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ or $\mathbb{R}$ equipped with a scalar product $(v|w)$ (positive, nondegenerate, sesquilinear form). It defines a metric on $\mathcal{V}$ by

$$\|v - w\| := \sqrt{(v - w|v - w)}. \tag{4.2}$$

We say that $\mathcal{V}, (\cdot|\cdot)$ is a Hilbert space if $\mathcal{V}$ is complete.

If $\mathcal{V}, (\cdot|\cdot)$ is not necessarily complete, then we can always complete it, that is find a larger complete space $\mathcal{V}^{\mathrm{cpl}}, (\cdot|\cdot)$ in which $\mathcal{V}$ is embedded as a dense subspace. $\mathcal{V}^{\mathrm{cpl}}$ is uniquely defined and is called the completion of $\mathcal{V}$.

For instance, if we take $C_{\mathrm{c}}(\mathbb{R})$, $C_{\mathrm{c}}^{\infty}(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$ with the usual scalar product $(f|g) = \int \overline{f(x)} g(x) \mathrm{d}x$, then its completion is $L^2(\mathbb{R})$.

If $\mathcal{V}$ is a Hilbert space, then $\{e_i \;:\; i \in I\}$ is called an orthonormal basis (o.n.b.) if it is a maximal orthonormal set. Note that every $v \in \mathcal{V}$ can be written as a linear combination $v = \sum_{i \in I} \lambda_i e_i$, where $\sum_{i \in I} |\lambda_i|^2 < \infty$, in a unique way

Note that in a finite dimensional Hilbert space every orthonormal basis is a basis. This is not true in infinite dimensional Hilbert spaces.

4.2 Direct sum

Let $(\mathcal{V}_i)_{i\in I}$ be a family of vector spaces. The *algebraic direct sum of* $\mathcal{V}_i$ will be denoted

$$\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i, \tag{4.3}$$

It consists of sequences $(v_i)_{i\in I}$, which are zero for all but a finite number of elements.

If $(\mathcal{V}_i)_{i\in I}$ is a family of Hilbert spaces, then $\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i$ has a natural scalar product.

$$\Big((y_i)_{i\in I}\Big|(v_i)_{i\in I}\Big) = \sum_{i\in I}(y_i|v_i). \tag{4.4}$$

The *direct sum of* $\mathcal{V}_i$ *in the sense of Hilbert spaces* is defined as

$$\underset{i\in I}{\oplus} \mathcal{V}_i := \Big(\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i\Big)^{\mathrm{cpl}}.$$

If I is finite, then $\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i = \underset{i\in I}{\oplus} \mathcal{V}_i$

Let $(\mathcal{V}_i)$, $(\mathcal{W}_i)$, $i \in I$, be families of vector spaces. If $a_i \in L(\mathcal{V}_i, \mathcal{W}_i)$, $i \in I$, then their *direct sum* is denoted $\underset{i\in I}{\oplus} a_i$ and belongs to $L\Big(\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i, \overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{W}_i\Big)$. It is defined as

$$\Big(\underset{i\in I}{\oplus} a_i\Big)(v_i)_{i\in I} = (a_i v_i)_{i\in I} \tag{4.5}$$

Let $\mathcal{V}_i$, $\mathcal{W}_i$, $i \in I$ be families of Hilbert spaces, and $a_i \in B(\mathcal{V}_i, \mathcal{W}_i)$ with $\sup_{i\in I} \|a_i\| < \infty$. Then the operator $\underset{i\in I}{\oplus} a_i$ is bounded. Its extension in $B\Big(\underset{i\in I}{\oplus} \mathcal{V}_i, \underset{i\in I}{\oplus} \mathcal{W}_i\Big)$ will be denoted by the same symbol.

4.3 Tensor product

Let $\mathcal{V}, \mathcal{W}$ be vector spaces. The *algebraic tensor product of* $\mathcal{V}$ *and* $\mathcal{W}$ will be denoted $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$. Here is one of its definitions

Let $\mathcal{Z}$ be the space of finite linear combinations of vectors (v, w), $v \in \mathcal{V}$, $w \in \mathcal{W}$. In $\mathcal{Z}$ we define the subspace $\mathcal{Z}_0$ spanned by

$$\begin{aligned} (\lambda v, w) - \lambda(v, w), &\qquad (v, \lambda w) - \lambda(v, w), \\ (v_1 + v_2, w) - (v_1, w) - (v_2, w), &\qquad (v, w_1 + w_2) - (v, w_1) - (v, w_2). \end{aligned}$$

We set $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W} := \mathcal{Z}/\mathcal{Z}_0$. If $v \in \mathcal{V}$, $w \in \mathcal{W}$, we define $v \otimes w := (v, w) + \mathcal{Z}_0$.

Remark 4.1 *Note that* (v, w) *above is just a symbol and not an element of* $\mathcal{V} \oplus \mathcal{W}$. *Elements of the space* $\mathcal{Z}$ *have the form*

$$\sum_{j=1}^{n} \lambda_n(v_n, w_n). \tag{4.6}$$

In particular, in general

$$(v_1, w_1) + (v_2, w_2) \not\sim (v_1 + v_2, w_1 + w_2), \tag{4.7}$$
$$\lambda(v, w) \not\sim (\lambda v, \lambda w). \tag{4.8}$$

$\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$ is a vector space and $\otimes$ is an operation satisfying

$$(\lambda v) \otimes w = \lambda v \otimes w, \qquad v \otimes (\lambda w) = \lambda v \otimes w,$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \qquad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2.$$

Vectors of the form $v \otimes w$ are called *simple tensors.* Not all elements of $\mathcal{V} \otimes \mathcal{W}$ are simple tensors, but they span $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$.

If $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are bases of $\mathcal{V}$, resp. $\mathcal{W}$, then $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ is a basis of $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$,
Note that we can identify

$$\overset{\infty}{\underset{n=0}{\overset{\mathrm{al}}{\oplus}}} \mathcal{V}^{\overset{\mathrm{al}}{\otimes} n} \simeq \mathrm{Free}\{e_i \ : \ i \in I\}. \tag{4.9}$$

If $\mathcal{V}$, $\mathcal{W}$ are Hilbert spaces, then $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$ has a unique scalar product such that

$$(v_1 \otimes w_1 | v_2 \otimes w_2) := (v_1 | v_2)(w_1 | w_2), \quad v_1, v_2 \in \mathcal{V}, \quad w_1, w_2 \in \mathcal{W}.$$

To see this it is enough to choose o.n.b's $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ in $\mathcal{V}$, resp. $\mathcal{W}$. Then every element of $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$ can be written as an (infinite) linear combination of $e_i \otimes f_j$ and we can use them as an orthonormal set defining this scalar product.

We set
$$\mathcal{V} \otimes \mathcal{W} := (\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W})^{\mathrm{cpl}},$$
and call it the *tensor product of* $\mathcal{V}$ *and* $\mathcal{W}$ *in the sense of Hilbert spaces.* If $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are o.n.b's of $\mathcal{V}$, resp. $\mathcal{W}$, then $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ is an o.n.b. of $\mathcal{V} \otimes \mathcal{W}$,

If one of the spaces $\mathcal{V}$ or $\mathcal{W}$ is finite dimensional, then $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W} = \mathcal{V} \otimes \mathcal{W}$.

Proposition 4.2 *Let* $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2$ *be vector spaces. If* $a \in L(\mathcal{V}_1, \mathcal{V}_2)$ *and* $b \in L(\mathcal{W}_1, \mathcal{W}_2)$*, then there exists a unique operator* $a \otimes b \in L(\mathcal{V}_1 \overset{\mathrm{al}}{\otimes} \mathcal{W}_1, \mathcal{V}_2 \overset{\mathrm{al}}{\otimes} \mathcal{W}_2)$ *such that on simple tensors we have*

$$(a \otimes b)(y \otimes w) = (ay) \otimes (bw). \tag{4.10}$$

It is called the tensor product of a and b.

Proof. Choose bases $(e_i)_{i \in I}$ in $\mathcal{V}_1$ and $(f_j)_{j \in J}$ in $\mathcal{W}_1$. Define $a \otimes b$ on the basis $(e_i \otimes f_j)_{(i,j) \in I \times J}$ by
$$(a \otimes b) e_i \otimes f_j := (a e_i) \otimes (b f_j). \tag{4.11}$$
Then we check that thus defined operator satisfies (4.10). It is unique, because simple tensors span the whole tensor product. $\square$

Proposition 4.3 *If $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2$ are Hilbert spaces and $a \in B(\mathcal{V}_1, \mathcal{V}_2)$, $b \in B(\mathcal{W}_1, \mathcal{W}_2)$, then $a \otimes b$ is bounded. Hence it extends uniquely to an operator in $B(\mathcal{V}_1 \otimes \mathcal{W}_1, \mathcal{V}_2 \otimes \mathcal{W}_2)$, denoted by the same symbol.*

Proof. To prove the boundedness of $a \otimes b = a \otimes 1\!\!1\; 1\!\!1 \otimes b$, it is sufficient to consider the operator $a \otimes 1\!\!1$ from $\mathcal{V}_1 \overset{\mathrm{al}}{\otimes} \mathcal{W}$ to $\mathcal{V}_2 \overset{\mathrm{al}}{\otimes} \mathcal{W}$. Let $e_1, e_2, \dots$ and $f_1, f_2 \dots$ be orthonormal bases in $\mathcal{V}_1$, $\mathcal{W}$ resp. Consider a vector $\sum c_{ij} \mathrm{e}_i \otimes f_j$.

$$\begin{aligned} \Big\| a \otimes 1\!\!1 \sum_i c_{ij} e_i \otimes f_j \Big\|^2 &= \sum_j \Big\| \sum_i c_{ij} a e_i \Big\|^2 \\ = \sum_j \|a\|^2 \Big\| \sum_i c_{ij} e_i \Big\|^2 &= \sum_j \|a\|^2 \sum_i |c_{ij}|^2 \\ &= \|a\|^2 \Big\| \sum_{ij} c_{ij} e_i \otimes f_j \Big\|^2. \end{aligned}$$

□

4.4 Fock spaces

Let $\mathcal{Y}$ be a vector space. Let S_n denote the *permutation group of* n *elements* and $\sigma \in S_n$.

Proposition 4.4 *There exists a unique operator $\Theta(\sigma)$ on $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$ such that*

$$\Theta(\sigma) y_1 \otimes \cdots \otimes y_n = y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(n)}. \tag{4.12}$$

Proof. Choose a basis $\{e_i\}_{i \in I}$ of $\mathcal{Y}$. We define $\Theta(\sigma)$ on the corresponding basis of $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$:

$$\Theta(\sigma) e_{i_1} \otimes \cdots \otimes e_{i_n} = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}}.$$

Then we extend by linearity $\Theta(\sigma)$ to the whole $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$. It is easy to see that the operator defined in this way satisfies (4.12). The uniqueness is obvious. □

We can check that

$$S_n \ni \sigma \mapsto \Theta(\sigma) \in L(\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}) \tag{4.13}$$

is a group representation.

We say that a tensor $\Psi \in \overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$ is symmetric, resp. antisymmetric if

$$\Theta(\sigma)\Psi = \Psi, \quad \text{resp.} \qquad \Theta(\sigma)\Psi = \mathrm{sgn}(\sigma)\Psi. \tag{4.14}$$

We define the *symmetrization/antisymmetrization projections*

$$\Theta_{\mathrm{s}}^n := \frac{1}{n!} \sum_{\sigma \in S_n} \Theta(\sigma), \qquad \Theta_{\mathrm{a}}^n := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}\sigma \Theta(\sigma).$$

They project onto symmetric/antisymmetric tensors.

We will often write s/a to denote either s or a.

If $\mathcal{Y}$ is a Hilbert space, then $\Theta(\sigma)$ is unitary and $\Theta^n_{\mathrm{s/a}}$ are orthogonal projections.

Let $\mathcal{Y}$ be a vector space. The *algebraic n-particle bosonic/fermionic space* is defined as

$$\overset{\mathrm{al}}{\otimes}{}^n_{\mathrm{s/a}}\mathcal{Y} := \Theta^n_{\mathrm{s/a}}\overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}.$$

The *algebraic bosonic/fermionic Fock space* or the *symmetric/antisymmetric tensor algebra* is

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y}) := \overset{\infty}{\underset{n=0}{\overset{\mathrm{al}}{\oplus}}}\overset{\mathrm{al}}{\otimes}{}^n_{\mathrm{s/a}}\mathcal{Y}.$$

The *vacuum vector* is $\Omega := 1 \in \otimes^0_{\mathrm{s/a}}\mathcal{Y} = \mathbb{C}$.

If $\mathcal{Y}$ is a Hilbert space, then the *n-particle bosonic/fermionic space* is defined as

$$\otimes^n_{\mathrm{s/a}}\mathcal{Y} := \Theta^n_{\mathrm{s/a}} \otimes^n \mathcal{Y}.$$

The *bosonic/fermionic Fock space* is

$$\Gamma_{\mathrm{s/a}}(\mathcal{Y}) := \overset{\infty}{\underset{n=0}{\oplus}} \otimes^n_{\mathrm{s/a}}\mathcal{Y}.$$

4.5 Creation/annihilation operators

For $z \in \mathcal{Y}$ we define the *creation operator*

$$\hat{a}^*(z)\Psi := \Theta^{n+1}_{\mathrm{s/a}}\sqrt{n+1}z \otimes \Psi, \quad \Psi \in \otimes^n_{\mathrm{s/a}}\mathcal{Y},$$

and the *annihilation operator* $\hat{a}(z) := (\hat{a}^*(z))^*$. (We often omit the hat). We have

$$[a(z), a(w)]_\mp = [a^*(z), a^*(w)]_\mp = 0, \tag{4.15}$$
$$[a(z), a^*(w)]_\mp = (z|w). \tag{4.16}$$

We will sometimes write $(z|$ and $|z)$ for the following operators

$$\mathcal{V} \ni v \mapsto (z|v := (z|v) \in \mathbb{C}, \tag{4.17}$$
$$\mathbb{C} \ni \lambda \mapsto \lambda|z) := \lambda z \in \mathcal{V}. \tag{4.18}$$

Then on $\otimes^n_{\mathrm{s/a}}\mathcal{Y}$ we have

$$a^*(z) = \Theta^{n+1}_{\mathrm{s/a}}\sqrt{n+1}|z) \otimes \mathbb{1}^{n\otimes}, \tag{4.19}$$
$$a(z) = \sqrt{n}(z| \otimes \mathbb{1}^{(n-1)\otimes}. \tag{4.20}$$

Above we used the *compact notation* for creation/annihilation operators popular among mathematicians. Physicists commonly prefer the *traditional notation*, which is longer and less canonical.

One version of the traditional notation uses a fixed basis $\{e_i\}_{i\in I}$ of $\mathcal{Z}$ and set $a_i^* := a^*(e_i)$, $a_i := a(e_i)$. Then if $z = \sum_i z_i e_i$, we write

$$a^*(z) = \sum_i z_i a_i^*, \quad a(z) = \sum_i \overline{z}_i a_i, \tag{4.21}$$

$$[a_i, a_j^*]_\mp = \delta_{ij}, \quad [a_i, a_j]_\mp = 0. \tag{4.22}$$

If $\Phi \in \overset{\mathrm{al}}{\otimes}{}^n_{\mathrm{s/a}} \mathcal{Z}$, then it can be represented by a symmetric/antisymmetric matrix $\Phi_{i_1,\dots,i_n}$. The annihliation operator acts on Φ as

$$(a_i\Phi)_{j_1,\dots,j_{n-1}} = \sqrt{n}\Phi_{i,j_1,\dots,j_{n-1}}. \tag{4.23}$$

Alternatively, one often identifies $\mathcal{Z}$ with, say, $L^2(\mathbb{R}^d, \mathrm{d}\xi)$. If z equals a function $\Xi \ni \xi \mapsto z(\xi)$, then

$$a^*(z) = \int z(\xi) a_\xi^* \mathrm{d}\xi, \quad a(z) = \int \overline{z}(\xi) a_\xi \mathrm{d}\xi.$$

Note that formally

$$[a(\xi), a^*(\xi')]_\mp = \delta(\xi - \xi'), \quad [a(\xi), a(\xi')]_\mp = 0. \tag{4.24}$$

The space $\otimes^n_{\mathrm{s/a}}\mathcal{Z}$ can then be identified with the space of symmetric/antisymmetric square integrable functions $L^2(\mathbb{R}^{nd})$, and then

$$\big(a(\xi)\Phi\big)(\xi_1', \dots, \xi_{n-1}') = \sqrt{n}\Phi(\xi, \xi_1', \dots, \xi_{n-1}'). \tag{4.25}$$

4.6 Integral kernel of an operator

Every linear operator A on $\mathbb{C}^n$ can be represented by a matrix $[A_i^j]$.

One would like to generalize this concept to infinite dimensional spaces (say, Hilbert spaces) and continuous variables instead of a discrete variables i, j. Suppose that a given vector space is represented, say, as $L^2(\mathbb{R}^d)$, or more generally, $L^2(X)$ where X is a certain space with a measure. One often uses the representation of an operator A in terms of its *integral kernel* $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto A(x, y)$, so that

$$A\Psi(x) = \int A(x, y)\Psi(y)\mathrm{d}y.$$

Note that strictly speaking $A(\cdot, \cdot)$ does not have to be a function. E.g. in the case $X = \mathbb{R}^d$ it could be a distribution, hence one often says the *distributional kernel* instead of the *integral kernel*. (Note that we use the integral notation for distributions, thus writing for a test function Φ $\int F(x)\Phi(x)\mathrm{d}x$ often means $F(\Phi)$.)

Sometimes $A(\cdot, \cdot)$ is ill-defined anyway. At least formally, we have

$$AB(x, y) = \int A(x, z)B(z, y)\mathrm{d}z,$$

$$A^*(x, y) = \overline{A(y, x)}.$$

Here is a situation where there is a good mathematical theory of integral/distributional kernels:

Theorem 4.5 (The Schwartz kernel theorem) *B is a continuous linear transformation from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ iff there exists a distribution $B(\cdot,\cdot) \in \mathcal{S}'(\mathbb{R}^d \oplus \mathbb{R}^d)$ such that*

$$(\Psi|B\Phi) = \int \overline{\Psi(x)} B(x,y)\Phi(y)\mathrm{d}x\mathrm{d}y, \quad \Psi, \Phi \in \mathcal{S}(\mathbb{R}^d).$$

Note that $\Leftarrow$ is obvious. The distribution $B(\cdot,\cdot) \in \mathcal{S}'(\mathbb{R}^d \oplus \mathbb{R}^d)$ is called the *distributional kernel of the transformation B*. All bounded operators on $L^2(\mathbb{R}^d)$ satisfy the Schwartz kernel theorem.

Examples:

(1) $\mathrm{e}^{-\mathrm{i}xy}$ is the kernel of the Fourier transformation

(2) $\delta(x-y)$ is the kernel of identity.

(3) $\partial_x\delta(x-y)$ is the kernel of ∂_x.

4.7 Position and momentum representation

The standard definition of the Fourier transform of V is

$$\hat{V}(p) = \int \mathrm{e}^{-\mathrm{i}xp} V(x)\mathrm{d}x, \qquad V(x) = \frac{1}{(2\pi)^d}\int \hat{V}(p)\mathrm{d}p. \tag{4.26}$$

One uses the unitary Fourier transform

$$\mathcal{F}f(p) := \frac{1}{(2\pi)^{\frac{d}{2}}}\int \mathrm{e}^{-\mathrm{i}xp} f(x)\mathrm{d}x, \qquad \mathcal{F}^{-1}f(x) := \frac{1}{(2\pi)^{\frac{d}{2}}}\int \mathrm{e}^{\mathrm{i}xp} f(p)\mathrm{d}p. \tag{4.27}$$

to pass from the position to momentum representation. Thus if we have an operator K with integral kernel $K(x',x)$ in the position representation, then its kernel in the momentum representation is

$$K(p',p) = \frac{1}{(2\pi)^d}\int \mathrm{e}^{\mathrm{i}x'p' - \mathrm{i}xp} K(x',x)\mathrm{d}x'\mathrm{d}x. \tag{4.28}$$

For instance, the 1-body potential $V(x)$ acting on $L^2(\mathbb{R}^d)$ has the integral kernels

$$\delta(x'-x)V(x) \qquad \text{in the position representation} \tag{4.29}$$

$$\frac{\hat{V}(p'-p)}{(2\pi)^d} \qquad \text{in the momentum representation} \tag{4.30}$$

A 2-body potential $V(x_1 - x_2)$ acting on $L^2(\mathbb{R}^{2d})$ has the integral kernels

$$\delta(x_1'-x_1)\delta(x_2'-x_2)V(x_1-x_2) \qquad \text{in the position representation} \tag{4.31}$$

$$= \delta(p_1'+p_2'-p_1-p_2)\frac{\hat{V}(p_1'-p_1)}{(2\pi)^d} \qquad \text{in the momentum representation} \tag{4.32}$$

In fact, (4.32) equals

$$\frac{1}{(2\pi)^{2d}}\int\int \mathrm{e}^{\mathrm{i}(x_1'p_1'+x_2'p_2'-x_1p_1-x_2p_2)}\delta(x_1'-x_1)\delta(x_2'-x_2)V(x_1-x_2)\mathrm{d}x_1'\mathrm{d}x_2'\mathrm{d}x_1\mathrm{d}x_2. \tag{4.33}$$

If we replace $\mathbb{R}^d$ with $[0,L]^d$ with periodic boundary conditions, then the momentum space is $\frac{2\pi}{L}\mathbb{Z}^d$. The standard definition of the Fourier transform of V is

$$\hat{V}(p) = \int \mathrm{e}^{-\mathrm{i}xp} V(x)\mathrm{d}x, \qquad V(x) = \frac{1}{L^d}\sum_p \hat{V}(p). \tag{4.34}$$

The unitary Fourier transform is

$$\mathcal{F}f(p) := \frac{1}{L^{\frac{d}{2}}}\int \mathrm{e}^{-\mathrm{i}xp} f(x)\mathrm{d}x, \qquad \mathcal{F}^{-1}f(x) := \frac{1}{L^{\frac{d}{2}}}\sum_p \mathrm{e}^{\mathrm{i}xp} f(p). \tag{4.35}$$

4.8 Second quantization of operators

For a contraction q on $\mathcal{Z}$ the operator $q^{\otimes n}$ commutes with $\Theta(\sigma)$, $\sigma \in S_n$. Therefore, it preserves $\otimes^n_{\mathrm{s/a}}\mathcal{Z}$. We define the operator $\Gamma(q)$ on $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ by

$$\Gamma(q)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}} = q\otimes\cdots\otimes q\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}.$$

$\Gamma(q)$ is called the *second quantization of* q.

Similarly, for an operator h on $\mathcal{Z}$ the operator $h\otimes 1^{(n-1)\otimes}+\cdots+1^{(n-1)\otimes}\otimes h$ preserves $\otimes^n_{\mathrm{s/a}}\mathcal{Z}$. We define the operator $\mathrm{d}\Gamma(h)$ by

$$\mathrm{d}\Gamma(h)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}} = h\otimes 1^{(n-1)\otimes}+\cdots+1^{(n-1)\otimes}\otimes h\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}.$$

$\mathrm{d}\Gamma(h)$ is called the *(infinitesimal) second quantization of* h.

Note the identities

$$\begin{aligned}&\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}, \quad \Gamma(q)\Gamma(r)=\Gamma(qr), \quad [\mathrm{d}\Gamma(h),\mathrm{d}\Gamma(k)] = \mathrm{d}\Gamma([h,k]),\\ &\Gamma(q)\mathrm{d}\Gamma(h)\Gamma(q^{-1}) = \mathrm{d}\Gamma(qhq^{-1}).\end{aligned} \tag{4.36}$$

Let $\{e_i \mid i\in I\}$ be an orthonormal basis of $\mathcal{Z}$. Write $\hat{a}_i := \hat{a}(e_i)$. Let h be an operator on $\mathcal{Z}$ given by the matrix $[h_{ij}]$. Then

$$\mathrm{d}\Gamma(h) = \sum_{ij} h_{ij}\hat{a}_i^*\hat{a}_j. \tag{4.37}$$

Let us prove it in the bosonic case. Let $\Phi\in\Gamma^n_{\mathrm{s}}(\mathcal{Z})$.

$$\hat{a}_i^*\hat{a}_j\Phi = n\Theta^n_{\mathrm{s}}|e_i)\otimes \mathbb{1}^{(n-1)\otimes}(e_j|\otimes\mathbb{1}^{(n-1)\otimes}\Phi \tag{4.38}$$

$$= n\Theta^n_{\mathrm{s}}|e_i)(e_j|\otimes\mathbb{1}^{(n-1)\otimes}\Phi \tag{4.39}$$

$$= \frac{1}{(n-1)!}\sum_{\sigma\in S_n}\Theta(\sigma)|e_i)(e_j|\otimes\mathbb{1}^{(n-1)\otimes}\Theta(\sigma)^{-1}\Phi \tag{4.40}$$

$$= \sum_{k=1}^n \mathbb{1}^{(k-1)\otimes}|e_i)(e_j|\otimes\mathbb{1}^{(n-k)\otimes}\Phi. \tag{4.41}$$

More generally, if the integral kernel of an operator h is $h(x,y)$, then

$$\mathrm{d}\Gamma(h) = \int h(x,y)\hat{a}_x^*\hat{a}_y \mathrm{d}x\mathrm{d}y. \tag{4.42}$$

For instance, if h is the multiplication operator by $h(\xi)$, then $\mathrm{d}\Gamma(h) = \int h(\xi)\hat{a}_\xi^*\hat{a}_\xi \mathrm{d}\xi$.

4.9 Symmetric/antisymmetric tensor product

Let $\Psi \in \otimes_{\mathrm{s/a}}^p \mathcal{Z}$, $\Phi \in \otimes_{\mathrm{s/a}}^q \mathcal{Z}$. We set

$$\Psi \otimes_{\mathrm{s/a}} \Phi := \Theta_{\mathrm{s/a}}^{p+q} \Psi \otimes \Phi. \tag{4.43}$$

Note that

$$z \otimes \cdots \otimes z = z \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} z. \tag{4.44}$$

If there are n terms, it is often written as $z^{n\otimes}$. In the antisymmetric case one usually prefers

$$\Psi \wedge \Phi := \frac{(p+q)!}{p!q!} \Psi \otimes_{\mathrm{a}} \Phi. \tag{4.45}$$

The operations $\otimes_{\mathrm{s}}$, $\otimes_{\mathrm{a}}$, $\wedge$ are associative. We have

$$y_1 \wedge \cdots \wedge y_n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}, \tag{4.46}$$

$$y_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} y_n = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}. \tag{4.47}$$

Let $\{e_i\}_{i \in I}$ be a linearly ordered orthonormal basis in $\mathcal{Z}$. Then

$$\sqrt{n!} e_{i_1} \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e_{i_n}, \quad i_1 < \cdots < i_n, \tag{4.48}$$

forms an o.n.b of $\otimes_{\mathrm{a}}^n(\mathcal{Z})$.

$$\frac{\sqrt{n!}}{\sqrt{k_1! \cdots k_n!}} e_{i_1}^{\otimes k_1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} e_{i_m}^{\otimes k_m}, \quad k_1 + \cdots + k_m = n, \tag{4.49}$$

forms an o.n.b of $\otimes_{\mathrm{s}}^m(\mathcal{Z})$.

If $\dim \mathcal{Z} = d$, then

$$\dim \otimes_{\mathrm{s}}^n \mathcal{Z} = \frac{(d+n-1)!}{(d-1)!n!}, \quad \dim \otimes_{\mathrm{a}}^n \mathcal{Z} = \frac{d!}{n!(d-n)!}. \tag{4.50}$$

4.10 Exponential law

Let $\mathcal{Z}, \mathcal{W}$ be Hilbert spaces. We can treat them as subspaces of $\mathcal{Z} \oplus \mathcal{W}$. Let $\Phi \in \otimes_{\mathrm{s/a}}^n \mathcal{Z}$, $\Psi \in \otimes_{\mathrm{s/a}}^m \mathcal{W}$. We can identify $\Phi \otimes \Psi$ with

$$U\Phi \otimes \Psi := \sqrt{\frac{(n+m)!}{n!m!}} \Phi \otimes_{\mathrm{s/a}} \Psi \in \otimes_{\mathrm{s/a}}^{n+m}(\mathcal{Z} \oplus \mathcal{W}). \tag{4.51}$$

Theorem 4.6 *The map* (4.51) *extends to a unitary map*

$$U : \Gamma_{\mathrm{s/a}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s/a}}(\mathcal{W}) \to \Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \mathcal{W}). \tag{4.52}$$

It satisfies

$$\begin{aligned}
U\Omega \otimes \Omega &= \Omega, && (4.53)\\
\mathrm{d}\Gamma(h \oplus g)U &= U\big(\mathrm{d}\Gamma(h) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(g)\big), && (4.54)\\
\Gamma(p \oplus q)U &= U\Gamma(p) \otimes U\Gamma(q), && (4.55)\\
a^*(z \oplus w)U &= U\big(a^*(z) \otimes 1\!\!1 + 1\!\!1 \otimes a^*(w)\big), && (4.56)\\
a(z \oplus w)U &= U\big(a(z) \otimes 1\!\!1 + 1\!\!1 \otimes a(w)\big), \quad \textit{in the bosonic case}, && (4.57)\\
a^*(z \oplus w)U &= U\big(a^*(z) \otimes 1\!\!1 + (-1)^N \otimes a^*(z)\big), && (4.58)\\
a(z \oplus w)U &= U\big(a(z) \otimes 1\!\!1 + (-1)^N \otimes a(z)\big), \quad \textit{in the fermionic case}. && (4.59)
\end{aligned}$$

Proof. Let us prove the unitarity of this map in the symmetric case:

$$\begin{aligned}
\Phi \otimes_{\mathrm{s}} \Psi &= \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \Theta(\sigma)\Phi \otimes \Psi && (4.60)\\
&= \frac{n!m!}{(n+m)!} \sum_{[\sigma] \in S_{n+m}/S_n \times S_m} \Theta(\sigma)\Phi \otimes \Psi. && (4.61)
\end{aligned}$$

The terms on the right are mutually orthogonal. The maps $\Theta(\sigma)$ are unitary. The number of cosets in $S_{n+m}/S_n \times S_m$ is $\frac{(n+m)!}{n!m!}$. Therefore the square norm of (4.60) is

$$\frac{n!m!}{(n+m)!}\|\Phi \otimes \Psi\|^2. \tag{4.62}$$

□

4.11 Wick symbol

Suppose we fix a basis $\{e(i) \ : \ i \in I\}$ in the space $\mathcal{Z}$. Recall that

$$e(i_1) \otimes \cdots \otimes e(i_k), \quad i_1, \ldots, i_k \in I \tag{4.63}$$

is a basis of $\overset{\mathrm{al}}{\otimes}{}^k \mathcal{Z}$. Every linear map $b : \overset{\mathrm{al}}{\otimes}{}^k \mathcal{Z} \to \overset{\mathrm{al}}{\otimes}{}^m \mathcal{Z}$ can be represented by a matrix

$$b(i_1, \cdots i_m; i'_k, \cdots, i'_1), \tag{4.64}$$

Thus if $\Phi \in \otimes^m \mathcal{Z}$ to $\Psi \in \otimes^k \mathcal{Z}$, then

$$(\Phi|b\Psi) = \sum \cdots \sum \overline{\Phi(i_m, \cdots i_1)} b(i_1, \cdots i_m, i'_k, \cdots, i'_1)\Psi(i'_k, \cdots, i'_1). \tag{4.65}$$

(Note that we invert the order of $i_m, \ldots, i_1$—this is just a convention).

We can restrict (4.65) to $\Phi \in \otimes^k_{\mathrm{s/a}}\mathcal{Z}$ to $\Psi \in \otimes^m_{\mathrm{s/a}}\mathcal{Z}$. Then (4.65) will depend only on the symmetrization/antisymmetrization of b, that is

$$b^{\mathrm{s/a}} := \Theta^m_{\mathrm{s/a}} b \Theta^k_{\mathrm{s/a}}. \tag{4.66}$$

Thus to describe operators from $\otimes^k_{\mathrm{s/a}}\mathcal{Z}$ to $\otimes^m_{\mathrm{s/a}}\mathcal{Z}$ it is enough to consider matrices symmetric/antisymmetric separately wrt the first m and the last k arguments.

In this subsection we will put "hats" on the creation/annihillation operators. The symbols $a^*(i)$, $a(i)$ without hats will be reserved for classical variables, which in the bosonic case commute and in the fermionic anticommute, that is

$$[a^*(i), a^*(j)]_\mp = [a(i), a(j)]_\mp = [a^(i), a^*(j)]_\mp = 0. \tag{4.67}$$

As usual, by a (commuting/anticommuting) *polynomial* in the variables a_i^*, a_j we mean a linear combination of the following expressions

$$b(a^*, a) = \sum \cdots \sum b(i_1, \cdots i_m, i_k', \cdots, i_1') a^*(i_1) \cdots a^*(i_m) a(i_k') \cdots a(i_1') \tag{4.68}$$

where b are symmetric/antisymmetric separately wrt the first m and the last k arguments. In the symmetric case this can be interpreted as a usual polynomial In the antisymmetric case it is an element of the Grassmann algebra.

The *Wick quantization of* $b(a^*, a)$ is defined as

$$b(\hat{a}^*, \hat{a}) \quad = \quad \sum \cdots \sum b(i_1, \cdots i_m, i_k', \cdots, i_1') \hat{a}^*(i_1) \cdots \hat{a}^*(i_m) \hat{a}(i_k') \cdots \hat{a}(i_1'). \tag{4.69}$$

(Actually, by (4.66), in (4.68) and (4.69) we can consider b which is not symmetric/antisymmetric.)

Here is an equivalent definition of $b(\hat{a}^*, \hat{a})$: Its only nonzero matrix elements are between $\Phi \in \otimes^{p+m}_{\mathrm{s/a}}\mathcal{Z}$, $\Psi \in \otimes^{p+k}_{\mathrm{s/a}}\mathcal{Z}$, and equal

$$(\Phi | b(\hat{a}^*, \hat{a}) \Psi) \quad = \quad \frac{\sqrt{(m+p)!(k+p)!}}{p!} (\Phi | b \otimes 1_{\mathcal{Z}}^{\otimes p} \Psi). \tag{4.70}$$

To see this it is enough to check

$$\begin{aligned}
&\big(\Phi | \hat{a}^*(i_1) \cdots \hat{a}^*(i_m) \hat{a}(i_k') \cdots \hat{a}(i_1') \Psi\big) && (4.71)\\
=&\big(\hat{a}(i_m) \cdots \hat{a}(i_1) \Phi | \hat{a}(i_k') \cdots \hat{a}(i_1') \Psi\big) && (4.72)\\
=&\sqrt{(m+p) \cdots (p+1)(k+p) \cdots (p+1)} && (4.73)\\
&\times \sum_{j_p} \cdots \sum_{j_1} \overline{\Phi(i_m, \ldots, i_1, j_p, \ldots, j_1)} \Psi(i_m', \ldots, i_1', j_p, \ldots, j_1). && (4.74)
\end{aligned}$$

Essentially every operator on a Fock space can be written as a linear combination of (4.69).

4.12 Wick symbol and coherent states

In the bosonic case, we have the identities

$$\mathrm{e}^{-\hat{a}^*(b)+\hat{a}(b)}\hat{a}(v)\mathrm{e}^{\hat{a}^*(b)-\hat{a}(b)} = \hat{a}(v) + (v|b), \tag{4.75}$$

$$\mathrm{e}^{-\hat{a}^*(b)+\hat{a}(b)}\hat{a}^*(v)\mathrm{e}^{\hat{a}^*(b)-\hat{a}(b)} = \hat{a}(v) + (v|b). \tag{4.76}$$

We also introduce the coherent state corresponding to $b \in \mathcal{Z}$:

$$\Omega_b := \mathrm{e}^{\hat{a}^*(b)-\hat{a}(b)}\Omega. \tag{4.77}$$

Note that $\hat{a}(v)\Omega_b = (v|b)\Omega_b$. We have the identity

$$(\Omega_b|c(\hat{a}^*, \hat{a})\Omega_b) = c(b^*, b). \tag{4.78}$$

5 Clifford algebras

5.1 Clifford algebras

Let $\phi_1, ..., \phi_n$ satisfy the relations

$$[\phi_i, \phi_j]_+ = 2\delta_{ij}\mathbb{1}. \tag{5.1}$$

The associative algebra over $\mathbb{R}$ generated by $\mathbb{1}, \phi_1, \dots, \phi_n$ satisfying these relations is called the *(real) Clifford algebra with positive signature* $\mathrm{Cl}^+(\mathbb{R}^n) = \mathrm{Cl}^+(n)$.

Let $\gamma_1, ..., \gamma_n$ satisfy the relations

$$[\gamma_i, \gamma_j]_+ = -2\delta_{ij}\mathbb{1}. \tag{5.2}$$

The associative algebra over $\mathbb{R}$ generated by $\mathbb{1}, \gamma_1, \dots, \gamma_n$ satisfying these relations is called the *(real) Clifford algebra with negative signature* $\mathrm{Cl}^-(\mathbb{R}^n) = \mathrm{Cl}^-(n)$.

The associative algebra over $\mathbb{C}$ generated by $\mathbb{1}, \phi_1, \dots, \phi_n$ and satisfying (5.1) is called the *complex Clifford algebra* and will be denoted by $\mathrm{Cl}(\mathbb{C}^n)$. Clearly, it is isomorphic to the algebra over $\mathbb{C}$ generated by $\mathbb{1}, \gamma_1, \dots, \gamma_n$ satisfying (5.2), where the isomorphism is given by

$$\gamma_i := \mathrm{i}\phi_i. \tag{5.3}$$

Both $\mathrm{Cl}^+(\mathbb{R}^n)$ and $\mathrm{Cl}^-(\mathbb{R}^n)$ are real subalgebras of $\mathrm{Cl}(\mathbb{C}^n) = \mathrm{Cl}(n, \mathbb{C})$. In what follows we will treat $\mathrm{Cl}^+(\mathbb{R}^n)$ and $\mathrm{Cl}(\mathbb{C}^n)$ as basic objects, because $\mathrm{Cl}^-(\mathbb{R}^n)$ can be obtained by (5.3).

More generally, we can consider Clifford algebras $\mathrm{Cl}(q, p)$ of an arbitrary signature, generated by γ_i,

$$[\gamma_i, \gamma_j]_+ = \begin{cases} -2, & i = j = 1, \dots q; \\ 2, & i = j = q+1, \dots, q+p; \\ 0, & i \neq j. \end{cases} \tag{5.4}$$

More abstractly, let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$ equipped with a bilinear form $\langle v|w\rangle$, $v, w \in \mathcal{V}$. Then we define $\mathrm{Cl}(\mathcal{V})$ as the associative algebra generated by $\phi(v)$, $v \in \mathcal{V}$, with relations

$$\phi(v + w) = \phi(v) + \phi(w), \quad \phi(\lambda v) = \lambda\phi(v), \quad [\phi(v), \phi(w)]_+ = 2\langle v|w\rangle\mathbb{1}. \tag{5.5}$$

Setting

$$\phi(v) = \sum_{i=1}^{n} v_i \phi_i, \quad v = (v_1, \dots, v_n), \quad \mathcal{V} = \mathbb{R}^n, \tag{5.6}$$

$$\langle v|w \rangle = \pm \sum_{i=1}^{n} v_i w_i, \tag{5.7}$$

we obtain $\mathrm{Cl}^{\pm}(n)$.

5.2 Even Clifford algebras

The map $\phi_i \mapsto -\phi_i$ (or equivalently $\gamma_i \mapsto -\gamma_i$) extends uniquely to an automorphism of a Clifford algebra denoted α. Elements fixed by this automorphism are called *even*. The subalgebra of even elements of $\mathrm{Cl}(\mathbb{C}^n)$ is denoted $\mathrm{Cl}_0(\mathbb{C}^n)$. Elements that flip the sign under α are called odd. The set of odd elements is denoted $\mathrm{Cl}_1(\mathbb{C}^n)$.

If we view $\mathrm{Cl}^+(\mathbb{R}^n)$ and $\mathrm{Cl}^-(\mathbb{R}^n)$ as subalgebras of $\mathrm{Cl}(\mathbb{C}^n)$, then the set of even elements in both algebras coincides. We will denote it by $\mathrm{Cl}_0(\mathbb{R}^n)$ (without indicating the sign $\pm$).

We have an isomorphism

$$\psi : \mathrm{Cl}^-(\mathbb{R}^{n-1}) \to \mathrm{Cl}_0(\mathbb{R}^n), \tag{5.8}$$

$$\psi(\gamma_j) := \phi_j \phi_n, \quad j = 1, \dots, n-1. \tag{5.9}$$

In fact,

$$[\psi(\gamma_j), \psi(\gamma_k)]_+ = -2\delta_{jk} \mathbb{1}.$$

Similarly,

$$\mathrm{Cl}(\mathbb{C}^{n-1}) \simeq \mathrm{Cl}_0(\mathbb{C}^n).$$

5.3 Bases

The set

$$\phi_{i_1} \cdots \phi_{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \phi_{i_{\sigma(1)}} \cdots \phi_{i_{\sigma(k)}}, \qquad i_1 < \cdots < i_k. \tag{5.10}$$

is a basis of $\mathrm{Cl}^+(n)$. Hence $\mathrm{Cl}^+(n)$, as well as $\mathrm{Cl}^-(n)$ have a real dimension 2^n. $\mathrm{Cl}(n, \mathbb{C})$ has a complex dimension 2^n. Clearly,

$$\alpha\big(\phi_{i_1} \cdots \phi_{i_k}\big) = (-1)^k \phi_{i_1} \cdots \phi_{i_k}. \tag{5.11}$$

One can introduce an identification of the Grassmann algebra and the Clifford algebra. It is the linear map defined by

$$\mathrm{Op} : \mathbb{C}[\phi_1, \dots, \phi_n] \to \mathrm{Cl}(n, \mathbb{C}), \tag{5.12}$$

$$\mathrm{Op}\big(\phi_{i_1} \cdots \phi_{i_k}\big) := \sum_{\sigma \in S_k} \phi_{i_{\sigma(1)}} \cdots \phi_{i_{\sigma(k)}}, \qquad i_1 < \cdots < i_k. \tag{5.13}$$

Clearly, Op is not a homomorphism. It plays a role of quantization for fermionic systems.

5.4 Involution

The algebras $\mathrm{Cl}^+(n)$ are equipped with the involution, which is a linear map defined by

$$\phi_i^* = \phi_i, \quad (AB)^* = B^* A^*, \tag{5.14}$$

and called the *(Clifford) conjugation.* Another acceptable notation for the conjugation on $\mathrm{Cl}^+(n)$ is $A^{\mathrm{T}} = A^*$, and another name is the *(Clifford) transposition.*

In the algebras $\mathrm{Cl}^-(n)$ there is an analogous involution defined by

$$\gamma_i^* = -\gamma_i, \quad (AB)^* = B^* A^*, \tag{5.15}$$

Note that on $\mathrm{Cl}^-(2) \simeq \mathbb{C}$, the transposition coincides with the complex conjugation, and on $\mathrm{Cl}^-(3) \simeq \mathbb{H}$ it coincides with the quaternionic conjugation.

In $\mathrm{Cl}(n, \mathbb{C})$ we have two natural maps that extend (5.15): one by linearity, and then we denote it by A^{T} and call the *(Clifford) transposition*, the other one by antilinearity, and then we denote it by A^*. Thus the action on basis elements is

$$\Big(\lambda \mathrm{Op}\big(\phi_{i_1} \cdots \phi_{i_k}\big)\Big)^{\mathrm{T}} = (-1)^{\frac{k(k-1)}{2}} \lambda \mathrm{Op}\big(\phi_{i_1} \cdots \phi_{i_k}\big), \tag{5.16}$$

$$\Big(\lambda \mathrm{Op}\big(\phi_{i_1} \cdots \phi_{i_k}\big)\Big)^{*} = (-1)^{\frac{k(k-1)}{2}} \overline{\lambda} \mathrm{Op}\big(\phi_{i_1} \cdots \phi_{i_k}\big). \tag{5.17}$$

$\mathrm{Cl}(n, \mathbb{C})$ equipped with the antilinear involution $A \mapsto A^*$ is a $*$-algebra, called the *CAR algebra*, denoted $CAR(n) = CAR(\mathbb{R}^n)$. It depends on the choice of the real subspace $\mathbb{R}^n$ inside $\mathbb{C}^n$.

The transposition on $\mathrm{Cl}(n, \mathbb{C})$ does not depend on the choice of a real subspace of $\mathbb{C}^n$.

The unitary group of $\mathrm{Cl}^{\pm}(n)$ is defined as

$$U\big(\mathrm{Cl}^{\pm}(n)\big) := \{U \in \mathrm{Cl}^{\pm}(n) \mid U^* U = 1\!\!1\}. \tag{5.18}$$

(The notation $O\big(\mathrm{Cl}^+(n)\big) = U\big(\mathrm{Cl}^+(n)\big)$ and the name *orthogonal group* is also possible.)

In the complex case we have two distinct groups: orthogonal and unitary:

$$U\big(CAR(n)\big) := \{U \in CAR(n) \mid U^* U = 1\!\!1\}, \tag{5.19}$$

$$O\big(\mathrm{Cl}(n, \mathbb{C})\big) := \{U \in \mathrm{Cl}(n, \mathbb{C}) \mid U^{\mathrm{T}} U = 1\!\!1\}. \tag{5.20}$$

5.5 Volume element

Sometimes, the following element of $\mathrm{Cl}^+(\mathbb{R}^n)$ is called the *volume element*:

$$\omega := \phi_1 \cdots \phi_n.$$

Clearly, $\mathrm{Cl}^+(\mathbb{R}^n)$ is generated by $\phi_1, \ldots, \phi_{n-1}, \omega$. We have

$$\omega^2 = (-1\!\!1)^{\frac{1}{2}n(n-1)}, \quad \omega \phi_i = -(-1)^n \phi_i \omega.$$

In $\mathrm{Cl}^-(\mathbb{R}^n)$ instead we may prefer to use

$$\mathrm{i}^n \omega = \gamma_1 \cdots \gamma_n.$$

If n is even, then ω (as well as $\mathrm{i}\omega$) implements the authomorphism α:

$$\omega A \omega^{-1} = \alpha(A), \quad A \in \mathrm{Cl}(\mathbb{C}^n). \tag{5.21}$$

If n is odd ω (or $\mathrm{i}\omega$ commutes with $\mathrm{Cl}(\mathbb{C}^n)$.

5.6 The Jordan-Wigner construction

If $k \leq m$ and $A \in L(\otimes^k \mathbb{C}^2)$, then we identify A with $A \otimes \mathbb{1}^{\otimes(m-k)} \in L(\otimes^m \mathbb{C}^2)$.

Recall that $\sigma_1, \sigma_2, \sigma_3$ denote the standard Pauli matrices. Note that $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3 = \mathrm{i}\sigma_1\sigma_2$ span $L(\mathbb{C}^2)$. Hence σ_1, σ_2 generate $L(\mathbb{C}^2)$.

Let $n = 2m$. Consider the space $\otimes^m \mathbb{C}^2$. Introduce the operators

$$\begin{aligned} \rho(\phi_1) &:= \sigma_1, & \rho(\phi_2) &: = \sigma_2, \\ &\cdots \\ \rho(\phi_{2m-1}) &:= \sigma_3^{\otimes(m-1)} \otimes \sigma_1, & \rho(\phi_{2m}) &:= \sigma_3^{\otimes(m-1)} \otimes \sigma_2. \end{aligned}$$

Theorem 5.1 *ρ extends uniquely to a homomorphism*

$$\rho : \mathrm{Cl}(2m) \to L(\otimes^m \mathbb{C}^2). \tag{5.22}$$

We have

$$\rho(\omega) = \mathrm{i}^m \sigma_3^{\otimes m}, \tag{5.23}$$

$$\rho\big(\phi_1 \cdots \phi_{2k}\phi_{2k+1}\big) = \mathrm{i}^k \mathbb{1}^{\otimes k} \otimes \sigma_1, \quad \rho\big(\phi_1 \cdots \phi_{2k}\phi_{2k+2}\big) = \mathrm{i}^k \mathbb{1}^{\otimes k} \otimes \sigma_2. \tag{5.24}$$

(5.22) *is an isomorphism.*

Proof. It is easy to check that $\rho(\phi_1), \ldots, \rho(\phi_{2m})$ satisfy the Clifford relations. Hence the map ρ extends to a homorphism.

By (5.24), the image of ρ contains $\mathbb{1}^{\otimes k} \otimes L(\mathbb{C}^2)$. Hence it contains the whole $L(\otimes^m \mathbb{C}^2)$. $\square$

Theorem 5.2 *For $n = 2m+1$ there exist two homorphisms extending $\rho : \mathrm{Cl}(2n, \mathbb{C}) \to L(\otimes^m \mathbb{C}^2)$:*

$$\rho_\pm : \mathrm{Cl}(2n+1, \mathbb{C}) \to L(\otimes^m \mathbb{C}^2), \tag{5.25}$$

$$\rho_\pm(\phi_{2m+1}) : = \pm \sigma_3^{\otimes(m+1)}. \tag{5.26}$$

The map

$$\mathrm{Cl}(2m+1) \ni A \mapsto \big(\rho_+(A), \rho_-(A)\big) \in L(\otimes^m \mathbb{C}^2) \oplus L(\otimes^m \mathbb{C}^2) \tag{5.27}$$

is an isomorphism of algebras.

Proof. First we check that $\rho_\pm(\phi_1), \ldots, \rho_\pm(\phi_{2m+1})$ satisfy the Clifford relations. Hence (5.26) defines two homorphisms $\rho_\pm$.

Let us prove that (5.26) is onto. Let $\tilde{A}, \tilde{B}) \in L(\otimes^m \mathbb{C}^2) \oplus L(\otimes^m \mathbb{C}^2)$. The maps ρ are onto, hence we will find $A_+, A_- \in \mathrm{Cl}(2m, \mathbb{C})$ such that $\rho(A_+) = \tilde{A}_+$, $\rho(A_-) = \tilde{A}_-$. Next we put

$$\pi := (-\mathrm{i})^m \phi_1 \cdots \phi_{2m} \phi_{2m+1} \in \mathrm{Cl}(2m+1, \mathbb{C}).$$

Then π commutes with $\mathrm{Cl}(\mathbb{C}^{2m+1})$ and $\rho_\pm(\pi) = \pm \mathbb{1}$. Therefore, for $A \in \mathrm{Cl}(2m, \mathbb{C})$,

$$\begin{aligned}
\rho_\pm\Big(A\frac{(\mathbb{1} \pm \pi)}{2}\Big) &= \rho_\pm(A)\rho_\pm\Big(\frac{(\mathbb{1} \pm \pi)}{2}\Big) = \rho(A), \\
\rho_\pm\Big(A\frac{(\mathbb{1} \mp \pi)}{2}\Big) &= \rho_\pm(A)\rho_\pm\Big(\frac{(\mathbb{1} \pm \pi)}{2}\Big) = 0.
\end{aligned}$$

Hence,

$$\rho_\pm\Big(A_+\frac{(\mathbb{1} + \pi)}{2} + A_-\frac{(\mathbb{1} - \pi)}{2}\Big) = \rho(A_\pm) = \tilde{A}_\pm. \tag{5.28}$$

which proves that (5.26) is onto. □

5.7 Fock representations of Clifford algebras

In this subsection we describe a representation of Clifford algebras seemingly different from the Jordan-Wigner contruction. Eventually, it will turn out to be essentially the same representation in disguise.

Consider the Fock space $\Gamma_{\mathrm{a}}(\mathbb{C}^m)$ with the standard creation and annihilation operators a_i^*, a_j satisfying

$$[a_1, a_j]_+ = [a_i^*, a_j^*]_+ = 0, \quad [a_i, a_j^*]_+ = \delta_{ij}\mathbb{1}.$$

Consider $\mathrm{Cl}(2m, \mathbb{C})$ with generators $\phi_1, \dots, \phi_{2m}$. We define

$$\rho(\phi_{2i-1}) := a_i^* + a_i, \qquad \rho(\phi_{2i}) := \mathrm{i}^{-1}(a_i^* - a_i), \qquad i = 1, \dots, m. \tag{5.29}$$

Clearly, the above operators satisfy Clifford relations and are self-adjoint. Hence ρ extends uniquely to a $*$-isomorphism $\rho : \mathrm{Cl}(2m, \mathbb{C}) \to L\big(\Gamma_{\mathrm{a}}(\mathbb{C}^m)\big)$.

We have also the number operator

$$N = \sum_{i=1}^m a_i^* a_i$$

and the parity operator

$$(-1)^N = (-1)^{\sum_i a_i^* a_i} = \prod_{i=1}^m (-1)^{a_i^* a_i} = \prod_i (\mathbb{1} - 2a_i^* a_i).$$

Now

$$[(-1)^N, a_i]_+ = [(-1)^N, a_i^*]_+ = 0, \quad \Big((-1)^N\Big)^2 = \mathbb{1}.$$

Hence setting

$$\rho_\pm(\phi_{2m+1}) := \pm(-1)^N, \tag{5.30}$$

we extend ρ to two isomorphisms $\rho_\pm : \mathrm{Cl}(2m+1,\mathbb{C}) \to L\big(\Gamma_{\mathrm{a}}(\mathbb{C}^m)\big)$.

$\mathrm{Cl}(2m+1,\mathbb{C})$ can be also represented on $\Gamma_{\mathrm{a}}(\mathbb{C}^{m+1})$, if we set

$$\rho(\phi_{2m+1}) := a_{m+1}^* + a_{m+1}. \tag{5.31}$$

Now ρ is not irreducible: $\rho(\mathrm{Cl}(2m+1,\mathbb{C}))$ commutes with

$$(a_{m+1}^* + a_{m+1})(-1)^{\sum_{i=1}^m a_i^* a_i}. \tag{5.32}$$

ρ is a direct sum of the representations ρ_+ and ρ_-.

Note that the above constructions are fully equivalent to the Jordan-Wigner construction. In fact, first let us check it for $\mathrm{Cl}(3,\mathbb{C})$. We identify $\Gamma_{\mathrm{a}}(\mathbb{C})$ with $\mathbb{C}^2$ by

$$a^*\Omega = \begin{bmatrix}1\\0\end{bmatrix}, \quad \Omega = \begin{bmatrix}0\\1\end{bmatrix}, \quad a^* = \begin{bmatrix}0&1\\0&0\end{bmatrix}, \quad a = \begin{bmatrix}0&0\\1&0\end{bmatrix},$$

$$a^* + a = \sigma_1 = \begin{bmatrix}0&1\\1&0\end{bmatrix}, \quad \mathrm{i}^{-1}(a^* - a) = \sigma_2 = \begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix}, \quad -(-1)^N = \sigma_3 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

Thus the Jordan-Wigner consstruction and the Fock representation coincide for $n = 1, 2, 3$.

Consider now $n = 2m$. We have $\mathbb{C}^m \simeq \oplus_{j=1}^m \mathbb{C}$. Hence, by the exponential property of Fock spaces,

$$\Gamma_{\mathrm{a}}(\mathbb{C}^m) \simeq \otimes^m \Gamma_{\mathrm{a}}(\mathbb{C}) = \otimes^m \mathbb{C}^2. \tag{5.33}$$

and we easily check that under this identification ρ of the Jordan-Wigner representation and of the Fock represention coincide.

5.8 Form of Clifford algebras

Remember that the standard notation for the space of linear maps on space $\mathcal{V}$ is $L(\mathcal{V})$. If $\mathcal{V} = \mathbb{K}^n$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, then $L(\mathcal{V})$ can be identified with $n \times n$ matrices with entries in $\mathbb{K}$. Below we will often use an alternative notation $\mathbb{K}(n)$ for $L(\mathbb{K}^n)$.

The following table describes the form of various Clifford algebras in tems of $\mathbb{R}(2^m)$, $\mathbb{C}(2^m)$ and $\mathbb{H}(2^m)$. Note that the validity of the column $\mathrm{Cl}(n,\mathbb{C})$ was proven in Theorems 5.1 and 5.2. It implies the column $\mathrm{Cl}_0(n,\mathbb{C})$. Both have an obvious period 2.

The real columns have a period 8, which involves multiplying the arguments of the entries by $2^4 = 16$. Clearly, the column $\mathrm{Cl}^-(n)$ implies the column $\mathrm{Cl}_0(n)$.

n	$\mathrm{Cl}^+(\mathbb{R}^n)$	$\mathrm{Cl}^-(\mathbb{R}^n)$	$\mathrm{Cl}_0(\mathbb{R}^n)$	$\mathrm{Cl}(\mathbb{C}^n)$	$\mathrm{Cl}_0(\mathbb{C}^n)$
0	$\mathbb{R}$	$\mathbb{R}$		$\mathbb{C}$	
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{C}$,	$\mathbb{R}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}$
2	$\mathbb{R}(2)$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}\oplus\mathbb{C}$
3	$\mathbb{C}(2)$	$\mathbb{H}\oplus\mathbb{H}$,	$\mathbb{H}$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{C}(4)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{H}(2)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(4)$
6	$\mathbb{H}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$
7	$\mathbb{C}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{C}(16)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$

Consider a few first entries from the column $\mathrm{Cl}^-(n)$:

$$\begin{aligned}
\mathrm{Cl}^-(1) = \mathbb{C}: &\quad \mathrm{i} := \gamma_1;\\
\mathrm{Cl}^-(2) = \mathbb{H}: &\quad \mathrm{i} := \gamma_1, \quad \mathrm{j} := \gamma_2;\\
\mathrm{Cl}^-(3) = \mathbb{H}\oplus\mathbb{H}: &\quad (\mathrm{i},\mathrm{i}) := \gamma_1, \quad (\mathrm{j},\mathrm{j}) := \gamma_2, \quad (1,-1) := \gamma_1\gamma_2\gamma_3;\\
\mathrm{Cl}^-(4) = \mathbb{H}(2): &\quad \begin{bmatrix}\mathrm{i} & 0\\ 0 & \mathrm{i}\end{bmatrix} := \gamma_1, \quad \begin{bmatrix}\mathrm{j} & 0\\ 0 & \mathrm{j}\end{bmatrix} := \gamma_2,\\
&\quad \begin{bmatrix}\mathrm{k} & 0\\ 0 & \mathrm{k}\end{bmatrix} = \gamma_3, \quad \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} := \gamma_4
\end{aligned}$$

Next, $\mathrm{Cl}^+(n)$:

$$\begin{aligned}
\mathrm{Cl}^+(4) = \mathbb{H}(2): &\quad \begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix} := \phi_1, \quad \begin{bmatrix}0 & -\mathrm{j}\\ \mathrm{j} & 0\end{bmatrix} := \phi_2,\\
&\quad \begin{bmatrix}0 & -\mathrm{k}\\ \mathrm{k} & 0\end{bmatrix} = \phi_3, \quad \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} := \phi_4
\end{aligned}$$

Let us now describe the relationship between $\mathrm{Cl}(2m)$ and $\mathrm{Cl}(2m+1)$. ω always belongs to the center of $\mathrm{Cl}^+(\mathbb{R}^{2m+1})$ and $\mathrm{i}\omega$ belongs to the center of $\mathrm{Cl}^-(\mathbb{R}^{2m+1})$.

We have $\omega^2 = (-1)^m$, $(\mathrm{i}\omega)^2 = (-1)^{m+1}$. Hence we have the isomorphisms

$$\begin{aligned}
m \equiv 0 \mod (2), &\quad \mathrm{Cl}^+(\mathbb{R}^{2m}) \oplus \mathrm{Cl}^+(\mathbb{R}^{2m}) \ni (A_1, A_2) \mapsto \frac{\mathbb{1}+\omega}{2}A_1 + \frac{\mathbb{1}-\omega}{2}A_2 \in \mathrm{Cl}^+(\mathbb{R}^{2m+1}),\\
m \equiv 2 \mod (2), &\quad \mathbb{C}\mathrm{Cl}^+(\mathbb{R}^{2m}) \ni (A_1 + \mathrm{i}A_2) \mapsto A_1 + \omega A_2 \in \mathrm{Cl}^+(\mathbb{R}^{2m+1}).\\
m \equiv 0 \mod (2), &\quad \mathbb{C}\mathrm{Cl}^-(\mathbb{R}^{2m}) \ni (A_1 + \mathrm{i}A_2) \mapsto A_1 + \mathrm{i}\omega A_2 \in \mathrm{Cl}^-(\mathbb{R}^{2m+1}),\\
m \equiv 2 \mod (2), &\quad \mathrm{Cl}^-(\mathbb{R}^{2m}) \oplus \mathrm{Cl}^-(\mathbb{R}^{2m}) \ni (A_1, A_2) \mapsto \frac{\mathbb{1}+\mathrm{i}\omega}{2}A_1 + \frac{\mathbb{1}-\mathrm{i}\omega}{2}A_2 \in \mathrm{Cl}^-(\mathbb{R}^{2m+1}).
\end{aligned}$$

Note also that the complexification of $\mathbb{R}$ is $\mathbb{C}$ and of $\mathbb{H}$ is $\mathbb{C}(2)$.

5.9 Charge conjugation

Consider the Jordan-Wigner representation of $\mathrm{Cl}(n, \mathbb{C})$, ρ for $n = 2m$ or $\rho_\pm$ for $n = 2m+1$. We drop $\rho, \rho_\pm$ from the notation. We have $\phi_i = \overline{\phi_i}$ for odd i, including ϕ_{2m+1}, and for $\overline{\phi_i} = -\phi_i$ for even i. Consider first $n = 2m$. Set

$$\eta_+ := \mathrm{i}^m \phi_2\phi_4 \cdots \phi_{2m}, \quad \eta_- := \phi_1\phi_3 \cdots \phi_{2m-1}, \quad .$$

Then η_+ and η_- are real. Besides,

$$\begin{aligned}
\eta_+^2 &= (-1)^{\frac{m(m+1)}{2}}, & \eta_-^2 &= (-1)^{\frac{m(m-1)}{2}};\\
\eta_+\phi_i &= (-1)^m \overline{\phi_i}\eta_+, & \eta_-\phi_i &= -(-1)^m \overline{\phi_i}\eta, \quad i = 1, 2, \ldots, 2m;\\
\eta_+\phi_{2m+1} &= (-1)^m \overline{\phi_{2m+1}}\eta_+, & \eta_-\phi_{2m+1} &= (-1)^m \overline{\phi_{2m+1}}\eta_-.
\end{aligned}$$

Hence for $A \in \mathrm{Cl}(n, \mathbb{C})$

$$\begin{array}{llll}
n \equiv 0 \mod (8), & \eta_+^2 = \eta_-^2 = 1\!\!1, & A = \eta_+ \overline{A} \eta_+^{-1} & \alpha(A) = \eta_- \overline{A} \eta_-^{-1}; \\
n \equiv 1 \mod (8), & \eta_+^2 = 1\!\!1, & A = \eta_+ \overline{A} \eta_+^{-1}; & \\
n \equiv 2 \mod (8), & -\eta_+^2 = \eta_-^2 = 1\!\!1, & \alpha(A) = \eta_+ \overline{A} \eta_+^{-1} & A = \eta_- \overline{A} \eta_-^{-1}; \\
n \equiv 3 \mod (8), & -\eta_+^2 = 1\!\!1, & \alpha(A) = \eta_+ \overline{A} \eta_+^{-1}; & \\
n \equiv 4 \mod (8), & -\eta_+^2 = -\eta_-^2 = 1\!\!1, & A = \eta_+ \overline{A} \eta_+^{-1} & \alpha(A) = \eta_- \overline{A} \eta_-^{-1}; \\
n \equiv 5 \mod (8), & -\eta_+^2 = 1\!\!1, & A = \eta_+ \overline{A} \eta_+^{-1}; & \\
n \equiv 6 \mod (8), & \eta_+^2 = -\eta_-^2 = 1\!\!1, & \alpha(A) = \eta_+ \overline{A} \eta_+^{-1} & A = \eta_- \overline{A} \eta_-^{-1}; \\
n \equiv 7 \mod (8), & \eta_+^2 = 1\!\!1, & \alpha(A) = \eta_+ \overline{A} \eta_+^{-1}; &
\end{array}$$

6 Matrix Lie groups

6.1 Quaternionic determinant

Identify $\mathbb{H} \simeq \mathbb{C}^2$ by $v = x + \mathrm{j}y$, $x, y \in \mathbb{C}$, so that $\mathrm{j}x = \overline{x}\mathrm{j}$, $\mathrm{j}y = -\overline{y}\mathrm{j}$. Similarly, if $V \in L(\mathbb{H}^n)$, then $V = X + \mathrm{j}Y$ with $\mathrm{j}X = \overline{X}\mathrm{j}$, $\mathrm{j}Y = -\overline{Y}\mathrm{j}$, where $X, Y \in L(\mathbb{C}^n)$. Writing

$$\pi(\mathrm{j}) = \begin{bmatrix} 0 & -1\!\!1_n \\ 1\!\!1_n & 0 \end{bmatrix} =: J_n \tag{6.1}$$

we can represent V as

$$\pi(V) = \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \in L(\mathbb{C}^{2n}). \tag{6.2}$$

Thus

$$\pi(L(\mathbb{H}^n)) = \{V \in L(\mathbb{C}^{2n}) \mid J_n V = \overline{V} J_n\}. \tag{6.3}$$

In what follows we will often identify $L(\mathbb{H}^n)$ with a subspace of $L(\mathbb{C}^{2n})$ through (6.2), dropping π.

It is well-known that $L(\mathbb{R}^n)$ and $L(\mathbb{C}^n)$ are equipped with the homomorphism into $\mathbb{R}$, resp. $\mathbb{C}$ called the determinant. Matrices with nonzero determinant are invertible.

If $V \in L(\mathbb{H}^n)$, then its *quaternionic determinant* is defined as

$$\det V := \det \pi(V),$$

where on the right we use the usual determinant (in the sense of a complex matrix) and the embedding π defined in (6.2), and earlier in (2.9). Note that $\det VW = \det V \det W$. $\det V$ does not depend on the embedding of $\mathbb{C}$ in $\mathbb{H}$ and always has a real value ≥ 0. $\det V$ is nonzero if V is invertible

We also have the quaternionic trace

$$\mathrm{Tr}(V) := \mathrm{Tr}\pi(V) = 2\mathrm{Re}\big(\mathrm{Tr}(X)\big). \tag{6.4}$$

Clearly, the quaternionic trace is always real and $\det(\mathrm{e}^V) = \mathrm{e}^{\mathrm{Tr}(V)}$.

6.2 Classical matrix Lie groups

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We define

$$GL(\mathbb{K}^n) = \{V \in L(\mathbb{K}^n) \mid \det V \neq 0, \}, \tag{6.5}$$
$$SL(\mathbb{K}^n) = \{V \in L(\mathbb{K}^n) \mid \det V = 1, \}. \tag{6.6}$$

By a classical matrix Lie group we mean a subgroup of $GL(\mathbb{K}^n)$.

We now define several series of such subgroups defined as the sets elements of $GL(\mathbb{K}^n)$ preserving a certain 2-argument form (bilinear or sesquilinear).

$$\mathbb{K}^n \times \mathbb{K}^n \ni (v, w) \mapsto B(v, w). \tag{6.7}$$

That is, the general form of these groups are

$$G_B(\mathbb{K}^n) := \{V \in GL(\mathbb{K}^n) \mid B(Vv, Vw) = B(v, w), \quad v, w \in \mathbb{K}^n\}. \tag{6.8}$$

Their Lie algebras are

$$g_B(\mathbb{K}^n) := \{V \in gl(\mathbb{K}^n) \mid B(Vv, w) + B(v, Vw) = 0 \quad v, w \in \mathbb{K}^n\}. \tag{6.9}$$

Especially important series are the following three, which consist of compact groups:

$$\begin{aligned}
&O(\mathbb{R}^n) && = O(n): \quad v_1 w_1 + \cdots v_n w_n, \\
&U(\mathbb{C}^n) && = U(n): \quad \overline{v}_1 w_1 + \cdots \overline{v}_n w_n, \\
&Sp(\mathbb{H}^n) && = Sp(n): \quad v_1^* w_1 + \cdots v_n^* w_n.
\end{aligned}$$

Here are all series of classical matrix Lie groups:

$$\begin{aligned}
&O(\mathbb{R}^{q,p}) && = O(q,p): \quad -v_1 w_1 - \cdots - v_q w_q + v_{q+1} w_{q+1} + \cdots v_n w_n, \\
&Sp(\mathbb{R}^{2n}) && = Sp(n,\mathbb{R}): \quad v_1 w_{n+1} + \cdots + v_n w_{2n} - v_{n+1} w_1 + \cdots + v_{n+1} w_1, \\
&U(\mathbb{C}^{q,p}) && = U(q,p): \quad -\overline{v}_1 w_1 - \cdots - \overline{v}_q w_q + \overline{v}_{q+1} w_{q+1} + \cdots + \overline{v}_{q+p} w_{q+p}, \\
&O(\mathbb{C}^n) && = O(n,\mathbb{C}): \quad v_1 w_1 + \cdots v_n w_n, \\
&Sp(\mathbb{C}^{2n}) && = Sp(n,\mathbb{C}): \quad v_1 w_{n+1} + \cdots + v_n w_{2n} - v_{n+1} w_1 + \cdots + v_{n+1} w_1, \\
&Sp(\mathbb{H}^{q,p}) && = Sp(q,p): \quad -v_1^* w_1 - \cdots - v_q^* w_q + v_{q+1}^* w_{q+1} + \cdots + v_{q+p}^* w_{q+p}, \\
&O(\mathbb{H}^n) && = O^*(n): \quad v_1^* \mathrm{j} w_1 + \cdots v_n^* \mathrm{j} w_n.
\end{aligned}$$

Note that $Sp(n,\mathbb{R})$, resp. $Sp(n,\mathbb{C})$ is sometimes also denoted $Sp(2n,\mathbb{R})$, resp. $Sp(2n,\mathbb{C})$ (which incidentally shows the superiority of the notation $Sp(\mathbb{R}^{2n})$, resp. $Sp(\mathbb{C}^{2n})$, which is unambiguos).

Clearly

$$\begin{aligned}
U(\mathbb{C}^n) &= U(\mathbb{C}^{n,0}) = U(\mathbb{C}^{0,n}), \\
O(\mathbb{R}^n) &= O(\mathbb{R}^{n,0}) = O(\mathbb{R}^{0,n}), \\
Sp(\mathbb{H}^n) &= Sp(\mathbb{H}^{n,0}) = Sp(\mathbb{H}^{0,n}).
\end{aligned}$$

We have

$$\begin{aligned} V \in Sp(\mathbb{R}^{2n}),\ Sp(\mathbb{C}^{2n}),\ Sp(\mathbb{H}^{q,p}) \text{ or } O(\mathbb{H}^n) \ &\Rightarrow\ \det V = 1;\\ V \in O(\mathbb{R}^{q,p}) \text{ or } O(\mathbb{C}^n) &\Rightarrow\ \det V \in \{1,-1\};\\ V \in U(\mathbb{C}^{q,p}) \ &\Rightarrow\ \det V \in \{z \in \mathbb{C}|\ |z| = 1\}. \end{aligned}$$

We set

$$\begin{aligned} SO(\mathbb{R}^{q,p}) &:= O(\mathbb{R}^{q,p}) \cap SL(\mathbb{R}^{q+p}),\\ SO(\mathbb{C}^n) &:= O(\mathbb{C}^n) \cap SL(\mathbb{C}^n),\\ SU(\mathbb{C}^n) &:= O(\mathbb{C}^n) \cap SL(\mathbb{C}^n). \end{aligned}$$

Let us make some remarks concerning the quaternionic groups identified as subgroups of complex groups. Clearly,

$$GL(\mathbb{H}^n) = \{V \in GL(\mathbb{C}^{2n}) \mid J_n A = \overline{A} J_n\}, \tag{6.10}$$

$$SL(\mathbb{H}^n) = \{V \in SL(\mathbb{C}^{2n}) \mid J_n A = \overline{A} J_n\}. \tag{6.11}$$

Writing $v_i = x_i + \mathrm{j} y_i \in \mathbb{H}$, $i = 1, 2$, $x_i, y_i \in \mathbb{C}$, note that

$$v_1^* v_2 = \overline{x}_1 x_2 + \overline{y}_1 y_2 + \mathrm{j}(x_1 y_2 - y_1 x_2) \tag{6.12}$$

$$v_1^* \mathrm{j} v_2 = -\overline{x}_1 y_2 + \overline{y}_1 x_2 + \mathrm{j}(x_1 x_2 + y_1 y_2). \tag{6.13}$$

Therefore, writing $v_{1i} = x_{1i} + \mathrm{j} y_{1i} \in \mathbb{H}^n$ $v_{2i} = x_{2i} + \mathrm{j} y_{2i} \in \mathbb{H}^n$, we can rewrite the forms defining $Sp(\mathbb{H}^n)$, resp. $O(\mathbb{H}^n)$ as

$$\overline{x}_{11} x_{21} + \cdots + \overline{x}_{1n} x_{2n} + \overline{y}_{11} y_{21} + \cdots + \overline{y}_{1n} y_{2n} \tag{6.14}$$

$$+\mathrm{j}\big(x_{11} y_{21} + \cdots + x_{1n} y_{2n} - y_{11} x_{21} - \cdots - y_{1n} x_{2n}\big), \tag{6.15}$$

$$\text{resp.} \qquad -\overline{x}_{11} y_{21} - \cdots \overline{x}_{1n} y_{2n} + \overline{y}_{11} x_{21} + \cdots + \overline{y}_{1n} x_{2n} \tag{6.16}$$

$$+\mathrm{j}\big(x_{11} x_{21} + \cdots + x_{1n} x_{2n} + y_{11} y_{21} + \cdots + y_{1n} y_{2n}\big). \tag{6.17}$$

Now $Sp(\mathbb{H}^n)$, resp. $O(\mathbb{H}^n)$ can be defined as the set of $V \in GL(\mathbb{C}^{2n})$ preserving separately the form (6.14) and 6.15, resp. (6.16) and (6.17). Note that we do not need to check the conditions (6.3). Thus, we obtain (using a simple change of variabkes in the case of $O(\mathbb{H}^n)$)

$$Sp(\mathbb{H}^n) = SU(\mathbb{C}^{2n}) \cap Sp(\mathbb{C}^{2n}), \tag{6.18}$$

$$O(\mathbb{H}^n) \simeq SU(\mathbb{C}^{n,n}) \cap O(\mathbb{C}^{2n}). \tag{6.19}$$

Similarly,

$$Sp(\mathbb{H}^{q,p}) \simeq SU(\mathbb{C}^{2q,2p}) \cap Sp(\mathbb{C}^{2n}). \tag{6.20}$$

6.3 Reflections

Let $v \in \mathbb{R}^n$. The *reflection wrt* v is the map $R_v \in L(\mathbb{R}^n)$

$$R_v y := y - 2\frac{\langle v|y\rangle}{\langle v|v\rangle}v.$$

Clearly, $R_v^2 = \mathbb{1}$ i $R_v \in O(\mathbb{R}^n)\backslash SO(\mathbb{R}^n)$.

Theorem 6.1 *Reflections generate $O(\mathbb{R}^n)$. The set of even products of reflections coincides with $SO(\mathbb{R}^n)$.*

Proof. Consider first $O(\mathbb{R}^2)$ and the rotation

$$A_\phi := \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{6.21}$$

Take a pair of normalized vectors v_1, v_2 with angle $\frac{\phi}{2}$. Then it is easy to see that $A_\phi = R_{v_1}R_{v_2}$.

Let $A \in O(\mathbb{R}^n)$. After complexification, we can use the spectral theorem, which yields that in an appropriate basis A is the direct sum of matrices of the form (6.21) and of 1 and -1. □

For $v \in \mathbb{R}^n$, define

$$\phi(v) := \sum_i v_i\phi_i, \quad \gamma(v) := \sum_i v_i\gamma_i.$$

It is an element o $\mathrm{Cl}^+(\mathbb{R}^n)$, resp. $\mathrm{Cl}^-(\mathbb{R}^n)$. Clearly,

$$\phi(v)^* = \phi(v), \quad \gamma(v)^* = -\gamma(v), \quad \phi(v)\phi(v)^* = \gamma(v)\gamma(v)^* = \langle v|v\rangle,$$

Assume $\langle v|v\rangle = 1$. Then $\pm\phi(v)$ and $\pm\gamma(v)$ are unitary odd elements of $\mathrm{Cl}^+(\mathbb{R}^n)$, resp. $\mathrm{Cl}^-(\mathbb{R}^n)$, and

$$\big(\pm\phi(v)\big)\phi(y)\big(\pm\phi(v)\big)^* = -\phi(R_v y), \tag{6.22}$$

$$\big(\pm\gamma(v)\big)\gamma(y)\big(\pm\gamma(v)\big)^* = -\gamma(R_v y). \tag{6.23}$$

6.4 Pin and Spin groups

Let $Pin^+(n) = Pin^+(\mathbb{R}^n)$ be the group of $U \in U\big(\mathrm{Cl}^+(\mathbb{R}^n)\big)$ satisfying

$$\{U\phi(v)U^* \ : \ v \in \mathbb{R}^n\} = \{\phi(v) \ : \ v \in \mathbb{R}^n\}.$$

Analogously, let $Pin^-(n) = Pin^-(\mathbb{R}^n)$ be the group of $U \in U\big(\mathrm{Cl}^-(\mathbb{R}^n)\big)$ satisfying

$$\{U\gamma(v)U^* \ : \ v \in \mathbb{R}^n\} = \{\gamma(v) \ : \ v \in \mathbb{R}^n\}.$$

We set

$$Spin(\mathbb{R}^n) = Spin(n) := Pin^+(n) \cap \mathrm{Cl}_0(n) = Pin^-(n) \cap \mathrm{Cl}_0(n). \tag{6.24}$$

Theorem 6.2 *1. $Pin^+(n)$ is generated by $\phi(v)$, $\langle v|v\rangle = 1$, and $Pin^-(n)$ generated by $\gamma(v)$, $\langle v|v\rangle = 1$.*

2. If $U \in Spin(\mathbb{R}^n)$, then there exists $R_U \in SO(\mathbb{R}^n)$ such that

$$U\phi(y)U^* = \phi(R_U y), \qquad U\gamma(y)U^* = \gamma(R_U y), \tag{6.25}$$

3. If $U \in Pin^+(\mathbb{R}^n)\backslash Spin(\mathbb{R}^n)$, then there exists $R_U \in O(\mathbb{R}^n)\backslash SO(\mathbb{R}^n)$ such that

$$U\phi(y)U^* = -\phi(R_U y), \tag{6.26}$$

Similarly, if $U \in Pin^-(\mathbb{R}^n)\backslash Spin(\mathbb{R}^n)$, then there exists $R_U \in O(\mathbb{R}^n)\backslash SO(\mathbb{R}^n)$ such that

$$U\gamma(y)U^* = -\gamma(R_U y), \tag{6.27}$$

4. The maps

$$Pin^\pm(n) \ni U \mapsto R_U \in O(n) \tag{6.28}$$

are surjective group homomorphisms with kernel $\{\mathbb{1}, -\mathbb{1}\}$, satisfying

$$\alpha(U)\phi(y)U^* = U\phi(y)\alpha(U^*) = \phi(R_U y), \qquad U \in Pin^\pm(n). \tag{6.29}$$

5. We have $R_U = R_{-U}$.

$$R_{\pm\mathbb{1}} = \mathbb{1}, \qquad R_{\pm\omega} = -\mathbb{1}. \tag{6.30}$$

Proof. Let G be the group by $\phi(v)$, $\langle v|v\rangle = 1$. It is clearly a subgroup of $Pin^+(n)$, and we easily check that it satisfies all properties listed in the theorem. Now suppose that $Pin^+(n)$ is larger than G. Then the kernel of the homomorphism $Pin^+ \to O(n)$ should be larger that $\{\mathbb{1}, \mathbb{1}\}$.

We check that the only unitary elements of $L(\mathbb{C}^{2^m})$ commuting with $\phi_i(v)$ are $\{c\mathbb{1} \mid |c| = 1\}$ and only $\pm\mathbb{1}$ belong to $\mathrm{Cl}^+(2m)$. This yields that the kernel is $\{\mathbb{1}, \mathbb{1}\}$ for $n = 2m$.

The only unitary elements of of $L(\mathbb{C}^{2^m}) \oplus L(\mathbb{C}^{2^m})$ commuting with with $\phi_i(v)$ are $\{c_1\mathbb{1} \oplus c_2\mathbb{1} \mid |c_1| = |c_2| = 1\}$ and only $\pm(\mathbb{1}, \mathbb{1})$ and $\pm(\mathbb{1}, -\mathbb{1})$ belong to $\mathrm{Cl}^+(2m+1)$. Then we use (6.30) and the fact that ω is odd. □

$U \mapsto R_U$ defines the 2-fold coverings

$$1 \to \mathbb{Z}_2 \to Spin(\mathbb{R}^n) \to SO(\mathbb{R}^n) \to 1,$$
$$1 \to \mathbb{Z}_2 \to Pin^\pm(\mathbb{R}^n) \to O(\mathbb{R}^n) \to 1.$$

6.5 Other Pin groups

The group $Pin(q,p) = Pin(\mathbb{R}^{q,p})$ is defined as the subgroup of $\mathrm{Cl}(q,p)$ generated by $\gamma(v)$ with $\gamma(v)^2 = 1$ or $\gamma(v)^2 = -1$. We have

$$1 \to \mathbb{Z}_2 \to Spin(\mathbb{R}^{q,p}) \to SO(\mathbb{R}^{q,p}) \to 1,$$
$$1 \to \mathbb{Z}_2 \to Pin^\pm(\mathbb{R}^{q,p}) \to O(\mathbb{R}^{q,p}) \to 1.$$

There are also $Pin^{\pm}(n,\mathbb{C}) = Pin^{\pm}(\mathbb{C}^n)$, which are the groups generated by $\phi(v)$ with $\phi(v)^2 = \pm 1$. We set $Spin(n,\mathbb{C}) = Pin^{+}(n,\mathbb{C}) \cap \mathrm{Cl}_0(n,\mathbb{C}) = Pin^{-}(n,\mathbb{C}) \cap \mathrm{Cl}_0(n,\mathbb{C})$. We have

$$\begin{aligned} 1 \to \mathbb{Z}_2 \to Spin(\mathbb{C}^n) \to SO(\mathbb{C}^n) \to 1, \\ 1 \to \mathbb{Z}_2 \to Pin^{\pm}(\mathbb{C}^n) \to O(\mathbb{C}^n) \to 1. \end{aligned}$$

Especially important in applications is the group

$$Pin^{\mathrm{c}}(n) := \{cU \;:\; c \in \mathbb{C},\ |c| = 1,\ U \in Pin(n)\}. \tag{6.31}$$

An equivalent characterization of $Pin^{\mathrm{c}}(n)$: it is the set of elements U of $U\big(CAR(\mathbb{R}^n)\big)$ such that

$$\{U\phi(v)U^* \;:\; v \in \mathbb{R}^n\} = \{\phi(v) \;:\; v \in \mathbb{R}^n\}.$$

We have

$$\begin{aligned} 1 \to U(1) \to Spin^{\mathrm{c}}(\mathbb{R}^n) \to SO(\mathbb{R}^n) \to 1, \\ 1 \to U(1) \to Pin^{\mathrm{c}}(\mathbb{R}^n) \to O(\mathbb{R}^n) \to 1. \end{aligned}$$

6.6 Quadratic Hamiltonians

Consider $\mathrm{Cl}(n)$. Let H_{ij} be a real antisymmetric matrix. The expressions

$$\mathrm{Op}(H) = \frac{1}{2}\sum_{ij} H_{ij}\phi_i\phi_j \tag{6.32}$$

form a Lie algebra, which is isomorphic to $o(n)$. In fact

$$\big[\mathrm{Op}(H), \mathrm{Op}(G)\big] = \mathrm{Op}\big([H,G]\big). \tag{6.33}$$

It is easy to see that $Spin(n)$ coincides with the set of $\mathrm{e}^{\mathrm{Op}(H)}$ where H are real antisymmetric matrices. In fact, it is enough to consider $\mathrm{Cl}(2)$:

$$\mathrm{e}^{t\phi_1\phi_2} = \cos t\mathbb{1} + \sin t\phi_1\phi_2 = \phi_1(\cos t\phi_1 + \sin t\phi_2). \tag{6.34}$$

6.7 Low-dimensional coincidences

Recall that $U\big(\mathrm{Cl}_0(\mathbb{R}^n)\big)$ denotes the set of unitary elements of $\mathrm{Cl}_0(\mathbb{R}^n)$, that is $V \in \mathrm{Cl}_0(n)$ satisfying $V^*V = \mathbb{1}$. Obviously,

$$Spin(\mathbb{R}^n) \subset U\big(\mathrm{Cl}_0(\mathbb{R}^n)\big) \tag{6.35}$$

Now

(6.36)

n	$\mathrm{Cl}_0(\mathbb{R}^n)$	$U\big(\mathrm{Cl}_0(\mathbb{R}^n)\big)$
1	$\mathbb{R}$	$O(\mathbb{R})$
2	$\mathbb{C}$	$U(\mathbb{C})$,
3	$\mathbb{H}$	$Sp(\mathbb{H}) = SU(\mathbb{C}^2)$,
4	$\mathbb{H} \oplus \mathbb{H}$	$Sp(\mathbb{H}) \times Sp(\mathbb{H}) = SU(\mathbb{C}^2) \times SU(\mathbb{C}^2)$,
5	$\mathbb{H}(2)$	$Sp(\mathbb{H}^2)$,
6	$\mathbb{C}(4)$	$U(\mathbb{C}^4)$,
7	$\mathbb{R}(8)$	$O(\mathbb{R}^8)$.

Proposition 6.3 *Here are the (real) dimensions of the basic classical groups:*

$$\dim SO(\mathbb{R}^n) = \frac{n(n-1)}{2}, \tag{6.37}$$
$$\dim SU(\mathbb{C}^n) = (n+1)(n-1), \quad U(\mathbb{C}^n) = n^2, \tag{6.38}$$
$$\dim Sp(\mathbb{H}^n) = n(2n+1). \tag{6.39}$$

Proof. Instead of the groups we will consider their Lie algebras.

$$o(\mathbb{R}^n) = \{A \in gl(\mathbb{R}^n) \mid A = -A^{\mathrm{T}}\}. \tag{6.40}$$

Hence each element of $o(\mathbb{R}^n)$ is determned by its strictly upper triangular part. Hence

$$\dim so(\mathbb{R}^n) = \dim o(\mathbb{R}^n) = \frac{n(n-1)}{2}. \tag{6.41}$$

Clearly, $\dim su(\mathbb{C}^n) = \dim u(\mathbb{C}^n) - 1$. Now

$$u(\mathbb{C}^n) = \{A \in gl(\mathbb{R}^n) \mid A = -A^*\}. \tag{6.42}$$

Hence each element of $u(\mathbb{C}^n)$ is determned by its (real) diagonal and (complex) strictly upper triangular part. Hence

$$\dim u(\mathbb{R}^n) = n + 2\frac{n(n-1)}{2} = n^2. \tag{6.43}$$

Finally,

$$sp(\mathbb{H}^n) = \left\{ \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \mid X^* = X,\ Y = Y^{\mathrm{T}} \right\}. \tag{6.44}$$

The dimension of possible X is n^2 by what we know about $u(\mathbb{C}^n)$. The dimension of possible Y is $2n + 2\frac{n(n-1)}{2} = n^2 + n$. Hence

$$\dim sp(\mathbb{H}^n) = 2n^2 + n. \tag{6.45}$$

□

Now table (6.36) and the above proposition yield

n	$\dim SO(n) = \dim Spin(n)$	$\dim U\big(\mathrm{Cl}_0(\mathbb{R}^n)\big)$
1	0	$0 = \dim O(1)$,
2	1	$1 = \dim U(1)$,
3	3	$3 = \dim Sp(\mathbb{H}) = \dim SU(2)$,
4	6	$6 = \dim Sp(\mathbb{H}) \times Sp(\mathbb{H}) = \dim SU(2) \times SU(2)$,
5	10	$10 = \dim Sp(\mathbb{H}^2)$,
6	15	$16 = \dim U(4)$,
7	21	$28 = \dim O(8)$.

Thus

$$\begin{aligned}
Spin(\mathbb{R}^n) &= U\big(\mathrm{Cl}_0(\mathbb{R}^n)\big), \quad n = 1, 2, 3, 4, 5, &(6.46)\\
Spin(\mathbb{R}^n) &\subsetneq U\big(\mathrm{Cl}_0(\mathbb{R}^n)\big), \quad n \geq 6. &(6.47)
\end{aligned}$$

Actually, if we consider the Jordan-Wigner representations (in the odd case, the irreducible ones), then $\phi(v)$, the generators of Clifford algebras $\mathrm{Cl}(n)$ have determinant 1 starting from $n = 4$. But they also generate the Pin group. So all elements of the $Pin(n)$ have determinant 1 for $n \geq 4$. Therefore, for $n = 6$ we can write $Spin(6) \subset SU(4)$. We have $\dim SU(4) = 15$. Hence $Spin(6) = SU(4)$. Thus we obtain the coincidences in low dimensions:

$$\begin{array}{rcl}
Spin(\mathbb{R}^2) & \simeq & SO(\mathbb{R}^2),\\
Spin(\mathbb{R}^3) & \simeq & SU(\mathbb{C}^2),\\
Spin(\mathbb{R}^4) & \simeq & SU(\mathbb{C}^2) \times SU(\mathbb{C}^2),\\
Spin(\mathbb{R}^5) & \simeq & Sp(\mathbb{H}^2),\\
Spin(\mathbb{R}^6) & \simeq & SU(\mathbb{C}^4).
\end{array}$$

6.8 $SL(\mathbb{R}^2) = Sp(\mathbb{R}^2)$

Let

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that for $A \in L(\mathbb{R}^2)$ we have

$$\begin{aligned}
A^{\mathrm{T}} J A &= (\det A) J,\\
A^{\mathrm{T}} J + J A &= (\mathrm{Tr} A) J.
\end{aligned}$$

Hence $SL(\mathbb{R}^2) = Sp(\mathbb{R}^2)$ and $sl(\mathbb{R}^2) = sp(\mathbb{R}^2)$. For later reference note the following identities for 2×2 matrices:

$$\frac{1}{2}\mathrm{Tr} J A^{\mathrm{T}} J A = -\det A, \quad \mathrm{Tr} A = 0 \quad \Rightarrow \quad J A^{\mathrm{T}} J = A. \tag{6.48}$$

6.9 $SU(2) \simeq Spin(3)$

We can show directly that $SU(2) \simeq Spin(3)$ using the Jordan-Wigner representation:

$$\begin{aligned}
Spin(\mathbb{R}^3) &= \{a_0 \mathbb{1} + a_1 \phi_2 \phi_3 + a_2 \phi_3 \phi_1 + a_3 \phi_1 \phi_2 \ : \ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\}, &(6.49)\\
SU(2) &= \{a_0 \mathbb{1} + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 \ : \ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\}. &(6.50)
\end{aligned}$$

The following construction allows to see directly the 2-fold covering $SU(2) \to SO(3)$. Identify $\mathbb{R}^3$ with Hermitian matrices 2×2 of trace 0:

$$\mathbb{R}^3 \ni (x, y, z) \mapsto X = \begin{bmatrix} z & x + \mathrm{i}y \\ x - \mathrm{i}y & -z \end{bmatrix}.$$

Note that

$$\frac{1}{2}\mathrm{Tr}X_1X_2 = x_1x_2 + y_1y_2 + z_1z_2$$

defines the standard scalar product. Alternatively, the scalar product can be defined through the determinant:

$$-\det X = x^2 + y^2 + z^2.$$

For $A \in SU(2)$ we set

$$\rho_A X := AXA^*.$$

Then

$$\det \rho_A X = \det X.$$

Hence ρ_A preserves the scalar product.

$$SU(2) \ni A \mapsto \rho_A \in SO(3).$$

is a surjective homomorpism. Its kernel is $\{\mathbb{1}, -\mathbb{1}\}$.

For $Y \in su(2)$ we set

$$\rho_Y X := YX + XY^* = [Y, X].$$

(Note that $Y = -Y^*$).

6.10 $SL(2,\mathbb{R}) \simeq Spin_0(1,2)$

We identify $\mathbb{R}^3$ with 2×2 matrices of trace 0:

$$\mathbb{R}^3 \ni (x, y, z) \mapsto X = \begin{bmatrix} z & x+y \\ -x+y & -z \end{bmatrix}.$$

Note that

$$\frac{1}{2}\mathrm{Tr}X_1X_2 = -x_1x_2 + y_1y_2 + z_1z_2$$

Hence for $\mathbb{R} = \mathbb{R}$ we obtain a pseudoscalar product of signature $(1, 2)$. Alternatively, we can use the determinant

$$-\det X = -x^2 + y^2 + z^2.$$

For $A \in SL(2,\mathbb{R})$ we set

$$\rho_A X := AXA^{-1}.$$

Then

$$\det \rho_A X = \det X.$$

Hence ρ_A preserves the (pseudo-)scalar product

$$SL(2,\mathbb{R}) \ni A \mapsto \rho_A \in SO_0(1,2),$$
$$SL(2,\mathbb{C}) \ni A \mapsto \rho_A \in SO(3,\mathbb{C}),$$

are surjective homomorphisms. Their kernel is $\{\mathbb{1}, -\mathbb{1}\}$.

For $Y \in sl(2,\mathbb{R})$,

$$\rho_Y X := [Y, X].$$

6.11 $SL(2,\mathbb{C}) \simeq Spin_0(1,3)$

We identify $\mathbb{R}^4$ with 2×2 Hermitian matrices

$$\mathbb{R}^4 \ni (t,x,y,z) \mapsto X = \begin{bmatrix} t+z & x+\mathrm{i}y \\ x-\mathrm{i}y & t-z \end{bmatrix}.$$

Note that

$$\begin{aligned} \frac{1}{2}\mathrm{Tr} JX_1JX_2 &= -t_1t_2 + x_1x_2 + y_1y_2 + z_1z_2, \\ -\det X &= -t^2 + x^2 + y^2 + z^2. \end{aligned}$$

Hence we obtain a pseudoscalar product of signature $(1,3)$.

For $A \in SL(2,\mathbb{C})$ we set

$$\rho_A X := AXA^*.$$

Then

$$\det \rho_A X = \det X.$$

Hence ρ_A preserves the pseudoscalar product.

$$SL(2,\mathbb{C}) \ni A \mapsto \rho_A \in SO_0(1,3).$$

is a surjective homorphism. Its kernel is $\{\mathbb{1}, -\mathbb{1}\}$.

6.12 $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \simeq Spin_0(2,2)$

We identify $\mathbb{R}^4$ with 2×2 matrices:

$$\mathbb{R}^4 \ni (t,x,y,z) \mapsto X = \begin{bmatrix} t+z & x+y \\ x-y & t-z \end{bmatrix}.$$

Note that

$$\begin{aligned} \frac{1}{2}\mathrm{Tr} JX_1JX_2 &= -t_1t_2 + x_1x_2 - y_1y_2 + z_1z_2, \\ -\det X &= -t^2 + x^2 - y^2 + z^2. \end{aligned}$$

Hence we obtain a pseudo-scalar product of signature $(2,2)$.

For $(A,B) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ we set

$$\rho_{(A,B)} X := AXB^{-1}.$$

Then

$$\det \rho_{(A,B)} X = \det X.$$

Hence $\rho_{(A,B)}$ preserves the pseudoscalar product.

$$\begin{aligned} SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \ni (A,B) &\mapsto \rho_{(A,B)} \in SO_0(2,2), \\ SL(2,\mathbb{C}) \times SL(2,\mathbb{R}) \ni (A,B) &\mapsto \rho_A \in SO(4,\mathbb{C}), \end{aligned}$$

are surjective homomorphsims with kernel $\{\mathbb{1}, -\mathbb{1}\}$.

6.13 $SU(2) \times SU(2) \simeq Spin(4)$

Let $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Identify $\mathbb{R}^4$ with complex matrices 2×2 satisfying $J\overline{X} = XJ$ (or quaternions) as follows:

$$\mathbb{R}^4 \ni (t, x, y, z) \mapsto X = \begin{bmatrix} t + \mathrm{i}z & \mathrm{i}x + y \\ \mathrm{i}x - y & t - \mathrm{i}z \end{bmatrix}.$$

Note that

$$\frac{1}{2}\mathrm{Tr}X_1^* X_2 = t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2,$$
$$\det X = t^2 + x^2 + y^2 + z^2,$$

defines the standard scalar product.

For $(A, B) \in SU(2) \times SU(2)$ we set

$$\rho_{(A,B)} X := AXB^*.$$

Then

$$\det \rho_{(A,B)} X = \det X.$$

Hence $\rho_{(A,B)}$ preserves the scalar product.

$$SU(2) \times SU(2) \ni (A, B) \mapsto \rho_{(A,B)} \in SO(4).$$

is a surjective homorphism. Its kernel is $\{(\mathbb{1}, \mathbb{1}), -(\mathbb{1}, \mathbb{1})\}$.

7 Slater determinants and CAR representations

7.1 Reminder about fermionic Fock spaces

Let $\mathcal{W}$ be a Hilbert space. We consider the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{W})$. Recall that for $\in \mathcal{W}$ we have creation/annihilation operators $a^*(w)$, $a(w) = \big(a^*(w)\big)^*$ satisfying

$$[a(w), a(w')]_+ = 0, \quad [a(w), a_j^*(w')]_+ = (w|w'), \quad a(w)\Omega = 0. \tag{7.51}$$

If $e_1, e_2, \dots$ is a basis of $\mathcal{W}$, then we often use a different notation

$$a_i := a(e_i), \quad a_i^* := a^*(e_i),$$

$$a(w) = \sum \overline{w}_i a_i, \quad a^*(w) = \sum w_i a_i^*,$$

$$[a_i, a_j]_+ = 0, \quad [a_i, a_j^*]_+ = \delta_{i,j}, \quad a_i\Omega = 0. \tag{7.52}$$

The vectors $a_{i_1}^* \cdots a_{i_n}^* \Omega$, $i_1 < \cdots < i_n$ form an orthonormal basis of $\Gamma_{\mathrm{a}}(\mathcal{W})$.

7.2 Slater determinants

Let $\mathcal{W}$ be a Hilbert space. We consider the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{W})$.

Consider an orthogonal finite dimensional projection π on $\mathcal{W}$ and let $e_1, \dots, e_m$ be an orthonormal basis of $\pi\mathcal{W}$. Then

$$\Phi = a^*(e_1)\cdots a^*(e_m)\Omega = \frac{1}{\sqrt{m!}}\sum_{\sigma\in S_m}\mathrm{sgn}\sigma e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(m)} \tag{7.53}$$

$$= \sqrt{m!}e_1\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_m = \frac{1}{\sqrt{m!}}e_1\wedge\cdots\wedge e_m \tag{7.54}$$

is a normalized vector. Such vectors are called *Slater determinants*. If $f_1, \dots, f_m$ is another basis of $\pi\mathcal{W}$, so that $e_i = \sum_j c_{ij} f_j$, then

$$a^*(e_1)\cdots a^*(e_m)\Omega = \det[c_{ij}]a^*(f_1)\cdots a^*(f_m)\Omega.$$

Thus the state

$$\omega_\pi(A) := (\Phi|A\Phi)$$

depends only on π.

7.3 Changing the vacuum

Consider a basis $e_1, e_2, ..$ and the Slater determinant made of the first m vectors:

$$\Phi := a_1^*\cdots a_m^*\Omega. \tag{7.55}$$

It satisfies

$$a_i^*\Phi = 0, \quad i = 1, \dots, m, \qquad a_j\Phi = 0, \quad j = m+1, \dots, \tag{7.56}$$

A *conjugation* is an antilinear operator C such that $C^2 = 1\!\!1$. Let us fix a conjugation on $\pi\mathcal{W}$ such that $Ce_i = e_i$, $i = 1, \dots, m$. Thus

$$\mathbb{C}^n \ni w = \sum w_n e_n \mapsto Cw := \sum \overline{w}_i e_i \in \mathcal{W}.$$

Then we can set

$$\tilde{a}(w) := a^*(C\pi w) + a((1-\pi)w), \tag{7.57}$$

$$\tilde{a}(w) := a(C\pi w) + a^*((1-\pi)w). \tag{7.58}$$

Then $\tilde{a}(w)$, $\tilde{a}^*(w)$ satisfy the usual commutation relations with vacuum Φ

$$[\tilde{a}(w), \tilde{a}(w')]_+ = 0, \quad [\tilde{a}(w), \tilde{a}_j^*(w')]_+ = (w|w'), \quad \tilde{a}(w)\Phi = 0. \tag{7.59}$$

The vectors $\tilde{a}_{i_1}^*\cdots\tilde{a}_{i_n}^*\Omega$, $i_1 < \cdots < i_n$ form an orthonormal basis of $\Gamma_{\mathrm{a}}(\mathcal{W})$. Thus instead the space $\Gamma_a(\mathcal{W})$ is isomorphic to the space $\Gamma\big(C\pi\mathcal{W}\oplus(1-\pi)\mathcal{W}\big)$.

Often one renames

$$b_i := a_i^*, \quad b_i^* := a_i, \quad i = 1, \dots, m, \tag{7.60}$$

so that

$$\tilde{a}_i := \begin{cases} b_i & i \leq n, \\ a_j & j > n; \end{cases} \quad \tilde{a}_i^* := \begin{cases} b_i^* & i \leq n, \\ a_j^* & j > n. \end{cases}, \tag{7.61}$$

We can implement this change by a unitary transformation: Set

$$U := \begin{cases} \prod_{i=1}^m (a_i^* - a_i), & m \text{ is even;} \\ \prod_{i=1}^m (a_i^* - a_i)(-1)^N, & m \text{ is odd.} \end{cases} \tag{7.62}$$

U is unitary and satisfies

$$Ua^*(w)U^* = \tilde{a}^*(w), \qquad Ua(w)U^* = \tilde{a}(w), \qquad U\Omega = \Phi. \tag{7.63}$$

In fact, using

$$(a^* - a)a(a - a^*) = -a^*aa^* = -a^*(aa^* + a^*a) = -a^*, \tag{7.64}$$

$$(-1)^N a_i (-1)^N = -a_i, \tag{7.65}$$

we see that

$$Ua_i^*U^* = b_i, \quad i = 1, \dots, m; \tag{7.66}$$

$$Ua_i^*U^* = a_i^*, \quad i = m+1, \dots \tag{7.67}$$

$$U\Omega = \Phi. \tag{7.68}$$

7.4 Free fermionic Hamiltonians

For simplicity, assume that $\mathcal{W}$ is finite dimensional. Consider a self-adjoint operator h on $\mathcal{W}$. It can be diagonalized, so that

$$h = \sum_i \lambda_i |e_i)(e_i|.$$

Consider the Hamiltonian

$$H = \mathrm{d}\Gamma(h) = \sum_i \lambda_i a_i^* a_i^*. \tag{7.69}$$

It is easy to see that $\mathrm{d}\Gamma(h)$ possesses a unique ground state iff $0 \notin \sigma(h)$. Indeed, let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m < 0 < \lambda_{m+1} \leq \dots$. Then the ground state of $\mathrm{d}\Gamma(h)$ is given by

$$\Phi := a_1^* \cdots a_m^* \Omega,$$

so that

$$H\Phi = E\Phi, \quad E = \lambda_1 + \cdots + \lambda_m.$$

Setting

$$b_i = a_i^*, \qquad b_i^* = a_i, \qquad i = 1, \dots, m, \tag{7.70}$$

the Hamiltonian H can be rewritten as

$$H = \sum_{i \leq m} |\lambda_i| b_i^* b_i + \sum_{i > m} \lambda_i a_i^* a_i + \sum_{i \leq m} \lambda_i.$$

We can view this as the Hamiltonian on the Fock space $\Gamma_{\mathrm{a}}(C\pi\mathcal{W} \oplus (\mathbb{1} - \pi)\mathcal{W})$, and treat Φ as the new vacuum, b_i, resp. b_i^*, $i = 1, \dots, m$, as new annihilation/creation operators.

Note that strictly speaking this construction makes sense only for a finite dimensional $\mathbb{1}_{]-\infty,0]}(h)$. However, it is often used also if this dimension is infinite. The constant E is usually dropped—it is often in fact infinite, and we use then the renormalized Hamiltonian

$$H_{\mathrm{ren}} = \sum_{i \leq m} |\lambda_i| b_i^* b_i + \sum_{i > m} \lambda_i a_i^* a_i.$$

Example 7.1 *Consider the free Fermi gas with the chemical potential μ in volume L.*

$$H = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} (k^2 - \mu) a_k^* a_k.$$

The ground state is called the Fermi sea $\prod_{k^2 < \mu} a_k^* \Omega$. *It has the energy*

$$E = \sum_{k^2 < \mu} (k^2 - \mu).$$

The renormalized Hamiltonian is

$$H_{\mathrm{ren}} = \sum_{k^2 < \mu} |k^2 - \mu| b_k^* b_k + \sum_{k^2 \geq \mu} (k^2 - \mu) a_k^* a_k.$$

In infinite volume the Hamiltonian is

$$H = \int (k^2 - \mu) a_k^* a_k \mathrm{d}k.$$

E is infinite and the Slater determinant is ill defined. However, we can change the representation of CAR replacing H with

$$H_{\mathrm{ren}} = \int_{k^2 < \mu} |k^2 - \mu| b_k^* b_k \mathrm{d}k + \int_{k^2 \geq \mu} (k^2 - \mu) a_k^* a_k \mathrm{d}k.$$

Example 7.2 *Let $\alpha_i, , i = 1, 2, 3, \beta$ satisfy Clifford relations. Consider the Dirac Hamiltonian*

$$h := \vec{\alpha}\vec{p} + \beta m.$$

It is a self-adjoint operator on $L^2(\mathbb{R}^3 \otimes \mathbb{C}^4)$. The operator h has spectrum $\sigma(h) =]-\infty, -m] \cup [m, \infty[$. In fact, one easily shows that $h^2 = p^2 + m^2$, hence $\sigma(h^2) = [m^2, \infty[$, and there exists a distinguished conjugation C, called the charge conjugation, such that $ChC = -h$.

The naive quantization of h, that is $\mathrm{d}\Gamma(h)$, acts on the space $\Gamma_{\mathrm{a}}\big(L^2(\mathbb{R}^3 \otimes \mathbb{C}^4)\big)$. It is however physically meaningless—it yields an operator unbounded from below. Formally, the ground state of $\mathrm{d}\Gamma(h)$ is the Slater determinant with all negative energy states present. This state is called the Dirac sea.

In practice, we change the representation of CAR. Set

$$\Lambda^{\pm} := \mathbb{1}_{[0,\infty[}(\pm h).$$

On can take

$$C\Lambda^{-}L^2(\mathbb{R}^3 \otimes \mathbb{C}^4) \oplus \Lambda^{+}L^2(\mathbb{R}^3 \otimes \mathbb{C}^4)$$

as the physical one particle space.

Example 7.3 *Let us continue with the previous example. Let add a scalar and magnetic potential:*

$$h := \vec{\alpha}\big(\vec{p} - A(x)\big) + \beta m + V(x).$$

$\sigma(h)$ is still unbounded from below. (One does not have $ChC = -h$ though). Suppose that $0 \notin \sigma(h)$. Then one can repeat the previous construction.

7.5 Representations of CAR

Let $\big(\mathcal{V}, \langle\cdot|\cdot\rangle\big)$ be a real Hilbert space. Let $\mathcal{H}$ be a complex Hilbert space. We say that

$$\mathcal{V} \ni v \mapsto \phi_{\bullet}(v) \in B(\mathcal{H}) \tag{7.71}$$

is a *representation of Canonical Anticommutation Relations* (over $\mathcal{V}$ in $\mathcal{H}$) or a *CAR representation* if

$$\phi_{\bullet}(v)^* = \phi_{\bullet}(v), \qquad [\phi_{\bullet}(v), \phi_{\bullet}(w)]_+ = 2\langle v|w\rangle. \tag{7.72}$$

We say that it is irreducible if there are no closed subspaces in $\mathcal{H}$ invariant wrt $\phi_{\bullet}(v)$. We say that two representations of CAR $\mathcal{V} \ni v \mapsto \phi_i(v) \in B(\mathcal{H}_i)$, $i = 1, 2$, are equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U\phi_1(v) = \phi_2(v)U, \quad v \in \mathcal{V}. \tag{7.73}$$

Example 7.4 *If $\dim \mathcal{V}$ is even, then all irreducible CAR representations over $\mathcal{V}$ are equivalent, and given by the Jordan-Wigner (or Fock) construction. If $\dim \mathcal{V}$ is odd, then there exist two inequivalent irreducible CAR representations over $\mathcal{V}$.*

Example 7.5 *Let $\mathcal{V}$ be infinite dimensional with an o.n. basis $f_{i,j}$, $i = 1, 2,$, $j = 1, 2, \dots$. Consider a space $\mathcal{W}$ with basis e_j, $j = 1, 2 \dots$. We set*

$$\phi_{\bullet}(f_{1,j}) := a_j^* + a_j, \quad \phi_{\bullet}(f_{2,j}) := \frac{1}{\mathrm{i}}(a_j^* - a_j). \tag{7.74}$$

Then we obtain a CAR representation over $\mathcal{V}$ in $\Gamma_{\mathrm{a}}(\mathcal{W})$.

Example 7.6 *Let us continue with the previous example. Let* $I \subset \mathbb{N}$.

$$\phi_\bullet(f_{1,j}) := a_j^* + a_j, \quad \phi_\bullet(f_{2,j}) := \begin{cases} \frac{1}{\mathrm{i}}(a_j - a_j^*), & j \in I; \\ \frac{1}{\mathrm{i}}(a_j^* - a_j), & j \notin I. \end{cases} \tag{7.75}$$

Then we also obtain a CAR representation. We will say that it is a representation in $\Gamma_{\mathrm{a}}\big(C\pi\mathcal{W} \oplus (\mathbb{1}-\pi)\mathcal{W}\big)$, *where* π *is the projection in* $\mathcal{W}$ *onto* $\mathrm{Span}\{e_j \mid j \in I\}$. *Note that both representations are equivalent iff* π *is finite dimensional.*

7.6 CAR C^*-algebra

$\mathfrak{A}$ is a C^*-algebra if it is a Banach $*$algebra satisfying $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2$. ω is a state on $\mathfrak{A}$ if it is a functional on $\mathfrak{A}$ such that $\omega(A^*A) \geq 0$ and $\omega(\mathbb{1}) = 1$. $\pi : \mathfrak{A} \to B(\mathcal{H})$ is a $*$-representation if it is a $*$-homomorphism.

Every closed $*$-algebra in $B(\mathcal{H})$ is a C^*-algebra. Every functional of the form $A \mapsto \mathrm{Tr}A\rho$, where $\mathrm{Tr}\rho = 1$, $\rho \geq 0$ is a state.

Let $\big(\mathcal{V}, \langle\cdot|\cdot\rangle\big)$ be a real Hilbert space. If $\mathcal{V}$ is finite dimensional, then the CAR algebra $\mathrm{CAR}(\mathcal{V})$ was defined before (as the Clifford algebra $\mathrm{Cl}(\mathbb{C}\mathcal{V})$ generated by $\phi(v)$ $v \in \mathcal{V}$, equipped with the involution such that $\phi(v)^* = \phi(v)$).

Assume now that $\mathcal{V}$ separable and infinite dimensional. We can associate with it the algebra $\mathrm{CAR}(\mathcal{V})$ as follows. We choose an o.n. basis f_i, $i = 1, 2, \dots$. Let $\mathcal{V}_m := \mathrm{Span}(f_1, \dots, f_{2m})$. The Jordan-Wigner construction yields $\mathrm{CAR}(\mathcal{V}_m) = B(\otimes^m \mathbb{C}^2)$, as the algebra generated by elements $\phi(v)$, $v \in \mathcal{V}_m$. We identify

$$\mathrm{CAR}(\mathcal{V}_m) = B(\otimes^m \mathbb{C}^2) \ni A \mapsto A \otimes \mathbb{1}_{\mathbb{C}^2} \in B(\otimes^{m+1}\mathbb{C}^2) = \mathrm{CAR}(\mathcal{V}_{m+1}). \tag{7.76}$$

Thus $\mathrm{CAR}(\mathcal{V}_m)$ is an ascending sequence of C^*-algebras. We can define the algebra

$$\mathrm{CAR}_0(\mathcal{V}) := \bigcup_{j=1}^{\infty} \mathrm{CAR}(\mathcal{V}_m). \tag{7.77}$$

It is a normed $*$-algebra. We can take its completion

$$\mathrm{CAR}(\mathcal{V}) := \mathrm{CAR}_0(\mathcal{V})^{\mathrm{cpl}}. \tag{7.78}$$

It is a C^*-algebra with distinguished elements $\phi(v)$ satisfying

$$\phi(v)^* = \phi(v), \qquad [\phi(v), \phi(w)]_+ = 2\langle v|w\rangle, \qquad v, w \in \mathcal{V} \tag{7.79}$$

If $\mathcal{V} \ni v \mapsto \phi_\bullet(v) \in B(\mathcal{H})$ is a CAR representation, then we have a $*$representation of the C^*algebra $\rho_\bullet : \mathrm{CAR}_0(\mathcal{V}) \to B(\mathcal{H})$ defined by

$$\rho_\bullet\big(\phi(v)\big) = \phi_\bullet(v). \tag{7.80}$$

This representation extends uniquely by continuity to a representation $\rho_\bullet : \mathrm{CAR}(\mathcal{V}) \to B(\mathcal{H})$.

Thus given the formalism of CAR representations is essentially equivalent to the formalism of representations of CAR C^*-algebras.

8 Quantum gases and the Hartree-Fock method

8.1 Particle number preserving operators

Let $b : \otimes^k \mathcal{Z} \to \otimes^m \mathcal{Z}$. (We do not require that b preserves the symmetric/antisymmetric tensor product). Recall that the Wick quantization of b, denoted $b(\hat{a}^*, \hat{a})$, can be defined as follows. Its only nonzero matrix elements are between $\Phi \in \otimes_{\rm s/a}^{p+m} \mathcal{Z}$, $\Psi \in \otimes_{\rm s/a}^{p+k} \mathcal{Z}$, $p = 0, 1, \dots$ and are equal

$$(\Phi|b(\hat{a}^*, \hat{a})\Psi) \quad = \quad \frac{\sqrt{(m+p)!(k+p)!}}{p!}(\Phi|b \otimes 1_{\mathcal{Z}}^{\otimes p}\Psi). \tag{8.81}$$

Clearly, $b(\hat{a}^*, \hat{a})$ depends only on $\Theta_{\rm s/a}^m b \Theta_{\rm s/a}^k$, but it is convenient to allow for b that are not (anti-)symmetric.

Suppose now that $k = m$, that is, $b : \otimes_{\rm s/a}^m \mathcal{Z} \to \otimes_{\rm s/a}^m \mathcal{Z}$. Then the operator $b(\hat{a}^*, \hat{a})$ preserves the number of particles. For $\Phi \in \otimes_{\rm s/a}^n \mathcal{Z}$, $\Psi \in \otimes_{\rm s/a}^n \mathcal{Z}$ (8.81) can be rewritten as

$$(\Phi|b(\hat{a}^*, \hat{a})\Psi) \quad = \quad \frac{n!}{(n-m)!}(\Phi|b \otimes 1_{\mathcal{Z}}^{\otimes (n-m)}\Psi). \tag{8.82}$$

Suppose $1 \leq i_1 < \cdots i_m < n$. We will write $b_{i_1,\dots,i_m}$ for the operator b acting on $\otimes^n \mathcal{Z}$ whose "legs" are put at the slots $i_1, \dots, i_m$. For instance,

$$b_{1,\dots,m} = b \otimes \mathbb{1}^{\otimes(n-m)}. \tag{8.83}$$

Now $\frac{n!}{(n-m)!m!}$ is the number of m-element subsets of $\{1, 2, \dots, n\}$. Therefore we can rewrite (8.82) as

$$\frac{1}{m!}\big(\Phi|b(\hat{a}^*, \hat{a})\Psi\big) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \big(\Phi|b_{i_1,\dots,i_m}\Psi\big). \tag{8.84}$$

If in addition $b = \Theta(\sigma) b \Theta(\sigma)^{-1}$ for $\sigma \in S_n$, then b preserves $\otimes_{\rm s/a}^m \mathcal{Z}$ and we can write

$$\frac{1}{m!}b(\hat{a}^*, \hat{a}) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} b_{i_1,\dots,i_m} \quad \text{restricted to } \otimes_{\rm s/a}^m \mathcal{Z}. \tag{8.85}$$

In particular, for $m = 1$, we obtain the identity that we know:

$$b(\hat{a}^*, \hat{a}) = \sum_{1 \leq i \leq n} b_i = \mathrm{d}\Gamma(b). \tag{8.86}$$

The case $m = 2$ is especially important in applications:

$$\frac{1}{2}b(\hat{a}^*, \hat{a}) = \sum_{1 \leq i < j \leq n} b_{ij}. \tag{8.87}$$

8.2 N-body Schrödinger Hamiltonians

Consider the *N body Schrödinger Hamiltonian* and the corresponding *total momentum*. They are operators on $\otimes^n L^2(\mathbb{R}^d) \simeq L^2(\mathbb{R}^{dn})$

$$H_n = -\sum_{i=1}^{n} \frac{1}{2m_i}\Delta_i + \sum_{1\leq i<j\leq n} V_{ij}(x_i - x_j), \tag{8.88}$$

$$P_n = \sum_{i=1}^{n} \frac{1}{\mathrm{i}}\partial_{x_i}, \tag{8.89}$$

In the momentum representation

$$\begin{aligned} H_n &= \sum_{i=1}^{n} \frac{1}{2m_i} p_i^2 + (2\pi)^{-d} \sum_{1\leq i<j\leq N} \delta(p_i' + p_j' - p_j - p_i)\hat{V}_{ij}(p_i' - p_i). \\ P_n &= \sum_{i=1}^{n} p_i. \end{aligned}$$

Clearly, $[H_n, P_n] = 0$.

If the particles are identical, then m_i are the same, which for simplicity we assume to be $\frac{1}{2}$, and $V_{ij}(x) = V(x) = V(-x)$. We can then restrict the Hamiltonian and total momentum to $\otimes^n_{\mathrm{s/a}} L(\mathbb{R}^d) \simeq L^2_{\mathrm{s/a}}\big((\mathbb{R}^d)^N\big)$. Then we can use the 2nd quantized formalism on the Fock space $\Gamma_{\mathrm{s/a}}\big(L^2(\mathbb{R}^d)\big)$. We have the position representation, with the generic variables x, y and the momentum representation with the generic variables k, k'. We can pass from one representation to the other by

$$a^*(k) = (2\pi)^{-\frac{d}{2}} \int a^*(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x, \qquad a^*(x) = (2\pi)^{-\frac{d}{2}} \int a^*(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k, \tag{8.90}$$

$$a(k) = (2\pi)^{-\frac{d}{2}} \int a(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x, \qquad a(x) = (2\pi)^{-\frac{d}{2}} \int a(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k. \tag{8.91}$$

In the 2nd quantized notation we can rewrite all this as

$$H := \mathop{\oplus}_{n=0}^{\infty} H_n = -\int a_x^*\Delta_x a_x \mathrm{d}x + \frac{1}{2}\int\int \mathrm{d}x\mathrm{d}y V(x-y) a_x^* a_y^* a_y a_x$$

$$= \int p^2 a_p^* a_p \mathrm{d}p + \int\int\int\int \delta(p_1' + p_2' - p_1 - p_2)\frac{\hat{V}(p_1' - p_1)}{(2\pi)^d} a_{p_1'}^* a_{p_2'}^* a_{p_2} a_{p_1} \mathrm{d}p_1'\mathrm{d}p_2'\mathrm{d}p_1\mathrm{d}p_2 \tag{8.92}$$

$$= \int p^2 a_p^* a_p \mathrm{d}p + \frac{1}{2(2\pi)^d}\int\int\int \mathrm{d}p\mathrm{d}q\mathrm{d}k \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p, \tag{8.93}$$

$$P := \mathop{\oplus}_{n=0}^{\infty} P_n = \int a_x^* \frac{1}{\mathrm{i}}\partial_x a_x \mathrm{d}x \tag{8.94}$$

$$= \int p a_p^* a_p \mathrm{d}p. \tag{8.95}$$

Passing from (8.92) to (8.93) we evaluate the delta and set $p = p_1$, $q = p_2$, $k = p_1' - p_1$.

Let us now put our system in a box of size L with periodic boundary conditions. That is, we consider $L^2([0,L]^d) \simeq L^2\Big(\frac{2\pi}{L}\mathbb{Z}^d\Big)$ and its 2nd quantization. Again we use x, y in the position representation with periodic boundary conditions and k, k' in the momentum representation. We can pass from one representation to the other by

$$a^*(k) = L^{-\frac{d}{2}} \int a(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x, \qquad a^*(x) = L^{-\frac{d}{2}} \sum_k a(k) \mathrm{e}^{\mathrm{i}kx}, \tag{8.96}$$

$$a(k) = L^{-\frac{d}{2}} \int a(x) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x, \qquad a(x) = L^{-\frac{d}{2}} \sum_k a(k) \mathrm{e}^{-\mathrm{i}kx}. \tag{8.97}$$

Here are the analogs of (8.93) and (8.95):

$$\begin{aligned} H &= \sum_p p^2 a_p^* a_p + \frac{1}{2L^d} \sum_p \sum_q \sum_k \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p, \\ P &= \sum_p p a_p^* a_p. \end{aligned}$$

8.3 Hartree-Fock method for atomic systems

Suppose now that $V(x) = V(-x)$ and

$$H = -\int a_x^* \Delta_x a_x \mathrm{d}x + \int a_x^* W(x) a_x \mathrm{d}x + \frac{1}{2} \int\int a_x^* a_y^* V(x-y) a_x a_y \mathrm{d}x \mathrm{d}y. \tag{8.98}$$

Let π be an n-dimensional projection in $L^2(\mathbb{R}^d)$. Let $\pi(x,y)$ be its integral kernel and $\rho(x) := \pi(x,x)$ its diagonal. Let ω_π be the state defined by the Slater determinant corresponding to π. The Hartree-Fock functional is defined as

$$\mathcal{E}_{\mathrm{HF}}(\pi) := \omega_\pi(H). \tag{8.99}$$

Clearly, $\mathcal{E}_{\mathrm{HF}}(\pi)$ is an upper bound of the ground state energy of H. Here is an explicit formula for the Hartree-Fock functional:

$$\begin{aligned} \mathcal{E}_{\mathrm{HF}}(\pi) = &\int \partial_x \partial_y \pi(x,y)\Big|_{x=y} \mathrm{d}x + \int W(x)\rho(x)\mathrm{d}x \\ &+ \frac{1}{2}\int\int V(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int\int V(x-y)|\pi(x,y)|^2 \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{8.100}$$

To see this choose $f_1, \dots, f_n$, an o.n. basis of $\mathrm{Ran}\pi$. Then

$$\begin{aligned} &-\omega_\pi\Big(\int a_x^* \Delta_x a_x \mathrm{d}x\Big) \\ =&\int \big(\nabla_x a_x a^*(f_1)\cdots a^*(f_n)\Omega | \nabla_x a_x a^*(f_1)\cdots a^*(f_n)\Omega\big) \mathrm{d}x \\ =&\sum_{j=1}^n \int \overline{\nabla_x f_j(x)} \nabla_x f_j(x) \mathrm{d}x = \int \partial_x \partial_y \pi(x,y)\Big|_{x=y} \mathrm{d}x. \end{aligned}$$

$$
\begin{aligned}
&\omega_\pi\Big(\int\int a_x^* a_y^* V(x-y) a_y a_x \mathrm{d}x\Big)\\
=&\int\int \mathrm{d}x\mathrm{d}y V(x-y)\big(a_x a_y a^*(f_1)\cdots a^*(f_n)\Omega | \big(a_x a_y a^*(f_1)\cdots a^*(f_n)\Omega\big)\\
=&\sum_{i\neq j}\int\int \mathrm{d}x\mathrm{d}y V(x-y)\Big(\overline{f_i(x)f_j(y)}f_j(y)f_i(x)-\overline{f_i(x)f_j(y)}f_i(y)f_j(x)\Big)\\
=&\int\int \mathrm{d}x\mathrm{d}y V(x-y)\Big(\sum|f_i(x)|^2|f_j(y)|^2-\Big|\sum\overline{f_i(x)}f_i(y)\Big|^2\Big)\\
=&\int\int V(x-y)\mathrm{d}x\mathrm{d}y\big(\pi(x,x)\pi(y,y)-|\pi(x,y)|^2\big)
\end{aligned}
$$

Suppose π_{HF} is a minimizer of $\mathcal{E}_{\mathrm{HF}}$ and $\rho_{\mathrm{HF}}(x) := \pi_{\mathrm{HF}}(x,x)$. Then we can also define the Hartree-Fock Hamiltonian:

$$
h_{\mathrm{HF}} = -\Delta + W(x) + \int \rho_{\mathrm{HF}}(y)V(x-y)\mathrm{d}y - T_{\mathrm{ex}},
$$

where T_{ex} is a nonlocal operator with the kernel

$$
T_{\mathrm{ex}}(x,y) = V(x-y)\pi_{\mathrm{HF}}(x,y).
$$

We will show later that π_{HF} is the projection onto n lowest levels of h_{HF}.

8.4 Thomas-Fermi functional

A semiclassical argument implies that the first term in (8.100), that is the kinetic energy, can be approximated by

$$
(2\pi)^{-d}\frac{d}{d+2}c_d^{-2/d}\int \rho^{\frac{d+2}{d}}(x)\mathrm{d}x, \tag{8.101}
$$

where c_d is the volume of a unit ball in d dimensions. We also expect that the last term, that is the exchange energy is relatively small. This leads to the so-called Thomas-Fermi functional, which depends only on the density:

$$
\begin{aligned}
\mathcal{E}_{\mathrm{TF}}(\rho) :=&(2\pi)^{-d}\frac{d}{d+2}c_d^{-2/d}\int \rho^{\frac{d+2}{d}}(x)\mathrm{d}x\\
&+\int W(x)\rho(x)\mathrm{d}x + \frac{1}{2}\int\int V(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

In practice the Thomas-Fermi functional is often applied to atomic systems, where $d=3$, $W(x) = -\frac{Z}{|x|}$ and $V(x) = \frac{1}{|x|}$.

8.5 Expectation values of Slater determinants

The arguments from the previous subsection about the expectation values of Slater determinants can be generalized to a more abstract setting.

Theorem 8.1 *Let b be an operator on $\otimes^m \mathcal{W}$. Let π be a projection onto a subspace of $\mathcal{W}$. Then*

$$\omega_\pi\big(b(a^*, a)\big) = \sum_{\sigma \in S_m} \operatorname{Tr} b\, \pi^{\otimes m}\, \Theta(\sigma)\operatorname{sgn}(\sigma).$$

Proof. Suppose that ω is given by $a_1^* \cdots a_n^* \Omega$. It is enough to assume that

$$b = |e_{i_1}) \cdots |e_{i_m})(e_{j_m}| \cdots (e_{j_1}|,$$

corresponding to

$$b(a^*, a) = a_{i_1}^* \cdots a_{i_n}^* a_{j_n} \cdots a_{j_1}.$$

Now

$$(a_1^* \cdots a_n^* \Omega | a_{i_1}^* \cdots a_{i_m}^* a_{j_m} \cdots a_{j_1}\ a_1^* \cdots a_n^* \Omega) \tag{8.102}$$

is nonzero only if $i_1, \dots, i_m$ are distinct,

$$\{i_1, \dots, i_m\} = \{j_1, \dots, j_m\} \subset \{1, \dots, n\}.$$

Then it is ± 1, where its sign is determined by the unique permutation that maps $\{i_1, \dots, i_m\}$ onto $\{j_1, \dots, j_m\}$. Now

$$1 = \operatorname{Tr} \pi^{\otimes m} |e_{i_1}) \cdots |e_{i_m})(e_{j_m}| \cdots (e_{j_1}| \Theta(\sigma).$$

$\square$

In particular, we have the cases $n = 1, 2$:

$$\omega_\pi\big(\mathrm{d}\Gamma(h)\big) = \operatorname{Tr} \pi h, \tag{8.103}$$

$$\omega_\pi\big(b(a^*, a)\big) = \operatorname{Tr} b\, \pi \otimes \pi (\mathbb{1} - \tau), \tag{8.104}$$

where $\tau : \mathcal{W} \otimes \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$ is the transposition of the factors in the tensor product.

8.6 The Hartree-Fock method

We are now going to consider the Hartree-Fock method from a more abstract point of view. Let h be a self-adjoint operator on $\mathcal{W}$ and b on $\mathcal{W} \otimes \mathcal{W}$. We assume that $\tau b \tau = b$. Consider the particle number preserving operator

$$H = \mathrm{d}\Gamma(h) + \frac{1}{2} b(a^*, a).$$

We would like to find the ground state energy of H in the n-body sector.

The Hartree-Fock functional is the expectation value of H in a Slater determinant:

$$\mathcal{E}_{\mathrm{HF}}(\pi) := \omega_\pi(H) = \operatorname{Tr} h \pi + \frac{1}{2} \operatorname{Tr} b\, \pi \otimes \pi\, (\mathbb{1} - \tau).$$

The ground state energy of H is clearly estimated from above by its Hartree-Fock energy

$$E_{\mathrm{HF}} := \inf\{\mathcal{E}_{\mathrm{HF}}(\pi)\ :\ \pi \text{ is an } n\text{-dimensional orthogonal projection}\}.$$

If a minimizer of $\mathcal{E}_{\rm HF}$ exists, we denote it by $\pi_{\rm HF}$. We define the Hartree-Fock Hamiltonian (called also the Fock Hamiltonian) by its expectation value in a trace class matrix γ:

$$\mathrm{Tr} h_{\rm HF}\gamma := \mathrm{Tr} h\gamma + \mathrm{Tr}\, b\, \pi_{\rm HF}\otimes\gamma\, (\mathbb{1} - \tau).$$

Notice the absence of $\frac{1}{2}$.

Theorem 8.2 *$\pi_{\rm HF}$ is a projection onto n lowest lying levels of $h_{\rm HF}$*

Proof. Write the integral kernel of π as

$$\pi(x, y) = \sum_{i=1}^{n} \overline{f_i(x)} f_i(y),$$

where $f_1, \dots, f_n$ is an orthonormal basis of $\mathrm{Ran}\pi$. The Hartree-Fock functional can be written as

$$\begin{aligned}\mathcal{E}_{\rm HF}(\pi) =: \mathcal{E}(f_1, \dots f_n) = &\sum_i (f_i | h f_i) \\ &+ \frac{1}{2}\sum_{ij}(f_i \otimes f_j | b\, f_i \otimes f_j) - \frac{1}{2}\sum_{ij}(f_i \otimes f_j | b\, f_j \otimes f_i).\end{aligned}$$

Using the method of Lagrange multipliers, $E_{\rm HF}$ is given as the infimum of

$$\mathcal{E}_{\rm HF}(f_1, \dots, f_n) - \sum_{ij} \epsilon_{ij}\big((f_i|f_j) - \delta_{ij}\big),$$

over all f_i and Hermitian $[\epsilon_{ij}]$. Writing $f_i + \delta f_i$, $\epsilon_{ij} + \delta\epsilon_{ij}$ for the variations, we find

$$\delta\mathcal{E}_{\rm HF} = \sum_i \big(f_i | h_{\rm HF}\delta f_i\big) + \big(\delta f_i | h_{\rm HF} f_i\big) \tag{8.105}$$

$$- \sum_{ij} \epsilon_{ij}(f_i|\delta f_j) - \sum_{ij} \epsilon_{ij}(\delta f_i | f_j) \tag{8.106}$$

$$+ \sum_{ij} \delta\epsilon_{ij}\big((f_i|f_j) - \delta_{ij}\big). \tag{8.107}$$

Comparing the coefficients at δf_i on the right of the scalar product and on the left of the scalar product independently, we obtain

$$h_{\rm HF} f_i = \sum_j \epsilon_{ij} f_j.$$

We can diagonalize the matrix $[\epsilon_{ij}]$ with a unitary transformation, so that $\epsilon_{ij} = \delta_{ij}\epsilon_i$, and we obtain

$$h_{\rm HF} f_i = \epsilon_i f_i.$$

Thus the minimizing sequence $f_1, \dots, f_n$ can consist of normalized eigenvectors of $h_{\rm HF}$.

Now assume that there is an eigenvector of $h_{\rm HF}$, say g, orthogonal to $f_1, \dots f_n$ and with an eigenvalue β lower than one of the eigenvalues $\epsilon_1, \dots, \epsilon_n$. For instance,

$$h_{\rm HF} g = \beta g, \quad \beta < \epsilon_1.$$

Then we can consider a variation $f_1 + \delta f_1 := \sqrt{1-t^2} f_1 + tg$. This variation is tangent to the constraints. The first variation is zero. We compute the second variation:

$$\begin{aligned}
&\delta \mathcal{E}_{\rm HF}(f_1 + \delta f_1, f_2, \dots, f_n) \\
\approx &\frac{\delta^2}{\delta f_1^2} \mathcal{E}_{\rm HF} \delta f_1 \delta f_1 + \frac{\delta^2}{\delta \overline{f}_1^2} \mathcal{E}_{\rm HF} \delta \overline{f}_1 \delta \overline{f}_1 + 2 \frac{\delta^2}{\delta \overline{f}_1 \delta f_1} \mathcal{E}_{\rm HF} \delta \overline{f}_1 \delta f_1 \\
= &\sum_{ij} (f_i \otimes f_j | b\, \delta f_1 \otimes \delta f_1) - \sum_{ij} (f_i \otimes f_j | b\, \delta f_1 \otimes \delta f_1) &(8.108)\\
&+ \sum_{ij} (\delta f_1 \otimes \delta f_1 | b\, f_i \otimes f_j) - \sum_{ij} (\delta f_1 \otimes \delta f_1 | b\, f_j \otimes f_i) &(8.109)\\
&+ \sum_{j} (\delta f_1 \otimes f_j | b\, \delta f_1 \otimes f_j) - \sum_{j} (\delta f_1 \otimes f_j | b\, f_j \otimes \delta f_1) &(8.110)\\
&= (\delta f_1 | h_{\rm HF} \delta f_1) = t^2 (\epsilon_1 - \beta^2) < 0. &(8.111)
\end{aligned}$$

((8.108) and (8.109) are zero). □

Note that the Hartree-Fock energy is in general not equal to the sum of the lowest n eigenvalues of $H_{\rm HF}$.

9 Squeezed states

9.1 1-mode squeezed vector

Consider $\Gamma_{\rm s}(\mathbb{C})$.

Theorem 9.1 *Let* $|c| < 1$. *Then*

$$\Omega_c := (1 - |c|^2)^{\frac14} {\rm e}^{\frac{c}{2} a^{*2}} \Omega$$

is a normalized vector satisfying

$$(a - c a^*) \Omega_c = 0. \tag{9.112}$$

Proof.

$$\begin{aligned}
\left({\rm e}^{\frac{c}{2} a^{*2}} \Omega | {\rm e}^{\frac{c}{2} a^{*2}} \Omega \right) &= \sum_{n=0}^{\infty} \frac{|c|^{2n} (2n)!}{(n!)^2 2^{2n}} \\
&= \sum \frac{(-1)^n |c|^{2n} (-\frac12)(-\frac12 - 1) \cdots (-\frac12 - n)}{n!} = \left(1 - |c|^2\right)^{-\frac12}.
\end{aligned}$$

Using

$$e^{-\frac{c}{2}a^{*2}} a e^{\frac{c}{2}a^{*2}} = a - \frac{c}{2}[a^{*2}, a] = a + ca^*,$$

we obtain (9.120). □

Theorem 9.2 *Set*

$$U_t := e^{\frac{t}{2}(-a^{*2}+a^2)}.$$

Then

$$\begin{aligned} U_t a U_t^{-1} &= a \cosh t + a^* \sinh t, && (9.113)\\ U_t a^* U_t^{-1} &= a^* \cosh t + a \sinh t, && (9.114)\\ U_t &= \frac{1}{\sqrt{\cosh t}} e^{-\frac{\tanh t}{2}a^{*2}} \Gamma\Big(\frac{1}{\cosh t}\Big) e^{\frac{\tanh t}{2}a^2}, && (9.115)\\ \Omega_{\tanh t} &= U_t \Omega. && (9.116) \end{aligned}$$

Proof. (9.113) and (9.114) are immediate. We next compute

$$\begin{aligned} \frac{d}{dt} U_t &= \frac{1}{2}(-a^{*2} + a^2) U_t \\ &= -\frac{1}{2\cosh^2 t} a^{*2} U_t + \frac{1}{2\cosh^2 t} U_t a^2 - \frac{\sinh t}{\cosh^2 t} a^* U_t a - \frac{\sinh t}{2\cosh t} U_t. \end{aligned}$$

Then we use the identity concerning the derivative of $\Gamma(e^h) = e^{ha^*a}$ contained in (9.117). □

Lemma 9.3

$$\frac{d}{dt} e^{h(t)a^*a} = \dot{h}(t) e^{h(t)} a^* e^{h(t)a^*a} a. \tag{9.117}$$

Proof.

$$\begin{aligned} \frac{d}{dt} e^{ha^*a} &= \dot{h} e^{ha^*a} a^* a && (9.118)\\ &= \dot{h} e^{ha^*a} a^* e^{-ha^*a} e^{ha^*a} a = \dot{h} e^h a^* e^{ha^*a} a. && (9.119) \end{aligned}$$

□

9.2 Many-mode squeezed vector

Suppose c is a symmetric complex matrix on $\mathbb{C}^n$. One can show that then there exists an orthonormal basis such that c is diagonal where all terms on the diagonal are nonnegative. Therefore, we have the many-mode generalizations of the results of the previous subsection to $\Gamma_s(\mathbb{C}^n)$:

Theorem 9.4 *Let c be a symmetric $n\times n$ matrix such that $\|c\|<1$. Then*

$$\Omega_c := \det(1-|c|^2)^{\frac14}\mathrm{e}^{\frac12 c_{ij}a_i^*a_j^*}\Omega$$

is a normalized vector satisfying

$$(a_i - c_{ij}a_j^*)\Omega_c = 0. \tag{9.120}$$

where we write $|c| := \sqrt{c^*c}$.

Theorem 9.5 *Let θ be a symmetric $n\times n$ matrix. Set*

$$U_\theta := \mathrm{e}^{\frac12(-\theta_{ij}a_i^*a_j^*+\overline{\theta}_{ij}a_ja_i)}.$$

Then

$$U_\theta a_i U_\theta^{-1} = (\overline{\cosh|\theta|})_{ij}a_j + \left(\theta\frac{\sinh|\theta|}{|\theta|}\right)_{ij}a_j^*, \tag{9.121}$$

$$U_\theta a_i^* U_\theta^{-1} = (\cosh|\theta|)_{ij}a_j^* + \left(\overline{\theta}\frac{\sinh\overline{|\theta|}}{\overline{|\theta|}}\right)_{ij}a_j, \tag{9.122}$$

$$U_\theta = \frac{1}{\sqrt{\det\cosh|\theta|}}\mathrm{e}^{-\left(\theta\frac{\tanh|\theta|}{2|\theta|}\right)_{ij}a_i^*a_j^*}\Gamma\Big(\frac{1}{\cosh|\theta|}\Big)\mathrm{e}^{\left(\overline{\theta}\frac{\tanh\overline{|\theta|}}{2\overline{|\theta|}}\right)_{ij}a_ja_i}, \tag{9.123}$$

$$U_\theta\Omega = \Omega_{\frac{\tanh|\theta|}{|\theta|}\theta}. \tag{9.124}$$

9.3 Single-mode gauge-invariant squeezed vector

Consider $\Gamma_{\mathrm{s}}(\mathbb{C}^2)$. The creation/annihilation of first mode are denoted a^*, a, of the second b^*, b.

We assume that in our space there is a "charge operator"

$$Q := a^*a - b^*b,$$

and we are interested mostly in gauge invariant states, that is satisfying $Q=0$.

Theorem 9.6 *Let $|c|<1$. Then*

$$\Omega^c := (1-|c|^2)^{\frac12}\mathrm{e}^{ca^*b^*}\Omega$$

is a normalized vector satisfying

$$(a - cb^*)\Omega^c = 0, \tag{9.125}$$

$$(b - ca^*)\Omega^c = 0. \tag{9.126}$$

Proof.

$$\begin{aligned}\left(\mathrm{e}^{ca^*b^*}\Omega|\mathrm{e}^{ca^*b^*}\Omega\right) &= \sum_{n=0}^\infty \frac{|c|^{2n}(n!)^2}{(n!)^2}\\ &= \left(1-|c|^2\right)^{-1}.\end{aligned}$$

Using

$$e^{-ca^*b^*}ae^{ca^*b^*} = a - c[a^*b^*, a] = a + cb^*,$$

we obtain (9.126). $\square$

Remark 9.7 *Clearly,*

$$e^{ca^*b^*} = \exp\Big(\frac{c}{4}(a^* + b^*)^2 - \frac{c}{4}(a^* - b^*)^2\Big).$$

Hence a single mode gauge-invariant squeezed vector can be also understood as a 2-mode squeezed state. However, it is often simple to deal with it directly.

Theorem 9.8 *Set*

$$U^t := e^{t(-a^*b^*+ab)}.$$

Then

$$\begin{aligned}
U^t a U^{-t} &= a\cosh t + b^* \sinh t, & (9.127)\\
U^t a^* U^{-t} &= a^* \cosh t + b \sinh t, & (9.128)\\
U^t b U^{-t} &= b\cosh t + a^* \sinh t, & (9.129)\\
U^t b^* U^{-t} &= b^* \cosh t + a \sinh t, & (9.130)\\
U^t &= \frac{1}{\cosh t} e^{-\tanh t a^* b^*} \Gamma\Big(\frac{1}{\cosh t}\Big) e^{\tanh t ba}, & (9.131)\\
U^t \Omega = \Omega^{-\tanh t} &= \frac{1}{\cosh t} e^{-\tanh t a^* b^*} \Omega. & (9.132)
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
\frac{d}{dt} U^t &= (-a^*b^* + ba)U^t \\
&= -\frac{1}{\cosh^2 t} a^*b^*U^t + \frac{1}{\cosh^2 t} U^t ba - \frac{\sinh t}{\cosh^2 t}\Big(a^*U^t a + b^*U^t b\Big) - \frac{\sinh t}{\cosh t} U^t.
\end{aligned}$$

10 Bose gas and superfluidity

n identical *bosonic* particles are described by the Hilbert space, the Hamiltonian and the total momentum

$$\mathcal{H}_n := L^2_s\Big((\mathbb{R}^d)^n\Big) = \otimes_s^n L^2(\mathbb{R}^d), \tag{10.133}$$

$$H_n : = -\sum_{i=1}^n \Delta_i + \lambda \sum_{1\leq i<j\leq n} V(x_i - x_j), \tag{10.134}$$

$$P_n : = -\sum_{i=1}^n i\partial_{x_i}. \tag{10.135}$$

We have

$$P_n H_n = H_n P_n,$$

which expresses the *translational invariance* of our system. The *potential* V is a real function on $\mathbb{R}^d$ that decays at infinity and satisfies $V(x) = V(-x)$.

We enclose these particles in a box of size L with fixed density $\rho := \frac{n}{L^d}$ and n large. Instead of the more physical Dirichlet boundary conditions, to keep translational invariance we impose the *periodic boundary conditions*, replacing the original V by the *periodized potential*

$$V^L(x) \quad := \quad \sum_{n\in\mathbb{Z}^d} V(x+Ln) = \frac{1}{L^d} \sum_{p\in(2\pi/L)\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}px}\hat{V}(p),$$

well defined on the torus $[-L/2, L/2[^d$. (Note that above we used the *Poisson summation formula*).

The original Hilbert space, Hamiltonian and total momentum are replaced by

$$\mathcal{H}_n^L := L_{\mathrm{s}}^2\Big(\big([-L/2, L/2[^d\big)^n\Big) = \otimes_{\mathrm{s}}^n \big(L^2([-L/2, L/2[^d)\big), \tag{10.136}$$

$$H_n^L := -\sum_{i=1}^n \Delta_i^L + \lambda \sum_{1\le i<j\le n} V^L(x_i - x_j), \tag{10.137}$$

$$P_n^L := -\sum_{i=1}^n \mathrm{i}\partial_{x_i}^L. \tag{10.138}$$

Because of the periodic boundary conditions we still have

$$P_n^L H_n^L = H_n^L P_n^L.$$

In the sequel we drop the superscript L.

We use the second quantized formalism

$$\begin{aligned}
\mathcal{H} = \mathop{\oplus}_{n=0}^{\infty} \mathcal{H}_n &= \Gamma_{\mathrm{s}}\Big(L^2[0,L]^d\Big) \\
&\simeq \Gamma_{\mathrm{s}}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\big)\Big), \\
H := \mathop{\oplus}_{n=0}^{\infty} H_n &= -\int a_x^* \Delta_x a_x \mathrm{d}x + \frac{\lambda}{2}\int\int \mathrm{d}x\mathrm{d}y a_x^* a_y^* V(x-y) a_y a_x \\
&= \sum_p p^2 a_p^* a_p + \frac{\lambda}{2L^d}\sum_{p,q,k} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p, \\
P := \mathop{\oplus}_{n=0}^{\infty} P_n &= \int a_x^* \frac{1}{\mathrm{i}} \partial_x a_x \mathrm{d}x \\
&= \sum_p p a_p^* a_p.
\end{aligned}$$

10.1 Bogoliubov's approximation in the canonical formalism

We assume that the potential is *repulsive*, more precisely,

$$\hat{V} \geq 0, \quad V \geq 0.$$

The Hamiltonian H commutes with N. We are interested in its low energy part for a large number of particles N.

We expect that for low energies most particles will be spread evenly over the whole box staying in the *zeroth mode*, so that $N \simeq N_0 := a_{\mathbf{0}}^* a_{\mathbf{0}}$. (The Bose statistics does not prohibit to occupy the same state). Following the arguments of N. N. Bogoliubov from 1947, we drop all terms in the Hamiltonian involving more than two creation/annihilation operators of a nonzero mode. We obtain

$$\begin{aligned} H &\approx \frac{\lambda \hat{V}(\mathbf{0})}{2L^d} a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_{\mathbf{0}} a_{\mathbf{0}} + \sum_{k \neq 0} \Big(k^2 + a_0^* a_0 \frac{\lambda}{L^d} \big(\hat{V}(k) + \hat{V}(\mathbf{0}) \big) \Big) a_k^* a_k \\ &\quad + \sum_{k \neq \mathbf{0}} \frac{\lambda}{2L^d} \hat{V}(k) \Big(a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_k a_{-k} + a_k^* a_{-k}^* a_{\mathbf{0}} a_{\mathbf{0}} \Big) \\ &= \frac{\lambda \hat{V}(\mathbf{0}) \rho}{2} (N-1) + H_{\mathrm{Bog}} + R, \end{aligned}$$

where we set

$$\begin{aligned} \rho &:= \frac{N}{L^d}, \\ H_{\mathrm{Bog}} &:= \sum_{k \neq \mathbf{0}} \big(k^2 + \lambda \rho \hat{V}(k) \big) a_k^* a_k + \frac{1}{2} \sum_{k \neq \mathbf{0}} \lambda \rho \hat{V}(k) \big(a_k^* a_{-k}^* + a_k a_{-k} \big), \\ R &:= -\frac{\lambda \hat{V}(0)}{2L^d} (N - N_0)(N - N_0 - 1) \\ &\quad + \sum_{k \neq \mathbf{0}} \frac{\lambda}{2L^d} \hat{V}(k) \Big((a_{\mathbf{0}}^* a_{\mathbf{0}}^* - N) a_k a_{-k} + a_k^* a_{-k}^* (a_{\mathbf{0}} a_{\mathbf{0}} - N) \Big). \end{aligned}$$

We used

$$\begin{aligned} a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_{\mathbf{0}} a_{\mathbf{0}} &= N_0 (N_0 - 1) \\ &= N(N-1) - 2N_0(N - N_0) - (N - N_0)(N - N_0 - 1). \end{aligned}$$

We argue that R is small because

$$a_{\mathbf{0}}^* a_{\mathbf{0}}^* \approx a_{\mathbf{0}} a_{\mathbf{0}} \approx N_0 \approx N.$$

A *Bogoliubov transformation*, is a linear transformation of creation/annihilation operators preserving the commutation relations. If we demand in addition that it should commute with translations, it should have the form

$$\tilde{a}_p := c_p a_p + \mathrm{s}_p a_{-p}^*, \tag{10.139}$$

$$\tilde{a}_p^* := c_p a_p^* + s_p a_{-p}, \quad p \neq \mathbf{0}, \tag{10.140}$$

where $c_p^2 - s_p^2 = 1$. We are looking for a Bogoliubov transformation that diagonalizes the quadratic Hamiltonian H_{Bog}. Set

$$A_k := k^2 + \lambda\rho\hat{V}(k), \quad B_k := \lambda\rho\hat{V}(k). \tag{10.141}$$

Then

$$\begin{aligned}
H_{\mathrm{Bog}} &= \frac{1}{2}\sum_{k\neq 0}\Big(A_k(a_k^* a_k + a_{-k}^* a_{-k}) + B_k(a_k^* a_{-k}^* + a_{-k}a_k)\Big) \\
&= \frac{1}{2}\sum_{k\neq 0}\Big(C_k a_k^* + S_k a_{-k})(C_k a_k + S_k a_{-k}^*) - S_k^2\Big) \\
&= \frac{1}{2}\sum_{k\neq 0}\Big((C_k^2 - S_k^2)(c_k a_k^* + s_k a_{-k})(c_k a_k + s_k a_{-k}^*) - S_k^2\Big)
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad C_k &:= \frac{1}{2}(\sqrt{A_k + B_k} + \sqrt{A_k - B_k}), \quad S_k := \frac{1}{2}(\sqrt{A_k + B_k} - \sqrt{A_k - B_k}), \\
c_p &:= \frac{C_p}{\sqrt{C_p^2 - S_p^2}} = \frac{\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)} + |p|}{2\sqrt{\omega(p)}}, \\
s_p &:= \frac{S_p}{\sqrt{C_p^2 - S_p^2}} = \frac{\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)} - |p|}{2\sqrt{\omega(p)}}.
\end{aligned}$$

Set

$$\begin{aligned}
\omega(k) &= C_k^2 - S_k^2 = \sqrt{A_k^2 - B_k^2} = |p|\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)}, \\
E_{\mathrm{Bog}} &:= -\frac{1}{2}\sum_{p\neq 0} S_p^2 = -\frac{1}{2}\sum_{p\neq \mathbf{0}}\left(|p|^2 + \lambda\rho\hat{V}(p) - |p|\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)}\right),
\end{aligned}$$

$\omega(p)$ is called the *Bogoliubov dispersion relation* and E_{Bog} the *Bogoliubov energy*. Using the rotated creaton/annihilation operators, the Hamiltonian and total momentum can be written as

$$\begin{aligned}
H_{\mathrm{Bog}} &= E_{\mathrm{Bog}} + \sum_{p\neq\mathbf{0}} \omega(p)\tilde{a}_p^*\tilde{a}_p, \\
P &= \sum_{p\neq\mathbf{0}} p\tilde{a}_p^*\tilde{a}_p,
\end{aligned}$$

We can introduce β_p by

$$\cosh\beta_p = c_p, \qquad \cosh\beta_p = s_p \tag{10.142}$$

and the unitary operator

$$U = \exp\Big(\sum_{p\neq\mathbf{0}} \frac{\beta_p}{2}\left(-a_p^* a_{-p}^* + a_p a_{-p}\right)\Big).$$

Then U implements the Bogoiubov transformation:

$$\begin{aligned}
\tilde{a}_p &= U a_p U^*, \\
\tilde{a}_p^* &= U a_p^* U^*, \\
H_{\mathrm{Bog}} &= E_{\mathrm{Bog}} + U \sum_{p \neq \mathbf{0}} \omega(p) a_p^* a_p U^*, \\
P &= U \sum_{p \neq \mathbf{0}} p a_p^* a_p U^*.
\end{aligned}$$

We have

$$\tanh(\beta_p) := \frac{|p|^2 + \lambda\rho\hat{V}(p) - |p|\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)}}{\lambda\rho\hat{V}(p)},$$

The ground state of the Bogoliubov Hamiltonian is a squeezed state in the non-zero mode sector:

$$\frac{a_0^{*n}}{\sqrt{n!}} U\Omega = \frac{a_0^{*n}}{\sqrt{n!}} \exp\Big(\frac{1}{2}\sum_{p\neq 0} \tanh(\beta_p) a_p^* a_{-p}^*\Big)\Omega.$$

The Bogoliubov dispersion relation depends on λ and ρ only through $\lambda\rho = \frac{\lambda n}{L^d}$.

The Bogoliubov Hamiltonian depends on L only through the choice of the lattice spacing $\frac{2\pi}{L}$. Note that formally we can even take the limit $L \to \infty$ obtaining

$$\begin{aligned}
H_{\mathrm{Bog}} - E_{\mathrm{Bog}} &= (2\pi)^{-d} \int \omega(p) \tilde{a}_p^* \tilde{a}_p \mathrm{d}p, \\
P &= (2\pi)^{-d} \int p \tilde{a}_p^* \tilde{a}_p \mathrm{d}p.
\end{aligned}$$

We expect that the low energy part of the excitation spectra of H_n and H_{Bog} are close to one another for large n, hoping that then $n - n_0$ is small. We expect some kind of uniformity wrt L.

10.2 Grand-canonical approach

Suppose that $H = \oplus_{n=0}^{\infty} H^n$ is a particle-preserving Hamiltonian decomposed in n-particle sectors. Let N denote the number operator. Instead of studying it inside the nth sector it is often useful to consider its *grand-canonical* vdersion, that is $H_\mu := H - \mu N$, where $\mu \in \mathbb{R}$ is the parameter called the *chemical potential*. Instead of looking for the ground state of H_n it is often more convenient to look for the ground state of H_μ. The following simple fact justifies partly this approach:

Theorem 10.1 *Suppose that E_n is a sequence with $E_0 = 0$ and $\mu_j := E_j - E_{j-1}$ increasing. Let $\mu \in [\mu_n, \mu_{n+1}]$. Then*

$$\inf_k (E_k - \mu k) = E_n - \mu n. \tag{10.143}$$

Proof. Clearly,

$$E_k - \mu k = \sum_{j=1}^{k} (E_j - E_{j-1} - \mu). \tag{10.144}$$

Now

$$E_j - E_{j-1} - \mu \le E_j - E_{j-1} - \mu_n \le 0 \quad \text{for } j \le n; \tag{10.145}$$
$$E_j - E_{j-1} - \mu \ge E_j - E_{j-1} - \mu_{n+1} \le 0 \quad \text{for } j \ge n. \tag{10.146}$$

Hence the choice of k that minimizes (10.144) is $k = n$. □

10.3 Bogoliubov's approximation in the grand-canonical approach

Let us present an alternative derivation of the Bogoliubov dispersion relation based on the grand-canonical approach. For a *chemical potential* $\mu > 0$, we define the *grand-canonical Hamiltonian*

$$H_\mu := H - \mu N \quad = \quad \sum_p (p^2 - \mu) a_p^* a_p + \frac{\lambda}{2L^d} \sum_{p,q,k} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p.$$

We will mostly set $\lambda = 1$.

If E_μ is the ground state energy of H_μ, then it is realized in the sector n satisfying

$$\partial_\mu E_\mu \quad = \quad -n.$$

In what follows we drop the subscript μ.

For $\alpha \in \mathbb{C}$, we define the *displacement* or *Weyl operator* of the zeroth mode: $W_\alpha := \mathrm{e}^{-\alpha a_0^* + \overline{\alpha} a_0}$. Let $\Omega_\alpha := W_\alpha \Omega$ be the corresponding *coherent vector*. Note that $P\Omega_\alpha = 0$. The *expectation of the Hamiltonian* in Ω_α is

$$(\Omega_\alpha | H \Omega_\alpha) = -\mu |\alpha|^2 + \frac{\hat{V}(0)}{2L^d} |\alpha|^4.$$

It is minimized for $\alpha = \mathrm{e}^{\mathrm{i}\tau} \frac{\sqrt{L^d \mu}}{\sqrt{\hat{V}(0)}}$, where τ is an *arbitrary phase*.

We apply the *Bogoliubov translation to the zero mode* of H by W_α:

$$\tilde{a}_k = W_\alpha^* a_k W_\alpha, \quad \tilde{a}_k^* = W_\alpha^* a_k^* W_\alpha,$$

and thus the operators with and without tildes satisfy the same commutation relations. This means making the substitution

$$\begin{aligned} a_0 = \tilde{a}_0 + \alpha, &\quad a_0^* = \tilde{a}_0^* + \overline{\alpha}, \\ a_k = \tilde{a}_k, &\quad a_k^* = \tilde{a}_k^*, \qquad k \neq 0. \end{aligned}$$

We drop the tildes.

Here is the translated Hamiltonian:

$$
\begin{aligned}
H \quad := \quad & -L^d \frac{\mu^2}{2\hat{V}(0)} \\
& + \sum_k \Big(\frac{1}{2} k^2 + \hat{V}(k) \frac{\mu}{\hat{V}(0)} \Big) a_k^* a_k \\
& + \sum_k \hat{V}(k) \frac{\mu}{2\hat{V}(0)} \left(\mathrm{e}^{-\mathrm{i}2\tau} a_k a_{-k} + \mathrm{e}^{\mathrm{i}2\tau} a_k^* a_{-k}^* \right) \\
& + \sum_{k,k'} \frac{\hat{V}(k)\sqrt{\mu}}{\sqrt{\hat{V}(0) L^d}} (\mathrm{e}^{-\mathrm{i}\tau} a_{k+k'}^* a_k a_{k'} + \mathrm{e}^{\mathrm{i}\tau} a_k^* a_{k'}^* a_{k+k'}) \\
& + \sum_{k_1+k_2=k_3+k_4} \frac{\hat{V}(k_2-k_3)}{2L^d} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}.
\end{aligned}
$$

If we (temporarily) replace the potential $V(x)$ with $\lambda V(x)$, where λ is a (small) positive constant, the translated Hamiltonian can be rewritten as

$$H^\lambda = \lambda^{-1} H_{-1} + H_0 + \sqrt{\lambda} H_{\frac{1}{2}} + \lambda H_1.$$

Thus the 3rd and 4th terms are in some sense small, which suggests dropping them. Thus

$$H \approx -L^d \frac{\mu^2}{2\hat{V}(\mathbf{0})} + \mu(\mathrm{e}^{\mathrm{i}\tau} a_{\mathbf{0}}^* + \mathrm{e}^{-\mathrm{i}\tau} a_{\mathbf{0}})^2 + H_{\mathrm{Bog}},$$

where

$$
\begin{aligned}
H_{\mathrm{Bog}} \quad = \quad & \sum_{k\neq\mathbf{0}} \Big(\frac{1}{2} k^2 + \hat{V}(k) \frac{\mu}{\hat{V}(0)} \Big) a_k^* a_k \\
& + \sum_{k\neq\mathbf{0}} \hat{V}(k) \frac{\mu}{2\hat{V}(0)} \left(\mathrm{e}^{-\mathrm{i}2\tau} a_k a_{-k} + \mathrm{e}^{\mathrm{i}2\tau} a_k^* a_{-k}^* \right)
\end{aligned}
$$

Then we proceed as before obtaining the *Bogoliubov dispersion relation*

$$\omega(p) = |p| \sqrt{|p|^2 + 2\mu \frac{\hat{V}(p)}{\hat{V}(\mathbf{0})}}.$$

and the *Bogoliubov energy*

$$E_{\mathrm{Bog}} := -\frac{1}{2} \sum_{p\neq\mathbf{0}} \left(|p|^2 + \mu \frac{\hat{V}(p)}{\hat{V}(\mathbf{0})} - |p| \sqrt{|p|^2 + 2\mu \frac{\hat{V}(p)}{\hat{V}(\mathbf{0})}} \right)$$

Thus, as compared with the canonical approach, we have μ in place of $\lambda\rho$.

Note that the grand-canonical Hamiltonian H_μ is invariant wrt the $U(1)$ *symmetry* $\mathrm{e}^{\mathrm{i}\tau N}$. The parameter α has an arbitrary phase. Thus we *broke the symmetry* when translating the Hamiltonian.

The *zero mode* is not a harmonic oscillator – it has continuous spectrum and it can be interpreted as a kind of a *Goldstone mode*.

10.4 Landau's argument for superfluidity

A translation invariant system such as homogeneous Bose gas is described by a family of commuting self-adjoint operators (H, P), where $P = (P_1, \dots, P_d)$ is the momentum. If the translation invariance is on $\mathbb{R}^d$, then the momentum spectrum is $\mathbb{R}^d$. If it is in a box of side length L with periodic boundary conditions then $\mathrm{e}^{\mathrm{i}P_i L} = \mathbb{1}$, therefore the momentum spectrum is $\frac{2\pi}{L}\mathbb{Z}^d$.

Thus the *energy-momentum spectrum* $\sigma(H, P)$ is

$$\sigma(H,P) \subset \begin{cases} \mathbb{R} \times \mathbb{R}^d, & L = \infty, \\ \mathbb{R} \times \frac{2\pi}{L}\mathbb{Z}^d, & L < \infty. \end{cases}$$

By general arguments the momentum of the ground state of a Bose gas is zero. Let E denote the *ground state energy* of H. We define the critical velocity by

$$c_{\mathrm{crit}} := \sup\{c \ : \ H \geq E + c|P|\}.$$

Suppose that our n-body system is described by (H, P) with critical velocity c_{crit}. We add to H a perturbation u travelling at a speed w:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t = \Big(H + \lambda \sum_{i=1}^n u(x_i - \mathrm{w}t)\Big)\Psi_t.$$

We go to the moving frame:

$$\Psi_t^{\mathrm{w}}(x_1, \dots, x_n) := \Psi_t(x_1 - \mathrm{w}t, \dots, x_n - \mathrm{w}t).$$

We obtain a Schrödinger equation with a time-independent Hamiltonian

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t^{\mathrm{w}} = \Big(H - \mathrm{w}P + \lambda \sum_{i=1}^n u(x_i)\Big)\Psi_t^{\mathrm{w}}.$$

Let Ψ_{gr} be the *ground state* of H. Is it *stable against a travelling perturbation*? We need to consider the *tilted Hamiltonian* $H - \mathrm{w}P$.

If $|\mathrm{w}| < c_{\mathrm{crit}}$, then $H - \mathrm{w}P \geq E$ and Ψ_{gr} is still a ground state of $H - \mathrm{w}P$. So Ψ_{gr} is stable.

If $|\mathrm{w}| > c_{\mathrm{crit}}$, then $H - \mathrm{w}P$ is unbounded from below. So Ψ_{gr} is not stable any more.

11 Fermionic Gaussian states

11.1 1-mode particle-antiparticle vector

Consider $\Gamma_{\mathrm{a}}(\mathbb{C}^2)$. The creation/annihilation of first mode are denoted a^*, a, of the second b^*, b.

We assume that in our space there is a "charge operator"

$$Q := a^*a - b^*b,$$

and we are interested mostly in states with $Q = 0$.

Theorem 11.1 *Let $c \in \mathbb{C}$. Then*

$$\Omega^c := (1+|c|^2)^{-\frac{1}{2}} \mathrm{e}^{ca^*b^*}\Omega = (1+|c|^2)^{-\frac{1}{2}} \big(\Omega + ca^*b^*\Omega\big)$$

is a normalized vector satisfying

$$\begin{aligned} (a - cb^*)\Omega^c &= 0, \\ (b + ca^*)\Omega^c &= 0. \end{aligned}$$

Theorem 11.2 *Set*

$$U^t := \mathrm{e}^{t(-a^*b^*+ba)}.$$

Then

$$U^t a U^{-t} = a\cos t + b^* \sin t, \tag{11.147}$$
$$U^t a^* U^{-t} = a^*\cos t + b \sin t, \tag{11.148}$$
$$U^t b U^{-t} = b\cos t - a^* \sin t, \tag{11.149}$$
$$U^t b^* U^{-t} = b^*\cos t - a \sin t, \tag{11.150}$$
$$U^t = \cos t \mathrm{e}^{-\tan t a^*b^*}\Gamma\Big(\frac{1}{\cos t}\Big)\mathrm{e}^{\tan t ba}, \tag{11.151}$$
$$\Omega^{-\tan t} = U^t\Omega. \tag{11.152}$$

Proof. First we derive (11.147)-(11.150). Then we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}U^t &= (-a^*b^* + ba)U^t \\ &= -\frac{1}{\cos^2 t}a^*b^*U^t + \frac{1}{\cos^2 t}U^t ba + \frac{\sin t}{\cos^2 t}\Big(a^*U^t a + b^*U^t b\Big) - \frac{\sin t}{\cos t}U^t. \end{aligned}$$

□

11.2 Fermionic oscillator

Let

$$H = (a^* + a)(b^* + b).$$

Theorem 11.3 *We have $H^2 = -\mathbb{1}$, $H^* = -H$*

$$\begin{aligned} \mathrm{e}^{tH} &= \cos t \mathbb{1} + \sin t H, \\ \mathrm{e}^{tH}(a^*+a)\mathrm{e}^{-tH} &= \cos 2t(a^*+a) - \sin 2t(b^*+b), \\ \mathrm{e}^{tH}(b^*+b)\mathrm{e}^{-tH} &= \cos 2t(b^*+b) + \sin 2t(a^*+a), \\ \mathrm{e}^{tH}(a^*-a)\mathrm{e}^{-tH} &= a^* - a, \\ \mathrm{e}^{tH}(b^*-b)\mathrm{e}^{-tH} &= b^* - b, \\ \Omega^{\tan t} &= \mathrm{e}^{tH}\Omega. \end{aligned}$$

In particular,

$$
\begin{aligned}
\mathrm{e}^{\pm\frac{\pi}{2}H} &= \pm H, \\
Ha^*H^{-1} &= -a, & HaH^{-1} &= -a^*, \\
Hb^*H^{-1} &= -b, & HbH^{-1} &= -b^*.
\end{aligned}
$$

12 Fermi gas and superconductivity

12.1 Fermi gas

We consider fermions with spin $\frac{1}{2}$ described by the Hilbert space

$$\mathcal{H}_n := \otimes_{\mathrm{a}}^n \left(L^2(\mathbb{R}^d, \mathbb{C}^2) \right).$$

We use the chemical potential from the beginning and we do not assume the locality of interaction, so that the Hamiltonian is

$$H_n = -\sum_{i=1}^{n} \left(\Delta_i - \mu \right) + \lambda \sum_{1\leq i<j\leq n} v_{ij}.$$

The interaction will be given by a 2-body operator on $\otimes^2 \left(L^2(\mathbb{R}^d, \mathbb{C}^2) \right)$ given by

$$(v\Phi)_{i_1,i_2}(x_1,x_2) \quad = \quad \int\int v(x_1,x_2,x_3,x_4)\Phi_{i_1,i_2}(x_3,x_4)\mathrm{d}x_3\mathrm{d}x_4.$$

We will assume that v is invariant wrt the exchange of particles, Hermitian, real and translation invariant:

$$
\begin{aligned}
v(x_1,x_2,x_3,x_4) &= v(x_2,x_1,x_4,x_3) \\
&= \overline{v(x_1,x_2,x_3,x_4)} \\
&= \overline{v(x_4,x_3,x_2,x_1)} \\
&= v(x_1+y,x_2+y,x_3+y,x_4+y).
\end{aligned}
$$

By the invariance wrt the exchange of particles v preserves $\otimes_{\mathrm{a}}^2 \left(L^2(\mathbb{R}^d, \mathbb{C}^2) \right)$. By translation invariance, v can be written as

$$
\begin{aligned}
v(x_1,x_2,x_3,x_4) = (2\pi)^{-4d} \int & \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}k_3x_3-\mathrm{i}k_4x_4} q(k_1,k_2,k_3,k_4) \\
& \times \delta(k_1+k_2-k_3-k_4)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4,
\end{aligned}
$$

where q is a function defined on the subspace $k_1+k_2=k_3+k_4$. An example of such interaction is a local 2-body potential $V(x)$ such that $V(x)=V(-x)$, which corresponds to

$$
\begin{aligned}
v(x_1,x_2,x_3,x_4) &= V(x_1-x_2)\delta(x_1-x_4)\delta(x_2-x_3), \\
q(k_1,k_2,k_3,k_4) &= \int \mathrm{d}p\hat{V}(p)\delta(k_1-k_4-p)\delta(k_2-k_3+p).
\end{aligned}
$$

Similarly, as before, we periodize the interaction

$$\begin{aligned}
& v^L(x_1,x_2,x_3,x_4) \\
= & \sum_{\mathrm{n}_1,\mathrm{n}_2,\mathrm{n}_3\in\mathbb{Z}^d} v(x_1+\mathrm{n}_1L, x_2+\mathrm{n}_2L, x_3+\mathrm{n}_3L, x_4) \\
= & \frac{1}{L^{3d}} \sum_{k_1+k_2=k_3+k_4} \mathrm{e}^{\mathrm{i}k_1\cdot x_1+\mathrm{i}k_2x_2-\mathrm{i}k_3x_3-\mathrm{i}k_4x_4} q(k_1,k_2,k_3,k_4),
\end{aligned}$$

where $k_i \in \frac{2\pi}{L}\mathbb{Z}^d$. The Hamiltonian

$$H^{L,n} = \sum_{1\le i\le n} \big(-\Delta_i^L-\mu\big) + \sum_{1\le i<j\le n} v_{ij}^L$$

acts on $\mathcal{H}^{n,L} := \otimes_{\mathrm{a}}^n \big(L^2([-L/2,L/2]^d,\mathbb{C}^2)\big)$. We drop the superscript L.

We will denote the spins by $i =\uparrow,\downarrow$. It is convenient to put all the n-particle spaces into a single Fock space

$$\overset{\infty}{\underset{n=0}{\oplus}} \mathcal{H}^n = \Gamma_{\mathrm{a}}\big(L^2([L/2,L/2]^d,\mathbb{C}^2)\big)$$

and rewrite the Hamiltonian and momentum in the language of 2nd quantization:

$$\begin{aligned}
H := \overset{\infty}{\underset{n=0}{\oplus}} H^n = & \sum_i \int a_{x,i}^*(\Delta_x-\mu)a_{x,i_2}\mathrm{d}x \\
& + \frac{1}{2}\sum_{i,j}\int\int a_{x_1,i}^*a_{x_2,j}^* v(x_1,x_2,x_3,x_4)a_{x_3,j}a_{x_4,i}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4, \\
P := \overset{\infty}{\underset{n=0}{\oplus}} P^n = & -\sum_i \mathrm{i}\int a_{x,i}^*\nabla_x a_{x,i}\mathrm{d}x.
\end{aligned}$$

In the momentum representation,

$$\begin{aligned}
H &= \sum_i\sum_k (k^2-\mu)a_{k,i}^*a_{k,i} \\
& + \frac{1}{2L^d}\sum_{i,j}\sum_{k_1+k_2=k_3+k_4} q(k_1,k_2,k_3,k_4)a_{k_1,i}^*a_{k_2,j}^*a_{k_3,j}a_{k_4,i}, \\
P &= \sum_i\sum_k k a_{k,i}^*a_{k,i}.
\end{aligned}$$

We also have the generators of the spin $su(2)$.

$$S_x = \frac{1}{2}\sum_k (a_{k\uparrow}^*a_{k\downarrow} + a_{k\downarrow}^*a_{k\uparrow}), \tag{12.153}$$

$$S_y = \frac{\mathrm{i}}{2}\sum_k (a_{k\uparrow}^*a_{k\downarrow} - a_{k\downarrow}^*a_{k\uparrow}), \tag{12.154}$$

$$S_z = \frac{1}{2}\sum_k (a_{k\uparrow}^*a_{k\uparrow} - a_{k\downarrow}^*a_{k\downarrow}). \tag{12.155}$$

The Hamiltonian is invariant with respect to the spin $su(2)$.

13 Hartree-Fock-Bogoliubov approximation with BCS ansatz

We try to compute the excitation spectrum of the Fermi gas by approximate methods. We look for a minimum of the energy among Gaussian states. We assume that a minimizer is invariant wrt translations and the spin $su(2)$. We use the Hartree-Fock-Bogoliubov approximation with the Bardeen-Cooper-Schrieffer ansatz.

For a sequence $\frac{2\pi}{L}\mathbb{Z}^d \ni k \mapsto \theta_k$ such that $\theta_k = \theta_{-k}$, set

$$U_\theta := \prod_k \mathrm{e}^{\frac{1}{2}\theta_k(-a_{k\uparrow}^* a_{-k\downarrow}^* + a_{-k\downarrow}a_{k\uparrow} - a_{-k\uparrow}^* a_{k\downarrow}^* + a_{k\downarrow}a_{-k\uparrow})}.$$

(Note the double counting for $k \neq \mathbf{0}$). We are looking for a minimizer of the form $U_\theta\Omega$.

Note that U_θ commutes with P and the spin $su(2)$. Therefore, $U_\theta\Omega$ is translation and $su(2)$ invariant.

We want to compute

$$(U_\theta\Omega|HU_\theta\Omega) = (\Omega|U_\theta^* HU_\theta\Omega).$$

To do this we can use the fact that U_θ implements Bogoliubov rotations:

$$\begin{aligned}
U_\theta^* a_{k\uparrow}^* U_\theta &= \cos\theta_k a_{k\uparrow}^* + \sin\theta_k a_{-k\downarrow}, \\
U_\theta^* a_{k\uparrow} U_\theta &= \cos\theta_k a_{k\uparrow} + \sin\theta_k a_{-k\downarrow}^*, \\
U_\theta^* a_{k\downarrow}^* U_\theta &= \cos\theta_k a_{k\downarrow}^* - \sin\theta_k a_{-k\uparrow}, \\
U_\theta^* a_{k\downarrow} U_\theta &= \cos\theta_k a_{k\downarrow} - \sin\theta_k a_{-k\uparrow}^*,
\end{aligned}$$

After inserting this into $U_\theta^* HU_\theta$ the resulting expression can be Wick ordered.

In practice, this is usually presented differently. One makes the substitution

$$\begin{aligned}
a_{k\uparrow} &= \cos\theta_k b_{k\uparrow}^* + \sin\theta_k b_{-k\downarrow}, \\
a_{k\uparrow} &= \cos\theta_k b_{k\uparrow} + \sin\theta_k b_{-k\downarrow}^*, \\
a_{k\downarrow}^* &= \cos\theta_k b_{k\downarrow}^* - \sin\theta_k b_{-k\uparrow}, \\
a_{k\downarrow} &= \cos\theta_k b_{k\downarrow} - \sin\theta_k b_{-k\uparrow}^*,
\end{aligned}$$

in the Hamiltonian. Note that

$$\begin{aligned}
U_\theta a_{k\uparrow}^* U_\theta^* &= b_{k\uparrow}^*, \\
U_\theta a_{k\uparrow} U_\theta^* &= b_{k\uparrow}, \\
U_\theta a_{k\downarrow}^* U_\theta^* &= b_{k\downarrow}^*, \\
U_\theta a_{k\downarrow} U_\theta^* &= b_{k\downarrow}.
\end{aligned}$$

Then one Wick orders wrt the operators b^*, b. Our Hamiltonian becomes

$$\begin{aligned} H = B &+ \sum_k D(k)\big(b^*_{k\uparrow} b_{k\uparrow} + b^*_{k\downarrow} b_{k\downarrow}\big) \\ &+ \frac{1}{2}\sum_k O(k)\big(b^*_{k\uparrow} b^*_{-k\downarrow} + b^*_{-k\uparrow} b^*_{k\downarrow}\big) + \frac{1}{2}\sum_k \overline{O}(k)\big(b_{-k\downarrow} b_{k\uparrow} + b_{k\downarrow} b_{-k\uparrow}\big) \\ &+ \text{terms higher order in } b\text{'s}. \end{aligned}$$

Note that

$$(\Omega_\theta | H \Omega_\theta) = B.$$

By the Beliaev Theorem, minimizing B is equivalent to $O(k) = 0$.

If we choose the Bogoliubov transformation according to the minimization procedure, the Hamiltonian equals

$$H = B + \sum_k D(k)\big(b^*_{k\uparrow} b_{k\uparrow} + b^*_{k\downarrow} b_{k\downarrow}\big) + \text{terms higher order in } b\text{'s}$$

with

$$\begin{aligned} B &= \sum_k (k^2 - \mu)(1 - \cos 2\theta_k) \\ &\quad + \frac{1}{4L^d} \sum_{k,k'} \alpha(k,k') \sin 2\theta_k \sin 2\theta_{k'} \\ &\quad + \frac{1}{4L^d} \sum_{k,k'} \beta(k,k')(1 - \cos 2\theta_k)(1 - \cos 2\theta_{k'}). \end{aligned}$$

Here,

$$\begin{aligned} \alpha(k,k') &:= \frac{1}{2}\big(q(k,-k,-k',k') + q(-k,k,-k',k')\big), \\ \beta(k,k') &= 2q(k,k',k',k) - q(k',k,k',k). \end{aligned}$$

In particular, in the case of local potentials we have

$$\begin{aligned} \alpha(k,k') &:= \frac{1}{2}\big(\hat{V}(k-k') + \hat{V}(k+k')\big), \\ \beta(k,k') &= 2\hat{V}(\mathbf{0}) - \hat{V}(k-k'). \end{aligned}$$

The condition $\partial_{\theta_k} B = 0$, or equivalently $O(k) = 0$, has many solutions. We can have

$$\sin 2\theta_k = 0, \qquad \cos 2\theta_k = \pm 1,$$

They correspond to *Slater determinants* and have a fixed number of particles. The solution of this kind minimizing B, is called a *normal* or *Hartree-Fock solution*.

Under some conditions the global minimum of B is reached by a non-normal configuration satisfying

$$\sin 2\theta_k = -\frac{\delta(k)}{\sqrt{\delta^2(k)+\xi^2(k)}}, \qquad \cos 2\theta_k = \frac{\xi(k)}{\sqrt{\delta^2(k)+\xi^2(k)}},$$

where

$$\begin{aligned}\delta(k) &= \frac{1}{2L^d}\sum_{k'}\alpha(k,k')\sin 2\theta_{k'},\\ \xi(k) &= k^2-\mu+\frac{1}{2L^d}\sum_{k'}\beta(k,k')(1-\cos 2\theta_{k'}),\end{aligned}$$

and at least some of $\sin 2\theta_k$ are different from 0. It is sometimes called a *superconducting solution.*

For a superconducting solution we get

$$D(k) = \sqrt{\xi^2(k)+\delta^2(k)}.$$

Thus we obtain a positive dispersion relation. One can expect that it is strictly positive, since otherwise the two functions δ and ξ would have a coinciding zero, which seems unlikely. Thus we expect that the dispersion relation $D(k)$ has a *positive energy gap.*

Conditions guaranteeing that a superconducting solution minimizes the energy should involve some kind of negative definiteness of the quadratic form α – this is what we vaguely indicated by saying that the interaction is *attractive.* Indeed, multiply the definition of $\delta(k)$ with $\sin 2\theta_k$ and sum it up over k. We then obtain

$$\begin{aligned}&\sum_k \sin^2 2\theta_k\sqrt{\delta^2(k)+\xi^2(k)}\\ =\ & -\frac{1}{2L^d}\sum_{k,k'}\sin 2\theta_k\alpha(k,k')\sin 2\theta_{k'}.\end{aligned}$$

The left hand side is positive. This means that the quadratic form given by the kernel $\alpha(k,k')$ has to be negative at least at the vector given by $\sin 2\theta_k$.

14 Basics of representations of $su(n)$

14.1 Contragradient representation

Let $\mathfrak{g}$ be a Lie algebra. Consider a representation $\mathfrak{g} \ni A \mapsto \pi(A) \in L(\mathcal{V})$ on a finite dimensional space $\mathcal{V}$. The representation contragradient to π is defined as

$$\pi^{\mathrm{ctg}}(A) := -\pi(A)^{\mathrm{T}} \in L(\mathcal{V}^{\mathrm{T}}), \tag{14.1}$$

where $\mathcal{V}^{\mathrm{T}}$ denotes the dual of $\mathcal{V}$.

Let $\mathcal{V}$ be in addition a Hilbert space. By saying that π is *infinitesimally unitary* we mean that the corresponding group representation is unitary. Equivalently, $\pi(A)$ are antiself-adjoint:

$$\pi(A) = -\pi(A)^* = -\overline{\pi(A)}^{\mathrm{T}}. \tag{14.2}$$

Thus for an infinitesimally unitary representation we have

$$\pi^{\mathrm{ctg}}(A) = \overline{\pi(A)}. \tag{14.3}$$

When speaking of representations we will usually omit the symbol π. Various representations will be recognized by the space on which they act.

14.2 $su(n)$ and $sl(n, \mathbb{C})$

We will mostly speak about representations of

$$su(n) := \{A \in L(\mathbb{C}^n) \ : \ A^* = -A, \quad \mathrm{Tr}A = 0\}.$$

The complexification of $su(n)$ is

$$sl(n, \mathbb{C}) := \{A \in L(\mathbb{C}^n) \ : \qquad \mathrm{Tr}A = 0\} = su(n) + \mathrm{i}su(n).$$

Every finite dimensional representation of $su(3)$ extends to a complex representation of $sl(n, \mathbb{C})$. Conversely, for every finite dimensional complex representation of $sl(n, \mathbb{C})$ we can choose a scalar product so that its restriction to $su(n)$ is infinitesimally unitary. A representation of $su(n)$ is irreducible iff so is the corresponding representation of $sl(n, \mathbb{C})$.

Thus we can pass from representations of $su(n)$ to complex representations of $sl(n, \mathbb{C})$ and back. It is often convenient to use the complexified version.

$sl(n, \mathbb{C})$ has the obvious representation on $\mathbb{C}^n$. It will be called *fundamental*. Its contragradient representation, acting on $\mathbb{C}^{n\mathrm{T}}$ will be called *antifundamental*. When restricted to $su(n)$ we can write $\overline{\mathbb{C}}^n$ instead of $\mathbb{C}^{n\mathrm{T}}$.

14.3 Cartan algebra

Let $|1\rangle, \ldots, |n\rangle$ denote the canonical basis of $\mathbb{C}^n$. Let $\langle 1|, \ldots, \langle n|$ denote its dual basis, which is a basis of $\mathbb{C}^{n\mathrm{T}}$. $sl(n, \mathbb{C})$ is embedded in the obvious way in $gl(n, \mathbb{C})$, which is spanned by the operators $A_{ij} := |i\rangle\langle j|$. $gl(n, \mathbb{C})$ has a natural scalar product

$$\langle A|B\rangle = \mathrm{Tr}A^{\mathrm{T}}B, \tag{14.4}$$

in which A_{ij} is an orthonormal basis.

The set of diagonal elements of $sl(n, \mathbb{C})$ is called the *Cartan algebra* of $sl(n, \mathbb{C})$ and denoted $\mathfrak{h}$. It is a maximal commutative algebra in $sl(n, \mathbb{C})$. It is spanned by $H_{i,j} = -H_{ji} = A_{ii} - A_{jj}$, $i \neq j$. Note that

$$\langle H_{i,j}|H_{i,k}\rangle = -1, \quad j \neq k; \qquad \langle H_{i,j}|H_{i,j}\rangle = 2. \tag{14.5}$$

Hence the angle between H_{ij} and H_{ik} is $\frac{2\pi}{3}$. $H_{i,i+1}$, $i = 1, \ldots, n-1$ is a (non-orthogonal) basis.

14.4 Representation weights

Suppose π is a representation of the Lie algebra $su(n)$ (or $sl(n,\mathbb{C})$) on a finite dimensional space $\mathcal{V}$. Elements of $\mathcal{V}$ that are eigenvectors jointly of all elements of the Cartan algebra are called *weight vectors* of this representation. Their eigenvalues depend linearly on $\mathfrak{h}$, hence they can be interpreted as elements of $\mathfrak{h}^{\mathrm{T}}$. They are called *weights*. Denote by $\mathcal{V}_\beta$ the space of eigenvectors for the weight $\beta \in \mathfrak{h}^{\mathrm{T}}$. We thus have

$$Hv = \langle \beta | H \rangle v, \quad v \in \mathcal{V}_\beta, \quad H \in \mathfrak{h}.$$

For instance, consider the fundamental representation on $\mathbb{C}^n$. We have

$$H_{ij}|i\rangle = |i\rangle, \quad H_{ji}|i\rangle = -|i\rangle, \quad H_{jk}|i\rangle = 0, \quad i \notin \{j,k\}. \tag{14.6}$$

Hence $|i\rangle$ is a weight vector and the corresponding weight, denoted L_i, satisfies

$$\langle L_i | H_{ij} \rangle = -\langle L_i | H_{ji} \rangle = 1, \quad \langle L_i | H_{jk} \rangle = 0, \quad i \notin \{j,k\}. \tag{14.7}$$

Note that $\mathfrak{h}$ is $n-1$-dimensional, so $L_1, \dots, L_n$ have to be linearly dependent. In fact,

$$L_1 + \cdots + L_n = 0. \tag{14.8}$$

For the antifundamental representation weight vectors are $\langle i|$, $i = 1, \dots, n$, and the corresponding weight is $-L_i$.

14.5 Representations of $su(2)$

It is easy to describe all representations of $su(2)$. For every $n \in \mathbb{N}_0$ there exists exactly one n-dimensional representation and it acts on $\otimes_{\mathrm{s}}^{n-1}\mathbb{C}^2$, where $\mathbb{C}^2$ is the fundamental representation. The antifundamental representation is equivalent to the fundamental, because for $A \in sl(2,\mathbb{C})$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = -A^{\mathrm{T}}. \tag{14.9}$$

The Cartan algebra of $su(2)$ is H_{12}. $su(2)$ is spanned by H_{12}, A_{12}, A_{21} satisfying

$$[A_{12}, A_{21}] = H_{12}, \ [H_{12}, A_{12}] = 2A_{12}, \ [H_{12}, A_{21}] = -2A_{21}.$$

Eigenvalues of H_{12} in all representations are integers. The n-dimensional representation, called also the spin $\frac{n-1}{2}$ representation, has weights $-n+1, -n+3, \dots, n-1$, e.g.

$$\begin{gathered} 0; \\ -1, 1; \\ -2, 0, 2. \end{gathered}$$

14.6 Roots

Every Lie algebra has a representation on itself

$$\mathfrak{g} \ni A \mapsto [A, \cdot] \in L(\mathfrak{g}). \tag{14.10}$$

This representation is called the *adjoint representation*. Let us describe it in the case of $su(n)$.

$su(n)$ is spanned by A_{ij}, $a \neq j$ and the Cartan algebra $\mathfrak{h}$. The Cartan algebra consists of weight vectors for the adjoint representation with weight 0. The operators A_{ij} are called *roots operators* and satisfy

$$[H, A_{ij}] = \alpha_{ij}(H) A_{ij}, \quad H \in \mathfrak{h},$$

where α_{ij} is a linear functional on $\mathfrak{h}$ called a *root*. If i, j, k are distinct, then

$$\alpha_{ij}(H_{ij}) = 2, \quad \alpha_{ij}(H_{jk}) = -1, \quad \alpha_{ij}(H_{ki}) = -1.$$

Identifying $\mathfrak{h}^{\mathrm{T}}$ with $\mathfrak{h}$ with the help of the scalar product (14.4) we obtain the identification $\alpha_{ij} = \langle H_{ij} | \cdot \rangle$, because

$$\langle H_{ij} | H \rangle = \alpha_{ij}(H). \tag{14.11}$$

Hence A_{ij} are weight vectors for the adjoint representation and α_{ij} are the corresponding weights.

A_{ij}, A_{ji} and H_{ij} satisfy the relations $sl(2, \mathbb{C})$

$$[A_{ij}, A_{ji}] = H_{ij}, \; [H_{ij}, A_{ij}] = 2A_{ij}, \; [H_{ij}, A_{ji}] = -2A_{ji}.$$

Hence eigenvalues of H_{ij} have to be integers.

The set of elements of $\mathfrak{h}^{\mathrm{T}}$ which are integer linear combinations of roots is called the *root lattice*. $\mathcal{U}$. Clearly, $\mathcal{U} \subset \mathcal{W}$.

The set of elements of $\mathfrak{h}^{\mathrm{T}}$, which on H_{ij} have integer values, is called the *weight lattice* $\mathcal{W}$. Weights of all representations belong to the weight lattice.

Let $\beta \in \mathcal{W}$ be a weight of a certain representation. We have

$$A_{ij} \mathcal{V}_\beta \subset \mathcal{V}_{\beta + \alpha_{ij}}.$$

Clearly, the Cartan algebra preserves $\mathcal{V}_\beta$. Therefore, if a representation is irreducible and has a weight $\beta \in \mathcal{W}$, then all other weights belong to $\beta + \mathcal{U}$.

14.7 Representations of $su(3)$

It is easy to describe all irreducble representations of $sl(3, \mathbb{C})$. We consider $(p, q) \in \mathbb{N}_0^2$. On the space

$$\otimes_{\mathrm{s}}^p \mathbb{C}^3 \otimes \otimes_{\mathrm{s}}^q \mathbb{C}^{3\mathrm{T}}.$$

we have the obvious representation

$$\pi_{p,q}(A) = \sum_{k=0}^{p-1} \mathbb{1}^{\otimes k} \otimes A \otimes \mathbb{1}^{\otimes (p-1-k)} \otimes \mathbb{1}^{\otimes q} - \mathbb{1}^{\otimes p} \otimes \sum_{k=0}^{p-1} \mathbb{1}^{\otimes k} \otimes A^{\mathrm{T}} \otimes \mathbb{1}^{\otimes (q-1-k)}. \tag{14.12}$$

Its elements are tensors

$$\sum |i_1\rangle \otimes \cdots \otimes |i_p\rangle \otimes \langle j_1| \otimes \cdots \otimes \langle j_q|\, t^{i_1,\dots,i_p}_{j_1,\dots,j_q},$$

which for brevity can be written as $[t^{i_1,\dots,i_p}_{j_1,\dots,j_q}]$. We can introduce the contraction

$$[t^{i_1,\dots,i_p}_{j_1,\dots,j_q}] \mapsto [t^{i_1,\dots,i_{p-1},k}_{j_1,\dots,j_{q-1},k}],$$

where we use the Einstein summation convention. The contraction operator intertwines the representation on na $\otimes_{\mathrm{s}}^{p}\mathbb{C}^3 \otimes \otimes_{\mathrm{s}}^{q}\mathbb{C}^{3\mathrm{T}}$ with a representation on $\otimes_{\mathrm{s}}^{p-1}\mathbb{C}^3 \otimes \otimes_{\mathrm{s}}^{q-1}\mathbb{C}^{3\mathrm{T}}$. Its kernel is an invariant subspace and is an irreducible representation of $sl(3,\mathbb{C})$, which will be called the representation of type (p,q). The representation contravariant to (p,q) is (q,p).

$sl(3,\mathbb{C})$ can be also represented on the antisymmetric tensor product. However, this does not lead to additional irreducible representations. In fact, $\otimes_{\mathrm{a}}^{3}\mathbb{C}^3$ and $\otimes_{\mathrm{a}}^{3}\mathbb{C}^{3\mathrm{T}}$ are one-dimensional, and hence $sl(3,\mathbb{C})$ acts on them trivially. Besides, the representation on $\otimes_{\mathrm{a}}^{2}\mathbb{C}^3$ is equivalent to the antifundamental representation and on $\otimes_{\mathrm{a}}^{2}\mathbb{C}^{3\mathrm{T}}$ —to the fundamental one.

14.8 Fundamental and antifundamental representation of $su(3)$

The Cartan algebra of $su(3)$ is spanned by $H_{12} = -H_{21}$ and $H_{13} = -H_{31}$. We also have $H_{23} = -H_{32} = H_{21} + H_{13}$,

$$\langle H_{12}|H_{12}\rangle = 2, \quad \langle H_{13}|H_{13}\rangle = \langle H_{12}|H_{13}\rangle = 1$$

In the fudamental representation

$$\langle L_1|H_{12}\rangle = 1, \quad \langle L_1|H_{13}\rangle = 1.$$

We check that $L_1 = \frac{1}{3}\langle (H_{12} + H_{13})|\cdot\rangle$. Thus we can identify

$$L_1 = \frac{1}{3}(H_{12} + H_{13}), \quad L_2 = \frac{1}{3}(H_{23} + H_{21}), \quad L_3 = \frac{1}{3}(H_{31} + H_{32}).$$

Thus

$$L_1 - L_2 = \frac{1}{3}(H_{12} + H_{13}) - \frac{1}{3}(H_{21} + H_{23}) = \frac{1}{3}(2H_{12} - H_{31} - H_{23}) = H_{12}.$$

Clearly, $L_1 + L_2 + L_3 = 0$. If we choose L_1, L_2 as a basis, then

$$\begin{aligned} H_{12} &= L_1 - L_2, \\ H_{23} &= L_2 - L_3 = L_1 + 2L_2, \\ H_{31} &= L_3 - L_1 = -2L_1 - L_2. \end{aligned}$$

The vectors L_i span the weight lattice. Together with $-L_i$ they are situated on vertices of a regular hexagon:

$$\begin{array}{ccc} & -L_3 & \\ L_2 & & L_1 \\ -L_1 & & -L_2 \\ & L_3 & \end{array}$$

14.9 Triality of $su(3)$

The lattice $\mathcal{W}$ can be partitioned into three sublattices:

$$\mathcal{W}_k \quad := \quad \{n_1 L_1 + n_2 L_2 \ : \ n_1 + n_2 \in 3\mathbb{Z} + k\}.$$

Equivalently,

$$\mathcal{W}_0 = \mathcal{U}, \quad \mathcal{W}_1 = L_1 + \mathcal{U}, \quad \mathcal{W}_2 = 2L_1 + \mathcal{U}.$$

$k \in \mathbb{Z}_3$ is called the *triality* of the sublattice. The weights of a representation of type (p, q) belong to $\mathcal{W}_{p-q}$. In particular, roots have triality 0.

The center of $SU(3)$ is $\{\mathrm{e}^{\mathrm{i}\frac{2\pi k}{3}} \mathbb{1} \ : \ k = 0, 1, 2\} \simeq \mathbb{Z}_3$. $\mathbb{Z}_3$ has three irreducible representations, also numbered by $\mathbb{Z}_3$. The triality of a given representation corresponds to the representation of the center.

14.10 Negative and positive roots

Among root operators we distinguish negative roots:

$$A_{12}, \ A_{13}, \ A_{23}$$

and positive roots:

$$A_{21}, \ A_{31}, \ A_{32}.$$

A highest weight vector is annihilated by negative roots. Every irreducible representation has up to a multiplier a unique highest weight vector. Let us denote it by Ψ. Then every vector is a linear combination of vectors of the form $B_1 \cdots B_n \Psi$, where $B_1, \dots$ are positive roots.

Let e_1, e_2, e_3 be a basis of $\mathbb{C}^3$ and e^1, e^2, e^3 its dual basis in $\mathbb{C}^{3\mathrm{T}}$. The representation of type (p, q) on $\otimes_{\mathrm{s}}^p \mathbb{C}^3 \otimes \otimes_{\mathrm{s}}^q \mathbb{C}^{3\#}$ has a highest weight vector $\otimes^p e_1 \otimes \otimes^q e^3$ with weight $pL_1 - qL_3 = (p+q)L_1 + qL_2$.

14.11 Examples of weight diagrams

Fundamental representation, that is $(1, 0)$: weights $\{L_i\}$, heighest weight L_1

$$\begin{array}{ccc} 1 & & \underline{1} \\ & 1 & \end{array}$$

$(2, 0)$: weights $\{L_i + L_j\}$, heighest weight $2L_1$

$$\begin{array}{ccccc} 1 & & 1 & & \underline{1} \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

$(3, 0)$: weights $\{L_i + L_j + L_k\}$, heighest weight $3L_1$

$$\begin{array}{ccccccc} 1 & & 1 & & 1 & & \underline{1} \\ & 1 & & 1 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array}$$

Antifundamental representation, that is, $(0,1)$: weights $\{-L_i\}$, heighest weight $-L_3$

$$\begin{array}{ccc} & \underline{1} & \\ 1 & & 1 \end{array}$$

$(0,2)$: weights $\{-L_i - L_j\}$, heighest weight $-2L_3$

$$\begin{array}{ccccc} & & \underline{1} & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{array}$$

$(0,3)$: weight $\{-L_i - L_j - L_k\}$, heighest weight $-3L_3$

$$\begin{array}{ccccccc} & & & \underline{1} & & & \\ & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{array}$$

Adjoint representation, that is $(1,1)$, acts in $\mathbb{C}^3 \otimes \mathbb{C}^{3\mathrm{T}}$, weight $\{L_i - L_j,\ i \neq j;\ \ 2 \times 0\}$, heighest weight $L_1 - L_3$

$$\begin{array}{ccccc} & 1 & & \underline{1} & \\ 1 & & 2 & & 1 \\ & 1 & & 1 & \end{array}$$

Representation $(2,1)$ has weights $\{2L_i - L_j,\ i \neq j;\ -2L_i;\ 2 \times L_i\}$ heighest weight $2L_1 - L_3$

$$\begin{array}{ccccccc} & 1 & & 1 & & \underline{1} & \\ 1 & & 2 & & 2 & & 1 \\ & 1 & & 2 & & 1 & \\ & & 1 & & 1 & & \end{array}$$

The set of weight and their multiplicities for any representation has to satisfy the following properties:

(1) It is symmetric wrt reflections in any axis determined byej z osi zadanej przez L_k.

(2) Intersecting with an arbitrary line passing through the origin and orthogonal to L_k we obtain multiplicities of a certain representation of $SU(2)$.

(3) If the representation is irreducible, its weights are contained in one of the sublattices $\mathcal{W}_0$, $\mathcal{W}_1$ or $\mathcal{W}_2$.

to see (1) note that $B \mapsto W_{ij} A W_{ij}^{-1}$ is an isomorphism of the Lie algebra $sl(3,\mathbb{C})$, where

$$W_{ij} = W_{ij}^{-1} := A_{kk} + A_{ij} + A_{ji}.$$

This isomorphism exchanges H_{ij} with $-H_{ji}$ and H_{ik} with H_{jk}.

To see (2) we note that if $\mathcal{H}_\beta$ is a weight space, then $\oplus \mathcal{H}_{\beta+\mu}$, where μ are certain multiples of α_{ij}, span a representation of $sl(2,\mathbb{C})$.

The weight multiplicities (the dimensions of weight spaces) for irreducible representations satisfy the following properties. The weights on the boundary have multiplicity 1. In every next level they are increased by 1 unless we reach a level of the form of a triangle, when we stop increasing the multiplicity. In particular, for representations (n, n), which have triangular boundaries, all multiplicities ar 1.

15 Applications of $su(3)$ to particle physics

15.1 Symmetries in quantum mechanics

Let $\mathcal{H}$ be a Hilbert space describing a quantum system and $G \ni g \mapsto U(g) \in U(\mathcal{H})$ a unitary representation of a group G. Assume that $A_1, \dots, A_n$ is a set of commuting self-adjoint observables. Let $U(g)$, $g \in G$, commute with $A_1, \dots, A_n$. Then eigenspaces of $A_1, \dots, A_n$ are invariant wrt G.

The most common application of the group theory to quantum mechanics involves *approximate symmetries*. Suppose that the observables A_i slowly change in time. For instance, if $H = H_0 + V$ is an unperturbed Hamiltonian and V is in an appropriate sense small, then one of these observables can be H_0.

A different application consists in assuming that G is a *gauge group*. This means that both the Hamiltonian H and all physical observables commute with $U(g)$, $g \in G$.

Instead of representations of Lie groups, we will usually speak about representations of the corresponding Lie algebras.

15.2 Conserved charges

Every elementary particle, if left alone, eventually will decay and split into photons, neutrinos, electrons, protons and their antiparticles.

The following quantities do not depend on the decay channel: the electric charge

$$Q := \#p + \#\overline{e} - \#\overline{p} - \#e,$$

and the barion number

$$B := \#p - \#\overline{p}.$$

They are always conserved.

15.3 Isospin

A proton p and a neutron n have similar masses and properties unrelated to electromagnetic interactions. Similarly mesons π^+, π^0, π^-.

Let us describe Heisenbergs proposal meant to explain this: The Hamiltonian has a decomposition

$$H = H_{\mathrm{strong}} + H_{\mathrm{em}},$$

where H_{strong} is describes strong interactions and is invariant wrt the *isospin group* $SU(2)$, unlike the Hamiltonian of electromagnetic interactions H_{em}. Denote by I_1, I_2, I_3 the generators of $su(2)$. The electromagnetic interaction commutes only with I_3.

A proton p and a neutron n are eigenvectors of I_3 in the fundamental representation of $SU(2)$, which has the isospin $\frac{1}{2}$:

$$I_3 p = \frac{1}{2} p, \quad I_3 n = -\frac{1}{2} n.$$

Similarly, mesons π belong to the isospin 1 representation:

$$I_3 \pi^+ = \pi^+, \quad I_3 \pi^0 = 0, \quad I_3 \pi^- = -\pi^-.$$

More generally, it has been noticed that particles can be arranged in isospin multiplets. Inside each isospin multiplet particles have a similar mass and some other properties, however they have a different charge and the value of I_3.

It was noticed that interactions among particles can be divided into strong, which occur very fast and weak, which are much slower, and electromagnetc. The isospin is conserved in strong interactions, but not in weak interactions. Here is an example of a weak interaction that violates the isospin conservation:

$$\pi^+ \to \pi^0 + \mu^+ + \nu_\mu.$$

Taking into account strong interactions one can asign to each particle a value of I_3.

Note that for the nucleon and pion multiplets we have the relation

$$Q = I_3 + \frac{1}{2} B. \tag{15.1}$$

15.4 Strangeness

It was noticed that there exists another number which is conserved in strong interactions and in weak interactions it changes by ± 1. It was called *strangeness* and denoted S. It was assumed that the "standard particles" such as p, n, π, e have a zero strangeness.

It turned out that strongly interacting particles can be grouped in larger multiplets containing particles not only with different I_3, but also S. Inside each multiplet particles the masses are quite similar and the barion number is the same. It was noticed that these multiplets have a symmetric form if as coordinates we use I_3 and the *hypercharge*

$$Y = B + S.$$

The following relation, called the *Gell-Mann – Nishijima formula*, generalizes (15.1):

$$Q = I_3 + \frac{1}{2} Y.$$

Hadrons with a zero barion number are called *mesons*. On the diagrams below the vertical axis is parametrized by Y, and the horizontal axis by I_3.

The pseudoscalar nonet consists of the octet

$$\begin{array}{ccccc} & K^0 & & K^+ & \\ \pi^- & & \pi^0, \eta & & \pi^+ \\ & K^- & & K^{\overline{0}} & \end{array}$$

and the singlet η'.

The pseudovector nonet consists of the octet

$$\begin{array}{ccccc} & K^{*0} & & K^{*+} & \\ \rho^- & & \rho^0, \omega & & \rho^+ \\ & K^{*-} & & K^{*\overline{0}} & \end{array}$$

the singlet ω'.

There are also two barion ($B = 1$) multiplets. The spin $\frac{1}{2}$ octet:

$$\begin{array}{ccccc} & n & & p & \\ \Sigma^- & & \Sigma^0, \Lambda^0 & xs & \Sigma^+ \\ & \Xi^- & & \Xi^0 & \end{array}$$

The spin $\frac{3}{2}$ decuplet:

$$\begin{array}{ccccccc} \Delta^- & & \Delta^0 & & \Delta^+ & & \Delta^{++} \\ & \Sigma^{*-} & & \Sigma^{*0} & & \Sigma^{*+} & \\ & & \Xi^{*-} & & \Xi^{*0} & & \\ & & & \Omega^- & & & \end{array}$$

Finally, there are two antibarion ($B = -1$) multiplets consisting of antiparticles of the barion multiplets.

15.5 Quarks

Here is how one can explain the above properties of elementary particles. Introduce 3 quarks: u, d and s. We treat them as weight vectors for the fundamental representation of $SU(3)$:

$$\begin{array}{ccc} d & & u \\ & s & \end{array}$$

We also have antiquarks, which correspond to the antifundamental representation:

$$\begin{array}{ccc} & \overline{s} & \\ \overline{u} & & \overline{d} \end{array}$$

We assume that they have the following quantum numbers:

$$Q = \frac{1}{3}(2\#u - \#d - \#s), \quad B = \frac{1}{3}(\#u + \#d + \#s), \quad S = -\#s. \tag{15.2}$$

Consequently

$$Y = B + S = \frac{1}{3}(\#u + \#d - 2\#s), \quad I_3 = Q - \frac{1}{2}(B + S) = \frac{1}{2}(\#u - \#d). \tag{15.3}$$

Then all the above described multiplets of hadrons correspond to weight diagrams of certain irreducible representations of $su(3)$ with triality 0. Let us try to understand why precisely these representations show up.

Consider the group $SU(3)_{\mathrm{fl}}$ describing the flavors u, d, s, the group $SU(2)_{\mathrm{spin}}$ describing the spin and $SU(3)_{\mathrm{col}}$ describing the color. Quarks can be treated as elements of $\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \mathbb{C}^3_{\mathrm{col}}$, and antiquarks as elements of $\overline{\mathbb{C}}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}} \otimes \overline{\mathbb{C}}^3_{\mathrm{col}}$. The group $SU(3)_{\mathrm{fl}} \times SU(2)_{\mathrm{spin}} \times SU(3)_{\mathrm{col}}$ acts on them.

We will ignore the position degrees of freedom of quarks, remembering only their flavor, color and spin degrees of freedom. They are fermions, hence we will describe them by elements of the Fock space

$$\Gamma_{\mathrm{a}}\Big(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \mathbb{C}^3_{\mathrm{col}} \ \oplus \ \overline{\mathbb{C}}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}} \otimes \overline{\mathbb{C}}^3_{\mathrm{col}}\Big). \tag{15.4}$$

We will often use the exponential property of (fermionic) Fock spaces, which implies

$$\otimes_{\mathrm{a}}^n(\mathcal{Z} \oplus \mathcal{W}) \simeq \mathop{\oplus}_{k=0}^{n} \otimes_{\mathrm{a}}^k \mathcal{Z} \otimes \otimes_{\mathrm{a}}^{n-k} \mathcal{W}. \tag{15.5}$$

Thus bound states of p quarks and q antiquarks are described by elements of

$$\otimes_{\mathrm{a}}^p\Big(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \mathbb{C}^3_{\mathrm{col}}\Big) \otimes \otimes_{\mathrm{a}}^q\Big(\overline{\mathbb{C}}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}} \otimes \overline{\mathbb{C}}^3_{\mathrm{col}}\Big). \tag{15.6}$$

The *confinement conjecture* says that in physics we have only "‘colorless"’ states, that is states on which the color group acts trivially. If we embed (15.6) in the space

$$\otimes^p\Big(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}}\Big) \otimes \otimes^q\Big(\overline{\mathbb{C}}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}}\Big) \otimes \Big(\otimes^p \mathbb{C}^3_{\mathrm{col}} \otimes \otimes^q \overline{\mathbb{C}}^3_{\mathrm{col}}\Big), \tag{15.7}$$

they will have the form $\Psi \otimes \Phi$, where Φ, corresponding to "‘color"’ degrees of freedom, is a singlet wrt $su_{\mathrm{col}}(3)$.

The smallest (p, q) for which $su_{\mathrm{col}}(3)$ has a singlet representation on $\otimes^p \mathbb{C}^3_{\mathrm{col}} \otimes \otimes^q \overline{\mathbb{C}}^3_{\mathrm{col}}$ are $(1, 1)$ (mesons), $(3, 0)$ (barions) and $(0, 3)$ (antibarions).

In particular, mesons are elements of

$$\begin{aligned} &\Big(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \mathbb{C}^3_{\mathrm{col}}\Big) \otimes \Big(\overline{\mathbb{C}}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}} \otimes \overline{\mathbb{C}}^3_{\mathrm{col}}\Big) \\ \simeq \quad &\Big(\mathbb{C}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^3_{\mathrm{fl}}\Big) \otimes \Big(\mathbb{C}^2_{\mathrm{spin}} \otimes \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}}\Big) \otimes \Big(\mathbb{C}^3_{\mathrm{col}} \otimes \overline{\mathbb{C}}^3_{\mathrm{col}}\Big). \end{aligned}$$

(Note that there is no antisymmetrization). The colorlessness condition yields

$$\Psi \otimes \frac{1}{\sqrt{3}}\big(|1, \overline{1}) + |2, \overline{2}) + |2, \overline{2})\big),$$

where 1,2,3 corresponds to the three colors and

$$\Psi \in \mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \overline{\mathbb{C}}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}} \simeq \left(\mathbb{C}^3_{\mathrm{fl}} \otimes \overline{\mathbb{C}}^3_{\mathrm{fl}}\right) \otimes \left(\mathbb{C}^2_{\mathrm{spin}} \otimes \overline{\mathbb{C}}^2_{\mathrm{spin}}\right).$$

For the representation of $SU(3)_{\mathrm{fl}}$we have $3 \otimes \overline{3} = 8 + 1$. For the representation of $SU(2)_{\mathrm{spin}}$ we have $2 \otimes 2 = 3 + 1$, which yields spin 0 and 1. Hence we obtain both meson nonets.

Here is the "‘quark content"’ of meson nonets:

$$\begin{array}{ccccc} & d\overline{s} & & u\overline{s} & \\ d\overline{u} & & d\overline{d}, u\overline{u}, s\overline{s} & & u\overline{d} \\ & s\overline{u} & & s\overline{d} & \end{array}$$

Mesons of zero charge differ with their quark content. Assuming exact $su(3)$ symmetry they are

$$\begin{aligned} \pi^0 &= \frac{1}{\sqrt{2}}(d\overline{d} - u\overline{u}), \\ \eta &= \frac{1}{\sqrt{6}}(2s\overline{s} - d\overline{d} - u\overline{u}), \\ \eta' &= \frac{1}{\sqrt{3}}(s\overline{s} + d\overline{d} + u\overline{u}). \end{aligned}$$

Barions are elements of

$$\otimes^3_{\mathrm{a}}\left(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}} \otimes \mathbb{C}^3_{\mathrm{col}}\right) \subset \otimes^3\left(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}}\right) \otimes \otimes^3\mathbb{C}^3_{\mathrm{col}}.$$

The colorlessness condition yields

$$\Psi \otimes \frac{1}{\sqrt{3!}}\Big(|1,2,3) + |2,3,1) + |3,1,2) - |1,3,2) - |3,2,1) - |1,3,2)\Big).$$

The color part of the vector is antisymmetric. Hence Ψ has to be an element of $\otimes^3_{\mathrm{s}}\left(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}}\right)$, whose dimension is $\frac{6\cdot 7\cdot 8}{1\cdot 2\cdot 3} = 56$. The action of the group $SU(3)_{\mathrm{fl}} \times SU(2)_{\mathrm{spin}}$ has inside $\otimes^3_{\mathrm{s}}\left(\mathbb{C}^3_{\mathrm{fl}} \otimes \mathbb{C}^2_{\mathrm{spin}}\right)$ a representation $\otimes^3_{\mathrm{s}}\mathbb{C}^3_{\mathrm{fl}} \otimes \otimes^3_{\mathrm{s}}\mathbb{C}^2_{\mathrm{spin}}$. Its dimension is 10×4. What remains is the representation of dimension $56-40 = 16$. It is equivalent to the adjoint representation of $SU(3)_{\mathrm{fl}}$ times the identity on $\mathbb{C}^2$. Thus we have the decomposition

$$\mathbb{C}^{10} \otimes \mathbb{C}^4 \oplus \mathbb{C}^8 \otimes \mathbb{C}^2,$$

The first is a $su_{\mathrm{fl}}(3)$ decuplet (the representation of type $(3,0)$, that is, on $\otimes^3_{\mathrm{s}}\mathbb{C}^3$), and its spin is $\frac{3}{2}$. The second is an $su_{\mathrm{fl}}(3)$ octet (the representation of type $(1,1)$, that is, the adjoint representation) and its spin is $\frac{1}{2}$.

Here is the "‘quark content"’ of the barion multiplets:

$$\begin{array}{ccccccc} ddd & & ddu & & duu & & uuu \\ & dds & & dus & & uus & \\ & & dss & & uss & & \\ & & & sss & & & \end{array}$$

Here are the spin states of the the barions in the middle of the diagram, where there is the greatest degeneracy:

$$\begin{array}{ll}
\Sigma^{*0} & d\uparrow u\uparrow s\uparrow, \\
& \frac{1}{\sqrt{3}}(d\uparrow u\uparrow s\downarrow + d\uparrow u\downarrow s\uparrow + d\downarrow u\uparrow s\uparrow), \\
& \frac{1}{\sqrt{3}}(d\downarrow u\downarrow s\uparrow + d\uparrow u\downarrow s\downarrow + d\downarrow u\uparrow s\downarrow), \\
& d\downarrow u\downarrow s\downarrow; \\
\Sigma^{0} & \frac{1}{\sqrt{6}}(2d\uparrow u\uparrow s\downarrow - d\uparrow u\downarrow s\uparrow - d\downarrow u\uparrow s\uparrow), \\
& \frac{1}{\sqrt{6}}(2d\downarrow u\downarrow s\uparrow - d\uparrow u\downarrow s\downarrow - d\downarrow u\uparrow s\downarrow); \\
\Lambda^{0} & \frac{1}{\sqrt{2}}(d\uparrow u\downarrow s\uparrow - d\downarrow u\uparrow s\uparrow), \\
& \frac{1}{\sqrt{2}}(d\uparrow u\downarrow s\downarrow - d\downarrow u\uparrow s\downarrow).
\end{array}$$

The states of Σ^{*-} and Σ^{-} are obtained from Σ^{*0}, resp Σ^{0} by replacing u with d.

All the physical representations of $SU(3)_{\mathrm{fl}}$ have the triality 0—this is essentially the meaning of the "‘colorlessness"’.

16 Aplications of group theory to Standard Model and Grand Unified Theories

16.1 Conventions

As usual, instead of representations of Lie groups, we will usually speak about representations of the corresponding Lie algebras.

Unitary representations of $u(1)$ are one-dimensional and are given by $q \in \mathbb{R}$, called the charge:

$$u(1) \simeq \mathbb{R} \ni \theta \mapsto \mathrm{e}^{\mathrm{i}\theta q}.$$

When we apply the tensor product, we add the charges.

An irreducible representation of $su(n)$, $so(n)$ are usually denoted by the number of their dimension. For the conjugate representation we add the bar. Thus the fundamental representation of $su(n)$ is denoted by n and the antifundamental by $\overline{n}$.

16.2 Standard Model

The Standard Model is based on the gauge group $SU(3) \times SU(2) \times U(1)$.

Suppose that the (self-adjoint) generators of $su(2)$ are denoted T_1, T_2, T_3. They are the generators of the so-called weak isospin. The self-adjoint generator of $u(1)$ will be denoted Y. It is the so-called weak hypercharge, which should not be confused with the so-called hypercharge, which has the same symbol.

The main assumption of the Weinberg-Salam model (which is the part of the standard model describing the weak and electromagnetic interactions) is the following: the electric charge Q comes partly from $SU(2)$ and partly from $U(1)$. This can be expressed as

$$Q = T_3 + Y. \tag{16.1}$$

(We use the convention from the book by S.Srednicki. Often one replaces Y with $2Y$, so that one obtains $Q = T_3 + \frac{Y}{2}$, which is analogous to the Gell-Mann–Nishijima formula).

Beside the gauge bosons, which correspond the Lie algebra $su(3) \oplus su(2) \oplus u(1)$, the Lagrangian contains charged particles corresponding to varrious irreducible representations (*multiplets*) of the group $SU(3) \oplus SU(2) \oplus U(1)$. Each particle has an antiparticle posessing the opposite chirality and charges. They can be divided as follows:

(1) A multiplet (or several multiplets) of complex scalar (Higgs) bosons needed to break the gauge symmetry $SU(2) \times U(1)$.

(2) Several multiplets of Weyl (chiral) fermions. Every multiplet appears in 3 generations. Fermionic multiplets can be divided in two families:

   (i) *Leptons*, which do not take part in strong interactions, in other words are singlets wrt $SU(3)$.

   (ii) *Quarks*, which are nontrivially transformed by $SU(3)$.

(By a multiplet we mean an irreducible, usually multidimensional representation of the gauge group.)

There exist two versions of the Standard Model: the original version, which we denote SM, does not contain the right-handed neutrinos. In a newer version, denoted νSM, there are additional right-handed neutrinos.

We will consistently use the terminology related to the first generation.

16.3 Leptons

Leptons can be divided into *electrons* and *neutrinos*. Electrons are both left- and right-handed. The left- and right-handed electrons have the same mass. From the point of view of electromagnetic and strong interactions they can be treated as Dirac fermions. They are denoted $e = (e_{\mathrm{L}}, e_{\mathrm{R}})$, They have $Q = -1$. The antiparticle for the electron is called the *positron* and denoted $\bar{e}$.

Neutrinos have $Q = 0$. Electronic neutrinos, denoted ν_e or $\nu_{e,\mathrm{L}}$, are in SM left-chiral and have a zero mass.

$(e_{\mathrm{L}}, \nu_{e,\mathrm{L}})$ form a doublet wrt $SU(2)$. We have

$$T_3 e_{\mathrm{L}} = -\frac{1}{2} e_{\mathrm{L}}, \quad T_3 \nu_{e,\mathrm{L}} = \frac{1}{2} \nu_{e,\mathrm{L}}.$$

Using (16.1), we obtain

$$Y e_{\mathrm{L}} = -\frac{1}{2} e_{\mathrm{L}}, \quad Y \nu_{e,\mathrm{L}} = -\frac{1}{2} \nu_{e,\mathrm{L}}.$$

e_{R} is a singlet for $SU(2)$. Thus $T_3 e_{\mathrm{R}} = 0$ and (16.1) implies

$$Y e_{\mathrm{R}} = -e_{\mathrm{R}}.$$

When describing the multiplets, it is convenient to restrict oneself to left-handed multiplets. Therefore, instead of the right-handed electron we take into account the left-handed positron. It has $Q = 1$ and $T_3 = 0$. Here is its hypercharge:

$$Y \overline{e}_{\mathrm{R}} = \overline{e}_{\mathrm{R}}.$$

In νSM additionally one introduces a right-handed neutrino $\nu_{e,\mathrm{R}}$, which transforms trivially under the gauge group. When describing multiplets we take into account its antiparticle $\overline{\nu}_{e,\mathrm{R}}$, which is left-handed.

Summing up, we have the following multiplets of left-handed leptons:

$$\begin{array}{ccc} L & := (e_{\mathrm{L}}, \nu_{e,\mathrm{L}}) & (1, 2, -\frac{1}{2}), \\ \overline{E} & := \overline{e}_{\mathrm{R}} & (1, 1, 1), \\ \overline{N} & := \overline{\nu}_{e,\mathrm{R}} & (1, 1, 0). \end{array}$$

16.4 Higgs scalar

In order to build invariant mass terms in the Lagrangian we need an additional scalar ϕ, which is a singlet for $SU(3)$ and a doublet for $SU(2)$. It has $Q = 0$ and the weak isospin $-\frac{1}{2}$. Hence $Y = \frac{1}{2}$. Therefore, its representation is

$$(1, 2, \frac{1}{2}).$$

16.5 Quarks

We have two quarks: u and d (recall that we consider a single generation). Proton and neutron, for instance, are built as follows

$$p = uud, \quad n = udd.$$

Therefore, the quarks have the following electric charge:

$$Qu = \frac{2}{3}u, \quad Qd = -\frac{1}{3}d.$$

They are triplets wrt $SU(3)$ – they transform according to the fundamental representation.

The antiquarks have the oposite electric charges

$$Q\overline{u} = -\frac{2}{3}\overline{u}, \quad Q\overline{d} = \frac{1}{3}\overline{d},$$

and transform according to the antifundamental representation.

Left-handed quarks are a doublet wrt $SU(2)$:

$$T_3 u_{\mathrm{L}} = \frac{1}{2}u_{\mathrm{L}}, \quad T_3 d_{\mathrm{L}} = -\frac{1}{2}d_{\mathrm{L}}.$$

Hence,

$$Yu_{\mathrm{L}} = \frac{1}{6}u_{\mathrm{L}}, \quad Yd_{\mathrm{L}} = \frac{1}{6}d_{\mathrm{L}}.$$

Right-handed quarks are singlets wrt $SU(2)$. Therefore,

$$T_3 u_{\mathrm{R}} = 0, \quad T_3 d_{\mathrm{R}} = 0.$$

Hence,

$$Yu_{\mathrm{R}} = \frac{2}{3}u_{\mathrm{R}}, \quad Yd_{\mathrm{R}} = -\frac{1}{3}d_{\mathrm{R}}.$$

Summing up, we have the following multiplets of left-handed quarks:

$$\begin{aligned} Q = (u_{\mathrm{L}}, d_{\mathrm{L}}) \qquad & (3, 2, \frac{1}{6}), \\ \overline{U} = \overline{u}_{\mathrm{R}} \qquad & (\overline{3}, 1, -\frac{2}{3}), \\ \overline{D} = \overline{d}_{\mathrm{R}} \qquad & (\overline{3}, 1, \frac{1}{3}). \end{aligned}$$

16.6 Standard Model Lagrangian

The Standard Model Lagrangian is a singlet wrt the gauge group. One could distinguish the following terms in the Lagrangian:

(1) The kinetic term for gauge fields.

(2) The kinetic terms for fermions.

(3) The kinetic term for scalar bosons.

(4) The scalar boson potential (a "Mexican hat"?)–because of renormalizabilty, it should be a polynomial of maximally degree 4. One also assumes it to be invariant wrt $\phi \to -\phi$.

(5) Mass terms, that is 2-linear terms in fermions without derivatives. They have to be singlets wrt the gauge group, and therefore most of them involve the scalar boson.

Let ψ, ψ' transform according to the fundamental representation of $SU(3)$. All invariant real 2-linear/antilinear expressions built out of ψ, ψ' have the form

$$\overline{\psi}^{\alpha}\psi'_{\alpha}$$

and their complex conjugates.

Let ψ, ψ' transform according to the fundamental representation of $SU(2)$. Then invariant two-linear/antilinear expressions built out of ψ, ψ' have the form

$$\begin{aligned} &\overline{\psi}^{i}\psi'_{i}, \\ &\epsilon^{ij}\psi_i\psi'_j, \end{aligned}$$

and their complex conjugates.

If $\psi_1, \dots, \psi_n$ have charges $y_1, \dots, y_n$ wrt $U(1)$, then $\psi_1 \cdots \psi_n$ is invariant iff $y_1 + \cdots + y_n = 0$. Therefore, possible non-kinetic terms in the ν SM Lagrangian involving only left-handed fermions are

$$\overline{\phi}_i \phi^i, \quad \left(\overline{\phi}_i \phi^i\right)^2, \tag{16.2}$$

$$\epsilon^{ij} \phi_i \overline{E} L_j, \ \epsilon^{ij} \phi_i \overline{D}^\alpha Q_{\alpha j}, \ \overline{\phi}^i \overline{U}^\alpha Q_{\alpha i}, \tag{16.3}$$

$$\overline{\phi}^i L_i \overline{N}, \quad \overline{N} C \overline{N}. \tag{16.4}$$

Right-handed fermions appear in expressions conjugate to (16.3) and (16.4). α runs over the color index, i, j runs over the indices $1, 2$. C is the charge conjugation matrix. SM contains only (16.2) and (16.3).

16.7 $SU(n)$

$SU(n)$ has a fundamental and antifundamental representation in $\mathbb{C}^n$, resp. $\overline{\mathbb{C}}^n$. We will need the following irreducible representations:

$$\otimes_{\mathrm{s}}^p \mathbb{C}^n, \quad p = 1, 2, \dots, \qquad \dim \otimes_{\mathrm{s}}^n \mathbb{C}^d = \frac{(d+n-1)!}{(d-1)! n!}.$$

$$\otimes_{\mathrm{a}}^q \mathbb{C}^n, \quad q = 1, \dots, n-1, \qquad \dim \otimes_{\mathrm{a}}^n \mathbb{C}^d = \frac{n!}{d!(n-d)!}.$$

We have

$$\otimes_{\mathrm{a}}^q \mathbb{C}^n \simeq \otimes_{\mathrm{a}}^{n-q} \overline{\mathbb{C}}^n.$$

$$\otimes^2 \mathcal{Z} = \otimes_{\mathrm{s}}^2 \mathcal{Z} \oplus \otimes_{\mathrm{a}}^2 \mathcal{Z}.$$

We will use the following relations for any pair of spaces $\mathcal{Z}, \mathcal{W}$:

$$\otimes_{\mathrm{s/a}}^p (\mathcal{Z} \oplus \mathcal{W}) \simeq \bigoplus_{j=0}^{p} \otimes_{\mathrm{s/a}}^j \mathcal{Z} \oplus \otimes_{\mathrm{s/a}}^{p-j} \mathcal{W}.$$

16.8 Extending $SU(3) \otimes SU(2) \times U(1)$ to $SU(5)$

The following analysis is based partly on the book by Srednicki and the article by Baez-Huerta.

Set

$$Y = \begin{bmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{bmatrix}.$$

Let $A \in su(3)$, $B \in su(2)$ and $s \in \mathbb{R} \simeq u(1)$. Then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + sY \in su(5).$$

Thus we have the inclusion $su(3) \oplus su(2) \oplus u(1) \subset su(5)$, where Y is the generator of $u(1)$.

The fundamental representation of $su(5)$ can be decomposed as follows:

$$5 \to (3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2}).$$

Hence,

$$\overline{5} \quad \to \quad (\overline{3}, 1, \frac{1}{3}) \oplus (1, 2, -\frac{1}{2}). \tag{16.5}$$

$\otimes_{\mathrm{a}}^2 5 = 10$, the representation of $su(5)$, can be decomposed as

$$\begin{aligned} \otimes_{\mathrm{a}}^2 5 \quad &\to \quad \otimes_{\mathrm{a}}^2 (3, 1, -\frac{1}{3}) \oplus (3, 1, -\frac{1}{3}) \otimes (1, 2, \frac{1}{2}) \oplus \otimes_{\mathrm{a}}^2 (1, 2, \frac{1}{2}) \\ &= \quad (\overline{3}, 1, -\frac{2}{3}) \oplus (3, 2, \frac{1}{6}) \oplus (1, 1, 1), \end{aligned} \tag{16.6}$$

where we used the property $\otimes_{\mathrm{a}}^2 3 = \overline{3}$ of the representation of $su(3)$.

All left-handed multiplets of SM wrt $SU(3) \times SU(2) \times U(1)$ can be found in two multiplets wrt $SU(5)$: (16.5) and (16.6):

$$\begin{aligned} \otimes_{\mathrm{a}}^4 5 = \overline{5} : &\quad \overline{D}, L; \\ \otimes_{\mathrm{a}}^2 5 = 10 : &\quad \overline{U}, Q, E. \end{aligned}$$

16.9 Fields in GUT based on $SU(5)$

In GUT based on $SU(5)$, without a right-handed neutrino, beside gauge bosons parametrized by $su(5)$, we have the following fields:

(1) Complex scalar bosons

   (1) The boson Φ in the adjoint representation of $SU(5)$, responsible for breaking $SU(5)$ to $SU(3) \times SU(2) \times U(1)$. It couples only to gauge bosons and to ϕ.

   (2) The boson ϕ in the antifundamental representation of $SU(5)$ responsible for breaking $SU(2) \times U(1)$ to $U(1)$.

(3) Weyl left-handed fermions (and their antiparticles):

   (1) The multiplet $\psi = (L, \overline{D}) = (e_{\mathrm{L}}, \nu_{\mathrm{L}}, \overline{d}_{\mathrm{R}})$ in $\overline{5}$ (the antifundamental representation).

   (2) The multiplet $\chi = (\overline{E}, Q, \overline{U}) = (\overline{e}_{\mathrm{R}}, u_{\mathrm{L}}, d_{\mathrm{L}}, \overline{u}_{\mathrm{R}})$ in 10 (the antisymmetric representation).

Possible non-kinetic terms in the Lagrangian:

$$\begin{aligned} &\mathrm{Tr}\Phi^2,\ \mathrm{Tr}\Phi^4,\ (\mathrm{Tr}\Phi)^2, \\ &\overline{\phi} \cdot \phi,\ (\overline{\phi} \cdot \phi)^2,\ \overline{\phi} \cdot \Phi^2 \phi, \\ &\phi^i \psi^j \chi_{ij}, \quad \epsilon^{ijklm} \overline{\phi}_i \chi_{jk} \chi_{lm}. \end{aligned}$$

If we want neutrinos to have a mass, we need to add the field ν_{R}, which is a singlet for $SU(5)$ and the term

$$\overline{\phi}_i \psi^i \overline{\nu}_{\mathrm{R}}.$$

16.10 Extending $SU(3) \otimes SU(2) \times U(1)$ to $Spin(10)$

All the left-handed multiplets of the Standard Model wrt $SU(3) \times SU(2) \times U(1)$ can be found in the following two multiplets wrt $SU(5)$: $\otimes_{\mathrm{a}}^4 5$ (16.5) and $\otimes_{\mathrm{a}}^2 5$ (16.6). To obtain antiparticles it suffices to add $\otimes_{\mathrm{a}}^1 5$ and $\otimes_{\mathrm{a}}^3 5$. To include right-handed neutrinos and their antiparticles it suffices to add $\otimes_{\mathrm{a}}^0 5$ and $\otimes_{\mathrm{a}}^5 5$. We obtain a space that naturally identifies with the Fock space $\Gamma_{\mathrm{a}}(\mathbb{C}^5)$. It decomposes in two irreducible representations of $Spin(10)$ corresponding to left- and right-handed particles.

We consider the fermionic Fock space with the basis u, d, r, g, b. (u, d are not up and down quarks, although they are related to them). The antiparticles of left-handed leptons are u, d, and the righthanded down quark is r, g, b, depending on the color. The antiparticle to the righthanded up quark is made out of missing colors. The left-handed quarks are made of the "color" and of u, d. The antiparticle to the righthanded positron is ud.

The righthanded neutrino is identified with the "ceiling vector". The antiparicles are always made out of the missing constituents.

In the following list c denotes one of the colors r, g, b, and c, c', c'' is one of cyclic permutations of r, g, b. We write $a_1 \cdots a_n$ instead of $\frac{1}{\sqrt{n!}} a_1 \wedge \cdots \wedge a_n$.

$1, \mathrm{L}$	$5, \mathrm{R}$	$10, \mathrm{L}$	$\overline{10}, \mathrm{R}$	$\overline{5}, \mathrm{L}$	$1, \mathrm{R}$
$\overline{\nu}_{\mathrm{R}} = 1$	$\overline{e}_{\mathrm{L}} = u$	$\overline{e}_{\mathrm{R}} = ud$	$e_{\mathrm{R}} = cc'c''$	$e_{\mathrm{L}} = cc'c''d$	$\nu_{\mathrm{R}} = cc'c''ud$
	$\overline{\nu}_{\mathrm{L}} = d$	$u_{\mathrm{L}}^c = cu$	$\overline{u}_{\mathrm{L}}^c = c'c''d$	$\nu_{\mathrm{L}} = cc'c''u$	
	$d_{\mathrm{R}}^c = c$	$d_{\mathrm{L}}^c = cd$	$\overline{d}_{\mathrm{L}}^c = c'c''u$	$\overline{d}_{\mathrm{R}}^c = c'c''ud$	
		$\overline{u}_{\mathrm{R}}^c = c'c''$	$u_{\mathrm{R}}^c = cud$		

16.11 Extending $SU(3) \otimes SU(2) \times U(1)$ to $SU(2) \times SU(2) \times SU(4)$

In the Pati-Salam Theory we assume the existence of the fourth color "'white"', denoted w representing leptons. $SU(4)$ acts in two representations: the fundamental with basis r, g, b, w and antifundamental with basis $\overline{r}, \overline{g}, \overline{b}, \overline{w}$. c will denote r, g, or b.

The "left" group $SU(2)$ acts on $\mathbb{C}^2$ with basis u_{L}, d_{L}. The "right" group $SU(2)$ acts on $\mathbb{C}^2$ with basis u_{R}, d_{R}. These are "prequarks". The notation u and d now corresponds to the "isospin". The charge conjugation switches the "isospin" and chirality. Therefore, $\overline{u}_{\mathrm{L}} = d_{\mathrm{R}}$, $\overline{d}_{\mathrm{L}} = u_{\mathrm{R}}$.

Leptons are obtained by multiplying prequarks with w or $\overline{w}$. Quarks are obtained by multiplying prequarks with colors. Particles (including the right neutrino) are organized into four representations of $SU(2) \times SU(2) \times SU(4)$:

$(2, 1, 4)$	$(1, 2, 4)$	$(2, 1, \overline{4})$	$(1, 2, \overline{4})$
$\nu_{\mathrm{L}} = u_{\mathrm{L}} \otimes w$	$\nu_{\mathrm{R}} = u_{\mathrm{R}} \otimes w$	$\overline{e}_{\mathrm{R}} = u_{\mathrm{L}} \otimes \overline{w}$	$\overline{e}_{\mathrm{L}} = u_{\mathrm{R}} \otimes \overline{w}$
$e_{\mathrm{L}} = d_{\mathrm{L}} \otimes w$	$e_{\mathrm{R}} = d_{\mathrm{R}} \otimes w$	$\overline{\nu}_{\mathrm{R}} = d_{\mathrm{L}} \otimes \overline{w}$	$\overline{\nu}_{\mathrm{L}} = d_{\mathrm{R}} \otimes \overline{w}$
$u_{\mathrm{L}}^c = u_{\mathrm{L}} \otimes c$	$u_{\mathrm{R}}^c = u_{\mathrm{R}} \otimes c$	$\overline{d}_{\mathrm{R}}^{\overline{c}} = u_{\mathrm{L}} \otimes \overline{c}$	$\overline{d}_{\mathrm{L}}^{\overline{c}} = u_{\mathrm{R}} \otimes \overline{c}$
$d_{\mathrm{L}}^c = d_{\mathrm{L}} \otimes c$	$d_{\mathrm{R}}^c = d_{\mathrm{R}} \otimes c$	$\overline{u}_{\mathrm{R}}^{\overline{c}} = d_{\mathrm{L}} \otimes \overline{c}$	$\overline{u}_{\mathrm{L}}^{\overline{c}} = d_{\mathrm{R}} \otimes \overline{c}$

Let us introduce the operators of "left and right isospin", and the "color operator":

$$T_3^{\mathrm{L}} u_{\mathrm{L}} = \frac{1}{2} u_{\mathrm{L}}, \quad T_3^{\mathrm{L}} d_{\mathrm{L}} = -\frac{1}{2} d_{\mathrm{L}}; \tag{16.7}$$

$$T_3^{\mathrm{R}} u_{\mathrm{R}} = \frac{1}{2} u_{\mathrm{R}}, \quad T_3^{\mathrm{L}} d_{\mathrm{R}} = -\frac{1}{2} d_{\mathrm{R}}; \tag{16.8}$$

$$Zw = -\frac{1}{2} w, \quad Zc = \frac{1}{6} c. \tag{16.9}$$

They can be used to express the usual weak isospin and the weak hypercharge:

$$T = T^{\mathrm{L}}, \quad Y = T^{\mathrm{R}} + Z. \tag{16.10}$$

Thus $SU(3) \times SU(2) \times U(1) \subset SU(2) \times SU(2) \times SU(4)$, where $SU(3)$ is embedded in $SU(4)$, the weak $SU(2)$ coincides with the first Pati-Salam $SU(2)$ and $U(1)$ is defined by Y in (16.10).

We have the isomorphism $SU(4) \simeq Spin(6)$. We can reorganize the representation space of $SU(4)$ as a representation space of $Spin(6)$ as follows:

$$\begin{aligned} \langle w, r, g, b \rangle \oplus \langle \overline{w}, \overline{r}, \overline{g}, \overline{b} \rangle \simeq \mathbb{C}^4 \oplus \overline{\mathbb{C}}^4 \\ \simeq \langle \overline{w} \rangle \oplus \langle r, g, b \rangle \oplus \langle \overline{r}, \overline{g}, \overline{b} \rangle \oplus \langle w \rangle \simeq \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C} \simeq \Gamma_{\mathrm{a}}(\mathbb{C}^3), \end{aligned}$$

using the dictionary

$$w = rgb, \quad \overline{w} = 1, \qquad \overline{r} = gb, \quad \overline{g} = br, \quad \overline{b} = rg. \tag{16.11}$$

We also have the isomorphism $SU(2) \times SU(2) \simeq Spin(4)$. We can reorganize the representation of $SU(2) \times SU(2)$ as a representation of $Spin(4)$ as follows:

$$\begin{aligned} \langle u_L, d_L \rangle \oplus \langle u_R, d_R \rangle \simeq \mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}^2 \\ \simeq \langle u_L \rangle \oplus \langle d_L, d_R \rangle \oplus \langle u_R \rangle \simeq \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \simeq \Gamma_{\mathrm{a}}(\mathbb{C}^2). \end{aligned}$$

using the dictionary

$$u_L = 1, \qquad u_R = d_L d_R.$$

Hence the group $SU(2) \times SU(2) \times SU(4)$ can be identified with $Spin(4) \times Spin(6)$. Clearly, $Spin(4) \times Spin(6)/\mathbb{Z}_2$ is a subgroup of $Spin(10)$.