ON SOME FAMILIES OF EXACTLY SOLVABLE SCHRÖDINGER OPERATORS

JAN DEREZIŃSKI

Department of Mathematical Methods in Physics

FACULTY OF PHYSICS

UNIVERSITY OF WARSAW

1. POTENTIALS $\frac{1}{x^2}$
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Schrödinger operators that can be solved in terms of well-understood special functions play an important role in applications.

It is natural to organize them in holomorphic families of closed operators. Clearly, for most values of parameters these operators are non-self-adjoint. Even if we are interested mostly in self-adjoint cases, the existence of (non-self-adjoint) analytic continuation plays sometimes an important role in applications. Let me mention e.g. Regge poles, which were famous in the 60's.

HOMOGENEOUS SCHRÖDINGER OPERATORS

(in collaboration with LAURENT BRUNEAU
and VLADIMIR GEORGESCU)

Consider a formal differential expression

$$L_\alpha = -\partial_x^2 + \Big(-\frac{1}{4} + \alpha \Big) \frac{1}{x^2}.$$

We would like to interpret it as a well-defined (unbounded) operator on $L^2[0,\infty[$. To do this we need to specify its domain.

L_α, and closely related operators H_m and $H_{m,\kappa}$, which we introduce shortly, appear in various important and iteresting applications:

- The radial part of the Laplacian in dimension d on spherical harmonics of order l is expressed by L_α with $\alpha = (l + \frac{d-2}{2})^2$.
- These operators appear in the decomposition of the Aharonov-Bohm Hamiltonian, in the membranes with conical singularities, in the theory of many body systems with contact interactions and in the Efimov effect.

- They have a surprisingly rich mathematical phenomenology, which should be close to physicists' hearts: the "running coupling constant" flows under the action of the "renormalization group", there are two "phase transitions", attractive and repulsive fixed points, limit cycles, breakdown of conformal symmetry, etc.
- They have rather subtle and rich properties illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators). They naturally appear in representations of $sl(2,\mathbb{R})$.

- Essentially all basic objects related to H_m, such as their resolvents, spectral projections, wave and scattering operators and the evolution, can be explicitly computed in terms of special functions. A number of nontrivial identities involving functions from the Bessel family find an appealing operator-theoretical interpretation in terms of the operators H_m. Eg. the Barnes identity leads to the formula for wave operators.

Let U_τ be the group of dilations on $L^2[0, \infty[$, that is

$$(U_\tau f)(x) = \mathrm{e}^{\tau/2} f(e^\tau x).$$

We say that B is homogeneous of degree ν if

$$U_\tau B U_\tau^{-1} = \mathrm{e}^{\nu\tau} B.$$

Clearly, L_α is homogeneous of degree -2.

Here are two natural questions:

1. If $\alpha \in \mathbb{R}$, how to interpret L_α as a self-adjoint operator on $L^2[0, \infty[$? When is it homogeneous of degree -2?
2. If $\alpha \in \mathbb{C}$, how to interpret L_α as a closed operator on $L^2[0, \infty[$? When is it homogeneous of degree -2?

Two naive interpretations of L_α:

1. The minimal operator $L_\alpha^{\min}$: We start from L_α on $C_{\mathrm{c}}^\infty]0,\infty[$, and then we take its closure.
2. The maximal operator $L_\alpha^{\max}$: We consider the domain consisting of all
   $f \in L^2[0,\infty[$ such that $L_\alpha f \in L^2[0,\infty[$.

Clearly, $\mathrm{Dom}(L_{\min}) \subset \mathrm{Dom}(L_{\max})$ and

$$L_{\max}\Big|_{\mathrm{Dom}(L_{\min})} = L_{\min}.$$

In other words $L_{\min} \subset L_{\max}$.

We will see that it is often natural to write $\alpha = m^2$

Theorem 1. .

1. *For* $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = L_{m^2}^{\max}$.
2. *For* $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, *and the codimension of their domains is* 2.
3. $(L_{\alpha}^{\min})^* = L_{\overline{\alpha}}^{\max}$. *Hence, for* $\alpha \in \mathbb{R}$, $L_{\alpha}^{\min}$ *is Hermitian.*
4. $L_{\alpha}^{\min}$ *and* $L_{\alpha}^{\max}$ *are homogeneous of degree* -2.

Notice that

$$L x^{\frac{1}{2} \pm m} = 0.$$

Let ξ be a compactly supported cutoff equal 1 around 0.

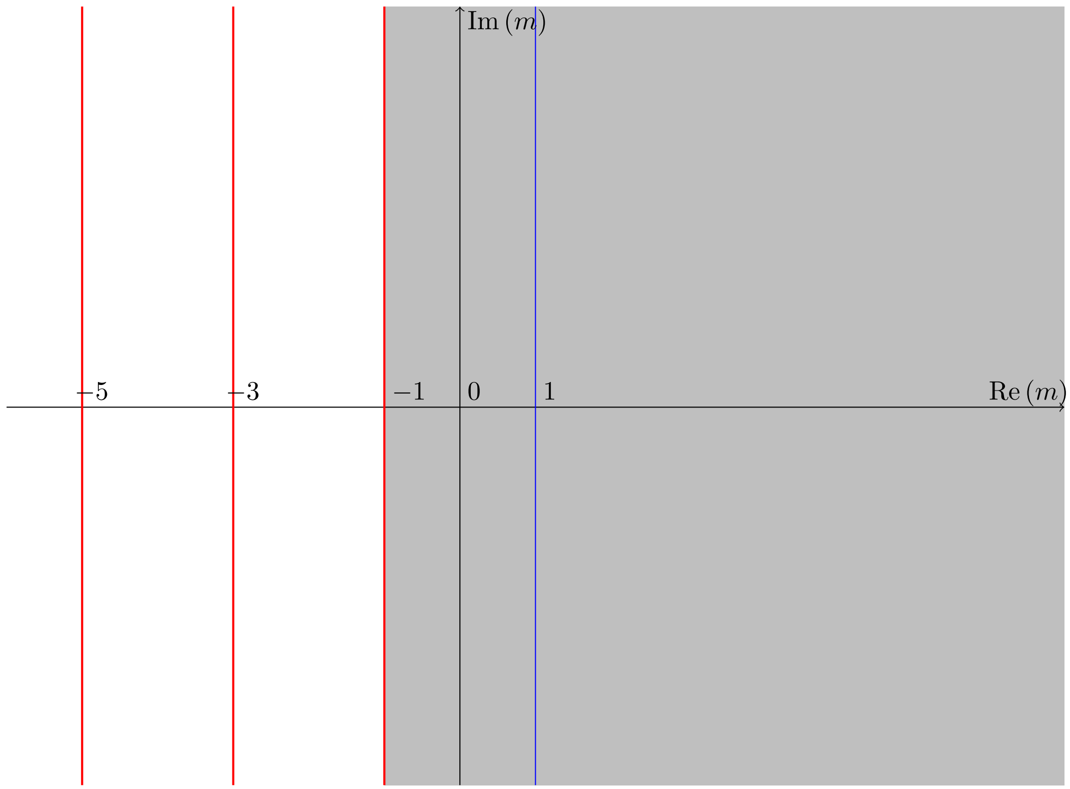
Let $-1 < \operatorname{Re} m$. Note that $x^{\frac{1}{2}+m}\xi$ belongs to $\operatorname{Dom} L_{m^2}^{\max}$.

This suggests to define the operator H_m to be the restriction of $L_{m^2}^{\max}$ to

$$\operatorname{Dom} L_{m^2}^{\min} + \mathbb{C} x^{\frac{1}{2}+m}\xi.$$

Theorem 2. .

1. *For* $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = H_m = L_{m^2}^{\max}$.
2. *For* $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}$ *and the codimension of the domains is* 1.
3. $H_m^* = H_{\overline{m}}$. *Hence, for* $m \in]-1, \infty[$, H_m *is self-adjoint.*
4. H_m *is homogeneous of degree* -2.
5. $\operatorname{sp} H_m = [0, \infty[$.
6. $\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$ *is a holomorphic family of closed operators.*



Im (m)
−5
−3
−1
0
1
Re (m)

Theorem 3. .

1. *For* $\alpha \geq 1$, $L_\alpha^{\min} = H_{\sqrt{\alpha}}$ *is essentially self-adjoint on* $C_{\mathrm{c}}^\infty]0, \infty[$.
2. *For* $\alpha < 1$, $L_\alpha^{\min}$ *is not essentially self-adjoint on* $C_{\mathrm{c}}^\infty]0, \infty[$.
3. *For* $0 \leq \alpha < 1$, *the operator* $H_{\sqrt{\alpha}}$ *is the Friedrichs extension and* $H_{-\sqrt{\alpha}}$ *is the Krein extension of* $L_\alpha^{\min}$.
4. $H_{\frac{1}{2}}$ *is the Dirichlet Laplacian and* $H_{-\frac{1}{2}}$ *is the Neumann Laplacian on halfline.*
5. *For* $\alpha < 0$, $L_\alpha^{\min}$ *has no homogeneous selfadjoint extensions.*

It is easy to see that

$$x^{-\frac{1}{2}}\Big(-\partial_x^2+\big(-\frac{1}{4}+\alpha\big)\frac{1}{x^2}\pm 1\Big)x^{\frac{1}{2}}$$
$$=-\partial_x^2-\frac{1}{x}\partial_x+\big(-\frac{1}{4}+\alpha\big)\frac{1}{x^2}\pm 1,$$

which is the (modified) Bessel operator.

Therefore, it is not surprising that various objects related to H_m can be computed with help of functions from the Bessel family.

Theorem 4. . *If* $R_m(\lambda; x, y)$ *is the integral kernel of the operator* $(\lambda - H_m)^{-1}$, *then for* $\operatorname{Re} k > 0$ *we have*

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if } x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if } x > y, \end{cases}$$

where I_m *is the modified Bessel function and* K_m *is the MacDonald function*.

Proposition 5.. *For* $0 < a < b < \infty$, *the integral kernel of* $\mathbb{1}_{[a,b]}(H_m)$ *is*

$$\mathbb{1}_{[a,b]}(H_m)(x,y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \mathrm{d}k,$$

where J_m *is the Bessel function.*

Let $\mathcal{F}_m$ be the operator on $L^2[0, \infty[$ given by

$$\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx)\sqrt{kx} f(x) \mathrm{d}x$$

$\mathcal{F}_m$ is the so-called Hankel transformation. Define also the operator $Xf(x) := xf(x)$.

Theorem 6.. *$\mathcal{F}_m$ is a bounded invertible involution on $L^2[0, \infty[$ diagonalizing H_m and anticommuting with the self-adjoint generator o dilations $A = \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x)$:*

$$\mathcal{F}^2 = \mathbb{1},$$

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = X^2,$$

$$\mathcal{F}_m A = -A\mathcal{F}_m.$$

Theorem 7. *Set*

$$\mathcal{I}f(x) = x^{-1}f(x^{-1}), \quad \Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$$

Then

$$\mathcal{F}_m = \Xi_m(A)\mathcal{I}.$$

Therefore, we have the identity

$$H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A)$$

(Result obtained independently by Bruneau, Georgescu, D, and by Richard and Pankrashkin).

Theorem 8. *The wave operators associated to the pair* H_m, H_k *exist and*

$$\begin{aligned}\Omega_{m,k}^{\pm} &:= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_m}\mathrm{e}^{-\mathrm{i}tH_k} \\ &= \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{F}_m\mathcal{F}_k \\ &= \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\frac{\Xi_k(A)}{\Xi_m(A)}.\end{aligned}$$

The formula

$$H_m := \Xi_m^{-1}(A) X^{-2} \Xi_m(A) \tag{1}$$

valid for $\operatorname{Re} m > -1$ can be used as an alternative definition of the family H_m also beyond this domain. It defines a family of closed operators for the parameter m that belongs to

$$\{m \mid \operatorname{Re} m \neq -1, -2, \dots\} \cup \mathbb{R}. \tag{2}$$

Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of (2).

In fact, $\Xi_m(A)$ is a unitary operator for all real values of m. Therefore, for $m \in \mathbb{R}$, (1) is well-defined and self-adjoint.

$\Xi_m(A)$ is bounded and invertible also for all m such that $\mathrm{Re}\, m \neq -1, -2, \dots$. Therefore, the formula (1) defines an operator for all such m.

One can then pose various questions:

- What happens with this operator along the lines $\mathrm{Re}\, m = -1, -2, \ldots$?
- What is the meaning of the operator to the left of $\mathrm{Re} = -1$? (It is not a differential operator!)

The definition (or actually a number of equivalent definitions) of a holomorphic family of bounded operators is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a holomorphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto B(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\mathrm{Ran}\, B(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)B(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 9.. *Suppose that* $\{H(z)\}_{z\in\Theta}$ *is a function whose values are closed operators on* $\mathcal{H}$. *Suppose in addition that for any* $z \in \Theta$ *the resolvent set of* $H(z)$ *is nonempty. Then* $z \mapsto H(z)$ *is a holomorphic family of closed operators if and only if for any* $z_0 \in \Theta$ *there exists* $\lambda \in \mathbb{C}$ *and a neighborhood* Θ_0 *of* z_0 *such that* λ *belongs to the resolvent set of* $H(z)$ *for* $z \in \Theta_0$ *and* $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ *is holomorphic on* Θ_0.

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set.

Conjecture 10.. *It is impossible to extend*

$$\{\mathrm{Re}\, m > -1\} \ni m \mapsto H_m$$

to a holomorphic family of closed operators on a larger connected open subset of $\mathbb{C}$.

RADIAL COULOMB SCHRÖDINGER OPERATORS

(in collaboration with SERGE RICHARD)

Consider

$$L_{\beta,\alpha} := -\partial_x^2 + \Big(\alpha - \frac{1}{4}\Big)\frac{1}{x^2} - \frac{\beta}{x},$$

where the parameters β, α are complex numbers. It is natural to write $\alpha = m^2$.

Its eigenvalue equation

$$\left(-\partial_x^2 + \left(\alpha - \frac{1}{4}\right)\frac{1}{x^2} - \frac{\beta}{x} - \frac{1}{4}\right) u(x) = 0$$

is known as the Whittaker equation. Therefore, its solutions can be expressed in terms of Whittaker functions, or equivalently, ${}_1F_1$ and ${}_2F_0$ confluent functions.

For any $m \in \mathbb{C}$ with $\mathrm{Re}\,(m) > -1$ we introduce the closed operator $H_{\beta,m}$ that equals L_{β,m^2} on functions that behave as

$$x^{\frac{1}{2}+m}\big(1 - \frac{\beta}{1+2m}x\big)$$

near zero. We obtain a family

$$\mathbb{C} \times \{m \in \mathbb{C} \mid \mathrm{Re}\,(m) > -1\} \ni (\beta, m) \mapsto H_{\beta,m}\,,$$

which is holomorphic except for a singularity at $(0, -\frac{1}{2})$.

If $\mathrm{Re}\,(m) \geq 1$ the boundary codition is not needed. If $\mathrm{Re}\,(m) \geq -\frac{1}{2}$, then we can use a simplified boundary condition

$$x^{\frac{1}{2}+m}.$$

For $-1 < \mathrm{Re}\,(m) < 1$ the family $H_{\beta,m}$ does not describe all possible well-posed extensions of $L^{\min}_{\beta,m^2}$ (extensions with a non-empty resolvent set). It describes all extensions with pure boundary conditions. I will not discuss mixed boundary conditions, which can also be analyzed (work in progress together with J. Faupin, Q. N. Nguyen and S. Richard).

Let us present

$$e^{-i2\phi}\sigma\left(H_{\beta,-0.75+3.2i}\right)$$

$$\beta = e^{i\phi} \quad \text{for} \quad \phi = \frac{n}{8}\pi \quad \text{with} \quad n = 0, \ldots, 15.$$

By dilation analyticity, the point spectrum does not move, the continuous spectrum, on the other hand, rotates as $e^{-i2\phi}$, like a giant hand of a clock. Eigenvalues hit by the continuous spectrum disappear and become resonances. Then they reappear when the hand of the clock comes again.

Spectrum is blue, resonances are red.

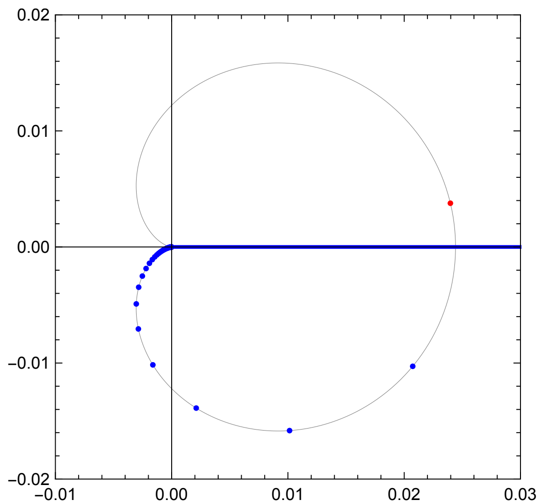


Fig. 9. $\phi = 0$

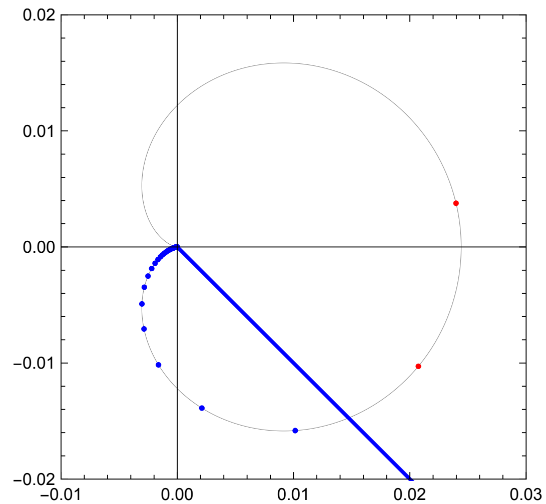


Fig. 10. $\phi = \frac{1}{8}\pi$

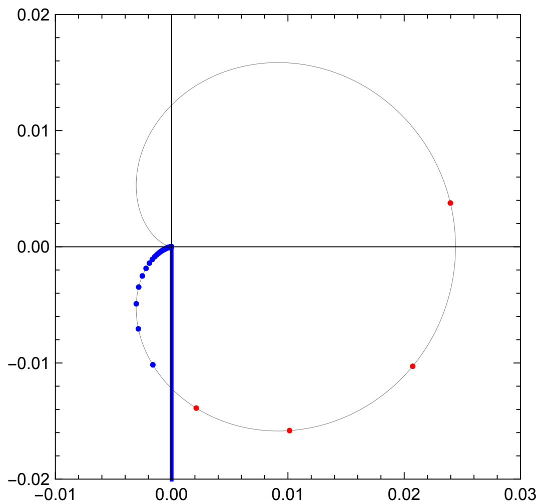


Fig. 11. $\phi = \frac{1}{4}\pi$

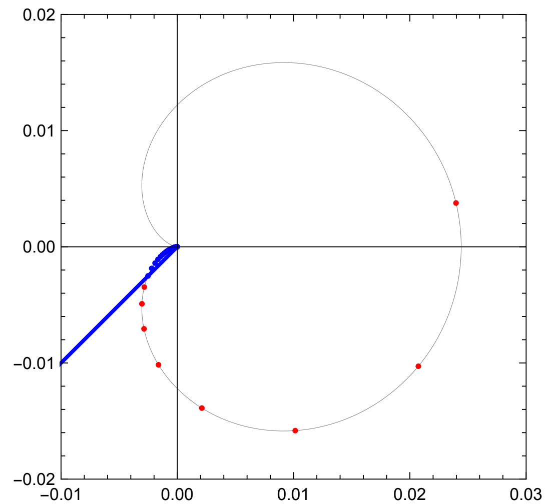


Fig. 12. $\phi = \frac{3}{8}\pi$

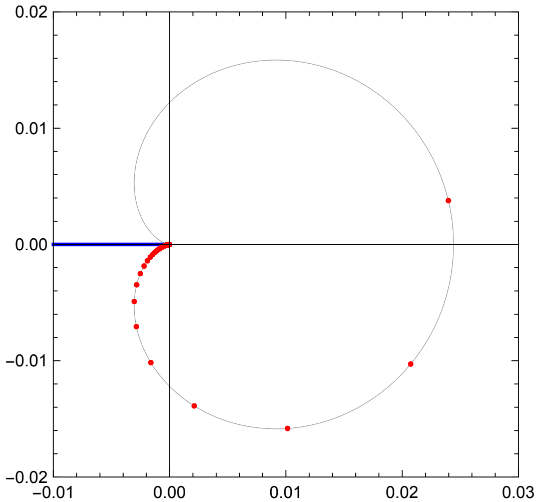


Fig. 13. $\phi = \frac{1}{2}\pi$

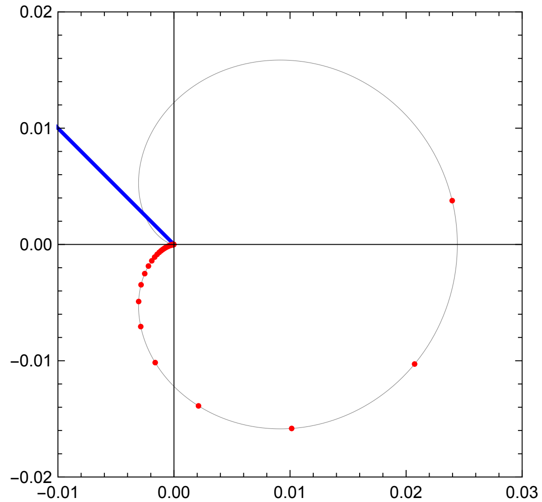


Fig. 14. $\phi = \frac{5}{8}\pi$

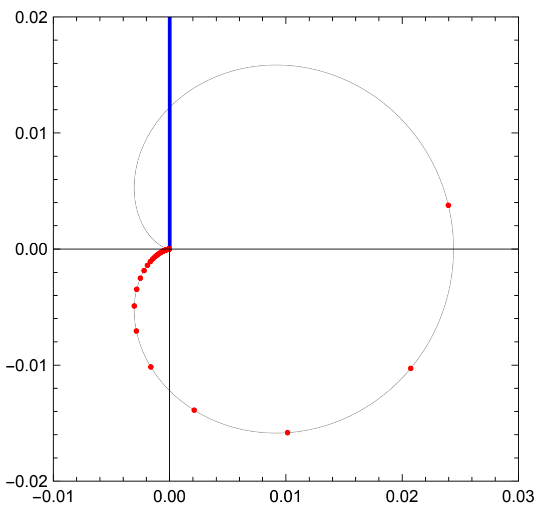


Fig. 15. $\phi = \frac{3}{4}\pi$

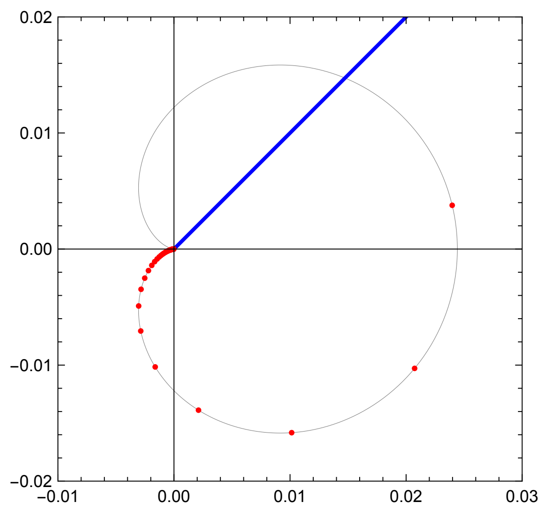


Fig. 16. $\phi = \frac{7}{8}\pi$

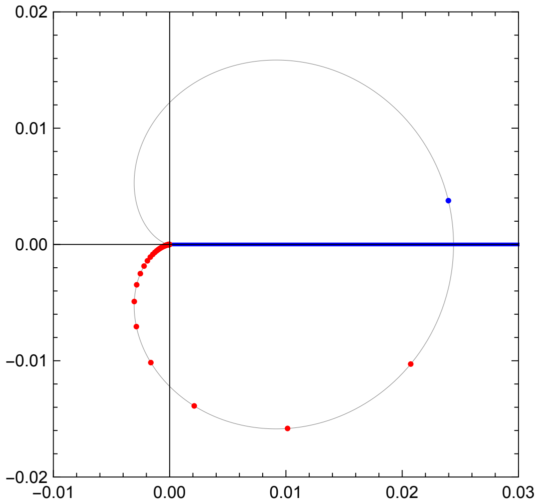


Fig. 17. $\phi = \pi$

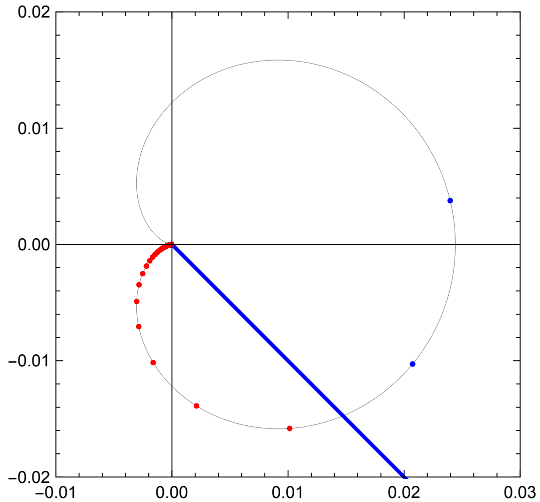


Fig. 18. $\phi = \frac{9}{8}\pi$

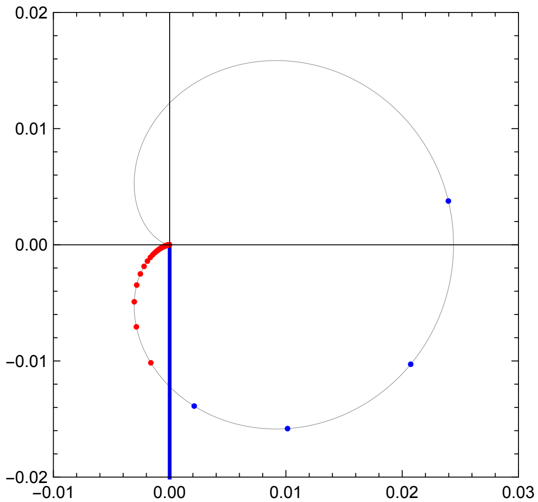


Fig. 19. $\phi = \frac{5}{4}\pi$

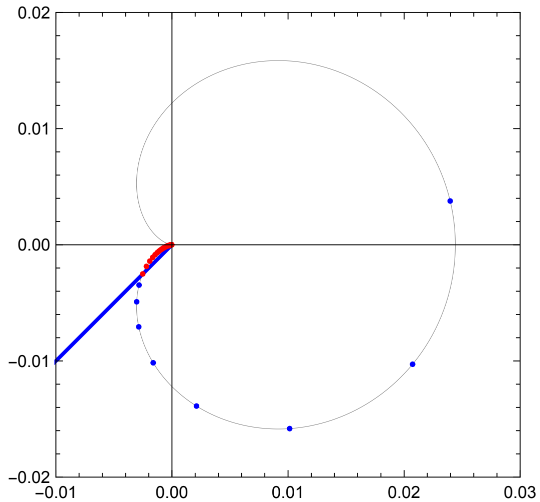


Fig. 20. $\phi = \frac{11}{8}\pi$

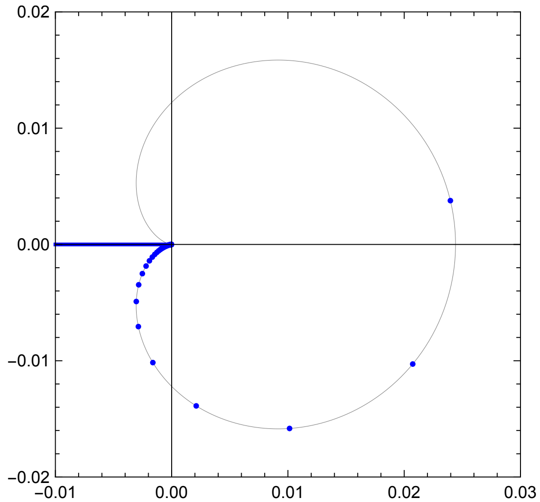


Fig. 21. $\phi = \frac{3}{2}\pi$

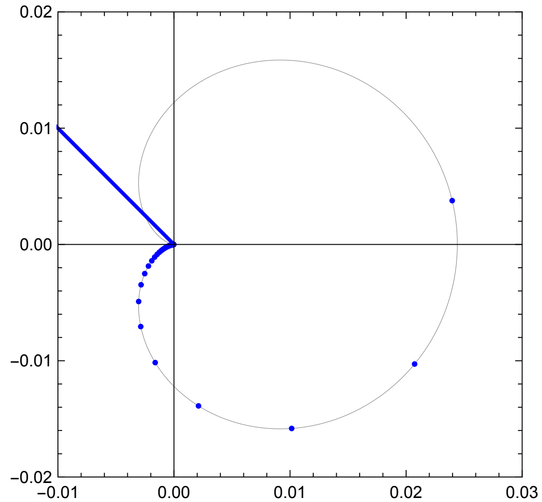


Fig. 22. $\phi = \frac{13}{8}\pi$

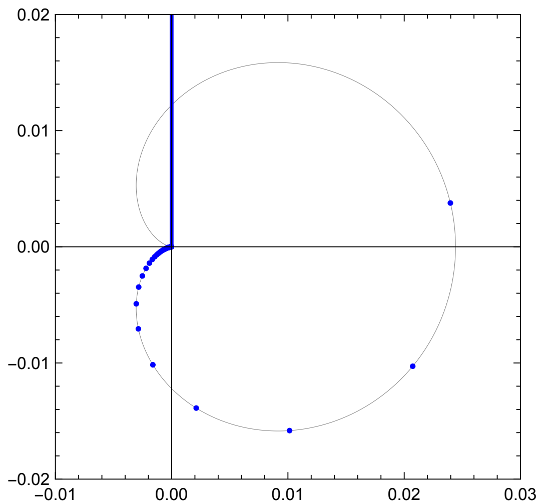


Fig. 23. $\phi = \frac{7}{4}\pi$

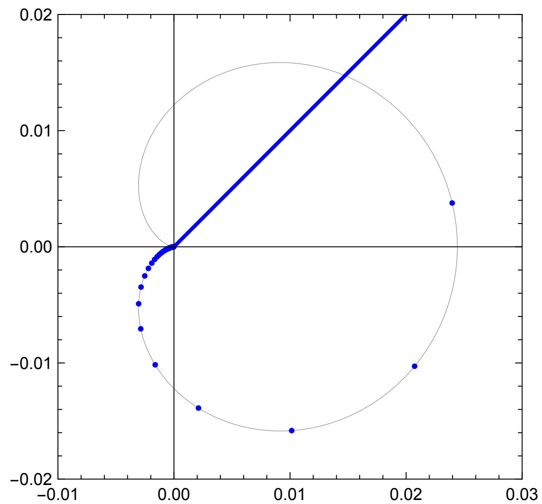


Fig. 24. $\phi = \frac{15}{8}\pi$

The singularity at $(\beta, m) = (0, -\frac{1}{2})$ is quite curious: it is invisible when we consider just the variable m. In fact,

$$m \mapsto H_m = H_{0,m}$$

is holomorphic around $m = -\frac{1}{2}$, and $H_{-\frac{1}{2}}$ has the Neumann boundary condition. It is also holomorphic around $m = \frac{1}{2}$, and $H_{\frac{1}{2}}$ has the Dirichlet boundary condition. Thus one has

$$H_{0,-\frac{1}{2}} \neq H_{0,\frac{1}{2}}.$$

If we introduce the Coulomb potential, then

$$\text{whenever } \beta \neq 0, \quad H_{\beta,-\frac{1}{2}} = H_{\beta,\frac{1}{2}}.$$

The function

$$(\beta, m) \mapsto H_{\beta,m} \qquad (*)$$

is holomorphic around $(0, \frac{1}{2})$, in particular,

$$\lim_{\beta \to 0} (\mathbb{1} + H_{\beta,\frac{1}{2}})^{-1} = (\mathbb{1} + H_{0,\frac{1}{2}})^{-1}.$$

But

$$\lim_{\beta \to 0} (\mathbb{1} + H_{\beta,-\frac{1}{2}})^{-1} = (\mathbb{1} + H_{0,\frac{1}{2}})^{-1} \neq (\mathbb{1} + H_{0,-\frac{1}{2}})^{-1}$$

Thus $(*)$ is not even continuous near $(0, -\frac{1}{2})$.

ALMOST HOMOGENEOUS SCHRÖDINGER OPERATORS

(in collaboration with SERGE RICHARD)

Let us go back to

$$L_\alpha = -\partial_x^2 + \Big(-\frac{1}{4} + \alpha\Big)\frac{1}{x^2}.$$

Recall that we defined $L_{m^2}^{\max}$ and $L_{m^2}^{\min}$.

For any $\kappa \in \mathbb{C} \cup \{\infty\}$ let $H_{m,\kappa}$ be the restriction of $L_{m^2}^{\max}$ to the domain

$$
\begin{aligned}
\mathrm{Dom}(H_{m,\kappa}) &= \Big\{ f \in \mathrm{Dom}(L_{m^2}^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\
&\quad f(x) - c\big(x^{1/2-m} + \kappa x^{1/2+m}\big) \in \mathrm{Dom}(L_{m^2}^{\min}) \\
&\quad \text{around } 0 \Big\}, \qquad \kappa \neq \infty; \\
\mathrm{Dom}(H_{m,\infty}) &= \Big\{ f \in \mathrm{Dom}(L_{m^2}^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\
&\quad f(x) - c x^{1/2+m} \in \mathrm{Dom}(L_{m^2}^{\min}) \text{ around } 0 \Big\}.
\end{aligned}
$$

For $\nu \in \mathbb{C} \cup \{\infty\}$, let H_0^ν be the restriction of $L_0^{\max}$ to

$$\begin{aligned}
\operatorname{Dom}(H_0^\nu) \;=\; & \Big\{ f \in \operatorname{Dom}(L_0^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\
& f(x) - c\big(x^{1/2}\ln x + \nu x^{1/2}\big) \in \operatorname{Dom}(L_0^{\min}) \\
& \text{around } 0 \Big\}, \qquad \nu \neq \infty; \\
\operatorname{Dom}(H_0^\infty) \;=\; & \Big\{ f \in \operatorname{Dom}(L_0^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\
& f(x) - c x^{1/2} \in \operatorname{Dom}(L_0^{\min}) \text{ around } 0 \Big\}.
\end{aligned}$$

Proposition 11..

1. *For any* $|\mathrm{Re}\,(m)| < 1$, $\kappa \in \mathbb{C} \cup \{\infty\}$

$$H_{m,\kappa} = H_{-m,\kappa^{-1}}.$$

2. $H_{0,\kappa}$ *does not depend on* κ, *and these operators coincide with* H_0^{∞}.

Proposition 12. .

$$H^*_{m,\kappa} = H_{\overline{m},\overline{\kappa}} \qquad \textit{and} \qquad H_0^{\nu *} = H_0^{\overline{\nu}}.$$

In particular,

(i) $H_{m,\kappa}$ *is self-adjoint for* $m \in]-1,1[$ *and* $\kappa \in \mathbb{R} \cup \{\infty\}$, *and for* $m \in i\mathbb{R}$ *and* $|\kappa| = 1$.

(ii) H_0^{ν} *is self-adjoint for* $\nu \in \mathbb{R} \cup \{\infty\}$.

Proposition 13.. *For any* m *with* $|\mathrm{Re}\,(m)| < 1$ *and any* $\kappa, \nu \in \mathbb{C} \cup \{\infty\}$, *we have*

$$U_\tau H_{m,\kappa} U_{-\tau} = \mathrm{e}^{-2\tau} H_{m,\mathrm{e}^{-2\tau m}\kappa},$$
$$U_\tau H_0^\nu U_{-\tau} = \mathrm{e}^{-2\tau} H_0^{\nu+\tau}.$$

In particular, only

$$\begin{aligned} H_{m,0} &= H_m, \\ H_{m,\infty} &= H_{-m}, \\ H_0^\infty &= H_0 \qquad \textit{are homogeneous.} \end{aligned}$$

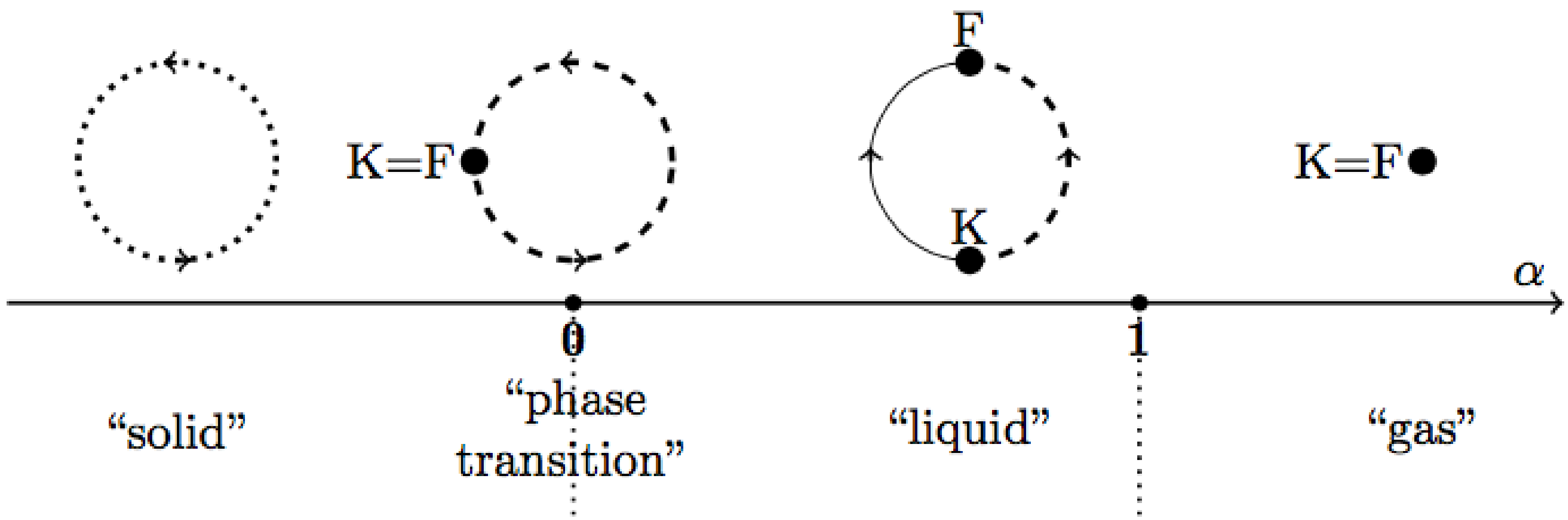
The renormalization group action: $R_\tau(A) := \mathrm{e}^{2\tau} U_\tau A U_{-\tau}$.

The homogeneous extensions are its only fixed points.

Self-adjoint extensions of the Hermitian operator

$$L_\alpha = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right)\frac{1}{x^2}.$$

K—Krein, F—Friedrichs, dashed line—single bound state, dotted line—infinite sequence of bound states.



The essential spectrum of $H_{m,\kappa}$ and H_0^ν is $[0,\infty[$.

Theorem 14. .

1. *$z \in \mathbb{C}\backslash[0,\infty[$ belongs to the point spectrum of $H_{m,\kappa}$ iff*

$$(-z)^{-m} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}.$$

2. *H_0^ν possesses an eigenvalue iff $-\pi < \operatorname{Im} 2\nu < \pi$, and then it is $z = -\mathrm{e}^{-2\nu}$.*

For a given m, κ all eigenvalues form a geometric sequence that lies on a logarithmic spiral, which should be viewed as a curve on the Riemann surface of the logarithm. Only its “physical sheet” gives rise to eigenvalues.

For m which are not purely imaginary, only a finite piece of the spiral is on the “physical sheet” and therefore the number of eigenvalues is finite.

If m is purely imaginary, this spiral degenerates to a half-line starting at the origin.

If m is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.

Theorem 15.. *Let* $m = m_{\rm r} + {\rm i} m_{\rm i} \in \mathbb{C}^{\times}$ *with* $|m_{\rm r}| < 1$.

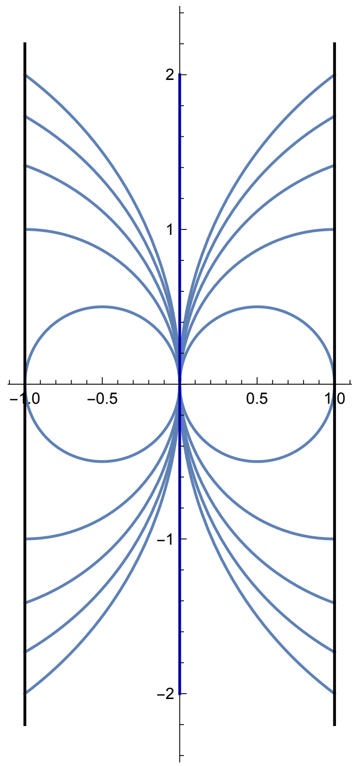
(i) Let $m_{\rm r} = 0$.

(a) If $\frac{\ln\left|\kappa\frac{\Gamma(m)}{\Gamma(-m)}\right|}{m_{\rm i}} \in]-\pi,\pi[$, *then* $\#\sigma_{\rm p}(H_{m,\lambda}) = \infty$,

(a) if $\frac{\ln\left|\kappa\frac{\Gamma(m)}{\Gamma(-m)}\right|}{m_{\rm i}} \notin]-\pi,\pi[$ *then* $\#\sigma_{\rm p}(H_{m,\lambda}) = 0$.

(ii) If $m_{\rm r} \neq 0$ *and if* $N \in \mathbb{N}$ *satisfies* $N < \frac{m_{\rm r}^2 + m_{\rm i}^2}{|m_{\rm r}|} \leq N+1$, *then*

$$\#\sigma_{\rm p}(H_{m,\lambda}) \in \{N, N+1\}.$$



2
1
−1.0
−0.5
0.5
1.0
−1
−2

HOMOGENEOUS RANK ONE PERTURBATIONS

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$ and operator X

$$Xf(x) := xf(x).$$

Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. We consider a family of operators formally given by

$$H_{m,\lambda} := X + \lambda |x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|.$$

Note that X is homogeneous of degree 1.

$|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is homogeneous of degree $1+m$. However strictly speaking, it is not a well defined operator, because $x^{\frac{m}{2}}$ is never square integrable.

If $-1 < \operatorname{Re} m < 0$, the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is form bounded relatively to X and then $H_{m,\lambda}$ can be defined. The perturbation is formally rank one. Therefore,

$$(z - H_{m,\lambda})^{-1} = (z - X)^{-1}$$
$$+ \sum_{n=0}^{\infty} (z - X)^{-1}|x^{\frac{m}{2}}\rangle(-\lambda)^{n+1}\langle x^{\frac{m}{2}}|(z - X)^{-1}|x^{\frac{m}{2}}\rangle^{n}\langle x^{\frac{m}{2}}|(z - X)^{-1}$$
$$= (z - X)^{-1}$$
$$+ \left(\lambda^{-1} - \langle x^{\frac{m}{2}}|(z - X)^{-1}|x^{\frac{m}{2}}\rangle\right)^{-1}(z - X)^{-1}|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|(z - X)^{-1}.$$

By straightforward complex analysis methods we obtain

$$\langle x^{\frac{m}{2}}|(z-X)^{-1}|x^{\frac{m}{2}}\rangle$$
$$=\int_0^\infty x^m(z-x)^{-1}\mathrm{d}x=(-z)^m\frac{\pi}{\sin\pi m}.$$

Therefore, the resolvent of $H_{m,\lambda}$ can be given in a closed form:

$$(z-H_{m,\lambda})^{-1} = (z-X)^{-1}$$
$$+\Big(\lambda^{-1}-(-z)^m\frac{\pi}{\sin\pi m}\Big)^{-1}(z-X)^{-1}|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|(z-X)^{-1}.$$

The above formula defines a resolvent of a closed operator for all $-1 < \operatorname{Re} m < 1$ and $\lambda \in \mathbb{C} \cup \{\infty\}$. This allows us to define a holomorphic family of closed operators $H_{m,\lambda}$.

Note that $H_{m,0} = X$.

$m = 0$ is special: $H_{0,\lambda} = X$ for all λ.

We introduce H_0^ρ for any $\rho \in \mathbb{C} \cup \{\infty\}$ by

$$(z - H_0^\rho)^{-1} = (z - X)^{-1}$$
$$-\big(\rho + \ln(-z)\big)^{-1}(z - X)^{-1}|x^0\rangle\langle x^0|(z - X)^{-1}.$$

In particular, $H_0^\infty = X$.

The group of dilations ("the renormalization group") acts on our operators in a simple way:

$$
\begin{aligned}
U_\tau H_{m,\lambda} U_\tau^{-1} &= \mathrm{e}^\tau H_{m,\mathrm{e}^{\tau m}\lambda}, \\
U_\tau H_0^\rho U_\tau^{-1} &= \mathrm{e}^\tau H_0^{\rho+\tau}.
\end{aligned}
$$

Define the unitary operator

$$(If)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}).$$

Its inverse is

$$(I^{-1}f)(x) := \left(\frac{y}{2}\right)^{\frac{1}{2}} f\left(\frac{y^2}{4}\right).$$

Note that

$$\begin{aligned} I^{-1}XI &= \frac{X^2}{4}, \\ I^{-1}AI &= \frac{A}{2}. \end{aligned}$$

We change slightly notation: the almost homogeneous Schrödinger operators H_m, $H_{m,\kappa}$ and H_0^ν will be denoted $\tilde{H}_m$, $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^\nu$

Recall that we introduced the Hankel transformation $\mathcal{F}_m$, which is a bounded invertible involution satisfying

$$
\begin{aligned}
\mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} &= X^2, \\
\mathcal{F}_m A \mathcal{F}_m^{-1} &= -A.
\end{aligned}
$$

Theorem 16. .

1.

$$\mathcal{F}_m^{-1} I^{-1} H_{m,\lambda} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_{m,\kappa},$$

where

$$\lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)},$$

2.

$$\mathcal{F}_m^{-1} I^{-1} H_0^{\rho} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_0^{\nu},$$

where $\rho = -2\nu$.