ON SOME FAMILIES OF EXACTLY SOLVABLE
SCHRÖDINGER OPERATORS

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1. **POTENTIALS** $\frac{1}{x^2}$
   (with LAURENT BRUNEAU and VLADIMIR GEORGESCU)

2. **POTENTIALS** $\frac{1}{x}$
   (with SERGE RICHARD)

3. **POTENTIALS** $\frac{1}{x^2}$ WITH BOUNDARY CONDITIONS
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4. **HOMOGENEOUS RANK ONE PERTURBATIONS**
Schrödinger operators that can be solved in terms of well-understood special functions play an important role in applications.

It is natural to organize them in holomorphic families of closed operators. Clearly, for most values of parameters these operators are non-self-adjoint. Even if we are interested mostly in self-adjoint cases, the existence of (non-self-adjoint) analytic continuation plays sometimes an important role in applications. Let me mention e.g. Regge poles, which were famous in the 60’s.
Consider a formal differential expression

$$L_\alpha = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right) \frac{1}{x^2}.$$ 

We would like to interpret it as a well-defined (unbounded) operator on $L^2[0, \infty[$. To do this we need to specify its domain.
$L_\alpha$, and closely related operators $H_m$ and $H_{m,\kappa}$, which we introduce shortly, are interesting for many reasons.

- They appear as the radial part of the Laplacian in all dimensions, in the decomposition of the Aharonov-Bohm Hamiltonian, in the membranes with conical singularities, in the theory of many body systems with contact interactions and in the Efimov effect.
• They have a surprisingly rich mathematical phenomenology, which should be close to physicists’ hearts: the “running coupling constant” flows under the action of the “renormalization group”, there are two ”phase transitions”, attractive and repulsive fixed points, limit cycles, breakdown of conformal symmetry, etc.

• They have rather subtle and rich properties illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators). They naturally appear in representations of $sl(2, \mathbb{R})$. 
Essentially all basic objects related to $H_m$, such as their resolvents, spectral projections, wave and scattering operators and the evolution, can be explicitly computed in terms of special functions. A number of nontrivial identities involving functions from the Bessel family find an appealing operator-theoretical interpretation in terms of the operators $H_m$. Eg. the Barnes identity leads to the formula for wave operators.
Let $U_\tau$ be the group of dilations on $L^2[0, \infty[$, that is

$$(U_\tau f)(x) = e^{\tau/2} f(e^\tau x).$$

We say that $B$ is homogeneous of degree $\nu$ if

$$U_\tau B U_\tau^{-1} = e^{\nu \tau} B.$$ 

Clearly, $L_\alpha$ is homogeneous of degree $-2$. 
Here are two natural questions:

1. If $\alpha \in \mathbb{R}$, how to interpret $L_\alpha$ as a self-adjoint operator on $L^2[0, \infty]$? When is it homogeneous of degree $-2$?

2. If $\alpha \in \mathbb{C}$, how to interpret $L_\alpha$ as a closed operator on $L^2[0, \infty]$? When is it homogeneous of degree $-2$?
Two naive interpretations of $L_\alpha$:

1. The **minimal** operator $L^{\min}_\alpha$: We start from $L_\alpha$ on $C_c^\infty[0, \infty[$, and then we take its closure.

2. The **maximal** operator $L^{\max}_\alpha$: We consider the domain consisting of all

\[ f \in L^2[0, \infty[ \text{ such that } L_\alpha f \in L^2[0, \infty[. \]

Clearly, $\text{Dom}(L^{\min}_\alpha) \subset \text{Dom}(L^{\max}_\alpha)$ and

\[ L^{\max}_\alpha \big|_{\text{Dom}(L^{\min}_\alpha)} = L^{\min}_\alpha. \]

In other words $L^{\min}_\alpha \subset L^{\max}_\alpha$. 
We will see that it is often natural to write $\alpha = m^2$

Theorem 1.

1. For $1 \leq \text{Re } m$, $L_{m^2}^{\text{min}} = L_{m^2}^{\text{max}}$.

2. For $-1 < \text{Re } m < 1$, $L_{m^2}^{\text{min}} \subsetneq L_{m^2}^{\text{max}}$, and the codimension of their domains is 2.

3. $(L_{\alpha}^{\text{min}})^* = L_{\alpha}^{\text{max}}$. Hence, for $\alpha \in \mathbb{R}$, $L_{\alpha}^{\text{min}}$ is Hermitian.

4. $L_{\alpha}^{\text{min}}$ and $L_{\alpha}^{\text{max}}$ are homogeneous of degree $-2$. 
Notice that
\[ L x^{\frac{1}{2} \pm m} = 0. \]

Let \( \xi \) be a compactly supported cutoff equal 1 around 0.

Let \( -1 < \text{Re} \ m \). Note that \( x^{\frac{1}{2} + m} \xi \) belongs to \( \text{Dom} L_{m^2}^{\text{max}} \).

This suggests to define the operator \( H_m \) to be the restriction of \( L_{m^2}^{\text{max}} \) to

\[ \text{Dom} L_{m^2}^{\text{min}} + \mathbb{C} x^{\frac{1}{2} + m} \xi. \]
Theorem 2.

1. For $1 \leq \text{Re} m$, \( L_{m^2}^{\min} = H_m = L_{m^2}^{\max} \).

2. For $-1 < \text{Re} m < 1$, \( L_{m^2}^{\min} \subset H_m \subset L_{m^2}^{\max} \) and the codimension of the domains is 1.

3. \( H_m^* = H_m \). Hence, for \( m \in ] -1, \infty [ \), \( H_m \) is self-adjoint.

4. \( H_m \) is homogeneous of degree $-2$.

5. \( \text{sp} \ H_m = [0, \infty [ \).

6. \( \{ \text{Re} m > -1 \} \ni m \mapsto H_m \) is a holomorphic family of closed operators.
Theorem 3.

1. For $\alpha \geq 1$, $L_{\alpha}^{\text{min}} = H^{\sqrt{\alpha}}$ is essentially self-adjoint on $C_c^\infty[0, \infty[$.

2. For $\alpha < 1$, $L_{\alpha}^{\text{min}}$ is not essentially self-adjoint on $C_c^\infty[0, \infty[$.

3. For $0 \leq \alpha < 1$, the operator $H^{\sqrt{\alpha}}$ is the Friedrichs extension and $H^{-\sqrt{\alpha}}$ is the Krein extension of $L_{\alpha}^{\text{min}}$.

4. $H^{\frac{1}{2}}$ is the Dirichlet Laplacian and $H^{-\frac{1}{2}}$ is the Neumann Laplacian on halfline.

5. For $\alpha < 0$, $L_{\alpha}^{\text{min}}$ has no homogeneous selfadjoint extensions.
It is easy to see that
\[
x^{-\frac{1}{2}} \left( -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right) \frac{1}{x^2} \pm 1 \right) x^{\frac{1}{2}}
\]
\[
= -\partial_x^2 - \frac{1}{x} \partial_x + \left(-\frac{1}{4} + \alpha\right) \frac{1}{x^2} \pm 1,
\]
which is the (modified) Bessel operator.

Therefore, it is not surprising that various objects related to $H_m$ can be computed with help of functions from the Bessel family.
Theorem 4. If $R_m(\lambda; x, y)$ is the integral kernel of the operator $(\lambda - H_m)^{-1}$, then for $\Re k > 0$ we have

$$R_m(-k^2; x, y) = \left\{ \begin{array}{ll} \sqrt{xy}I_m(kx)K_m(ky) & \text{if } x < y, \\ \sqrt{xy}I_m(ky)K_m(kx) & \text{if } x > y, \end{array} \right.$$

where $I_m$ is the modified Bessel function and $K_m$ is the MacDonald function.
Proposition 5. For \(0 < a < b < \infty\), the integral kernel of \(\mathbb{1}_{[a,b]}(H_m)\) is

\[
\mathbb{1}_{[a,b]}(H_m)(x, y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \, dk,
\]

where \(J_m\) is the Bessel function.
Let $\mathcal{F}_m$ be the operator on $L^2[0, \infty[$ given by

$$
\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx) \sqrt{kx} f(x) \, dx
$$

$\mathcal{F}_m$ is the so-called Hankel transformation. Define also the operator $X f(x) := xf(x)$.

**Theorem 6.** $\mathcal{F}_m$ is a bounded invertible involution on $L^2[0, \infty[$ diagonalizing $H_m$ and anticommuting with the self-adjoint generator of dilations $A = \frac{1}{2i}(x\partial_x + \partial_xx)$:

$$
\mathcal{F}^2 = 1,
$$

$$
\mathcal{F}_m H_m \mathcal{F}_m^{-1} = X^2,
$$

$$
\mathcal{F}_m A = -A \mathcal{F}_m.
$$
Theorem 7. Set

\[ \mathcal{I} f(x) = x^{-1} f(x^{-1}), \quad \Xi_m(t) = e^{i \ln(2)t} \frac{\Gamma\left(\frac{m+1+it}{2}\right)}{\Gamma\left(\frac{m+1-it}{2}\right)}. \]

Then

\[ \mathcal{F}_m = \Xi_m(A) \mathcal{I}. \]

Therefore, we have the identity

\[ H_m := \Xi_m^{-1}(A) X^{-2} \Xi_m(A) \]

(Result obtained independently by Bruneau, Georgescu, D, and by Richard and Pankrashkin).
Theorem 8. The wave operators associated to the pair $H_m, H_k$ exist and

$$\Omega_{m,k}^{\pm} := \lim_{t \to \pm \infty} e^{itH_m}e^{-itH_k}$$

$$= e^{\pm i(m-k)\pi/2} \mathcal{F}_m \mathcal{F}_k$$

$$= e^{\pm i(m-k)\pi/2} \frac{\Xi_k(A)}{\Xi_m(A)}.$$
The formula

\[
H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A)
\]  

valid for \(\text{Re} m > -1\) can be used as an alternative definition of the family \(H_m\) also beyond this domain. It defines a family of closed operators for the parameter \(m\) that belongs to

\[
\{m \mid \text{Re} m \neq -1, -2, \ldots \} \cup \mathbb{R}.
\]  

Its spectrum is always equal to \([0, \infty[\) and it is analytic in the interior of (2).
In fact, $\Xi_m(A)$ is a unitary operator for all real values of $m$. Therefore, for $m \in \mathbb{R}$, (1) is well-defined and self-adjoint.

$\Xi_m(A)$ is bounded and invertible also for all $m$ such that $\Re m \neq -1, -2, \ldots$. Therefore, the formula (1) defines an operator for all such $m$. 
One can then pose various questions:

- What happens with this operator along the lines \( \text{Re} m = -1, -2, \ldots \) ?
- What is the meaning of the operator to the left of \( \text{Re} = -1 \)? (It is not a differential operator!)
The definition (or actually a number of equivalent definitions) of a **holomorphic family of bounded operators** is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.
Suppose that $\Theta$ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a holo-
morphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood $\Theta_0$ of $z_0$, a Banach space $\mathcal{K}$ and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto B(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\text{Ran } B(z) = D(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)B(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.
We have the following practical criterion:

**Theorem 9.** Suppose that \( \{H(z)\}_{z \in \Theta} \) is a function whose values are closed operators on \( \mathcal{H} \). Suppose in addition that for any \( z \in \Theta \) the resolvent set of \( H(z) \) is nonempty. Then \( z \mapsto H(z) \) is a **holomorphic family of closed operators** if and only if for any \( z_0 \in \Theta \) there exists \( \lambda \in \mathbb{C} \) and a neighborhood \( \Theta_0 \) of \( z_0 \) such that \( \lambda \) belongs to the resolvent set of \( H(z) \) for \( z \in \Theta_0 \) and \( z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H}) \) is holomorphic on \( \Theta_0 \).
The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set.

**Conjecture 10.** It is impossible to extend

$$\{\Re m > -1\} \ni m \mapsto H_m$$

to a holomorphic family of closed operators on a larger connected open subset of $\mathbb{C}$. 
Consider

\[ L_{\beta,\alpha} := -\partial_x^2 + \left(\alpha - \frac{1}{4}\right) \frac{1}{x^2} - \frac{\beta}{x}, \]

where the parameters \( \beta, \alpha \) are complex numbers. It is natural to write \( \alpha = m^2 \).
Its eigenvalue equation

\[
\left( -\partial_x^2 + \left( \alpha - \frac{1}{4} \right) \frac{1}{x^2} - \frac{\beta}{x} - \frac{1}{4} \right) u(x) = 0
\]

is known as the **Whittaker equation**. Therefore, its solutions can be expressed in terms of **Whittaker functions**, or equivalently, \( _1F_1 \) and \( _2F_0 \) confluent functions.
For any $m \in \mathbb{C}$ with $\text{Re} \,(m) > -1$ we introduce the closed operator $H_{\beta,m}$ that equals $L_{\beta,m^2}$ on functions that behave as

$$x^{1+m} \left(1 - \frac{\beta}{1 + 2mx}\right)$$

near zero. We obtain a family

$$\mathbb{C} \times \{m \in \mathbb{C} \mid \text{Re} \,(m) > -1\} \ni (\beta, m) \mapsto H_{\beta,m},$$

which is holomorphic except for a singularity at $(0, -\frac{1}{2})$. 
If $\text{Re} (m) \geq 1$ the boundary condition is not needed. If $\text{Re} (m) \geq -\frac{1}{2}$, then we can use a simplified boundary condition

$$x^{\frac{1}{2}}+m.$$ 

For $-1 < \text{Re} (m) < 1$ the family $H_{\beta,m}$ does not describe all possible well-posed extensions of $L_{\beta,m}^{\min}$ (extensions with a non-empty resolvent set). It describes all extensions with pure boundary conditions. I will not discuss mixed boundary conditions, which can also be analyzed (work in progress together with J. Faupin, Q. N. Nguyen and S. Richard).
Let us draw the spectrum of $H_{\beta,m}$ for some values of parameters. We set $m_i := \text{Im}(m) = -2.4$ and $\beta = 1$, and we select

$$m_r := \text{Re}(m) = -0.75, -0.5, -0.25, 0, 0.25, 0.5, 1, 2.$$ 

Spectrum is blue, resonances are red.

Note that for any $-1 < m_r \leq -\frac{1}{2}$, we have the identity

$$\sigma(H_{1,m_r+i m_i}) = \sigma(H_{1,m_r+1+i m_i}).$$

Therefore, Fig. 1 and 2 have the same spectrum as Fig. 5 and 6. However Fig. 1 and 2 has additionally a resonance.
Fig. 1. $\sigma(H_{1,-0.75-2.4i})$

Fig. 2. $\sigma(H_{1,-0.5-2.4i})$

Fig. 3. $\sigma(H_{1,-0.25-2.4i})$

Fig. 4. $\sigma(H_{1,0-2.4i})$
Next we present

\[ e^{-i2\phi} \sigma(H_{\beta}, -0.75 + 3.2i) \]

\[ \beta = e^{i\phi} \quad \text{for} \quad \phi = \frac{n}{8\pi} \quad \text{with} \quad n = 0, \ldots, 15. \]

By **dilation analyticity**, the point spectrum does not move, the continuous spectrum, on the other hand, rotates as \( e^{-i2\phi} \), like a giant hand of a clock. Eigenvalues hit by the continuous spectrum disappear and become resonances. Then they reappear when the hand of the clock comes again.
Fig. 9. $\phi = 0$

Fig. 10. $\phi = \frac{1}{8}\pi$

Fig. 11. $\phi = \frac{1}{4}\pi$

Fig. 12. $\phi = \frac{3}{8}\pi$
Fig. 13. $\phi = \frac{1}{2}\pi$

Fig. 14. $\phi = \frac{5}{8}\pi$

Fig. 15. $\phi = \frac{3}{4}\pi$

Fig. 16. $\phi = \frac{7}{8}\pi$
Fig. 21. $\phi = \frac{3}{2}\pi$

Fig. 22. $\phi = \frac{13}{8}\pi$

Fig. 23. $\phi = \frac{7}{4}\pi$

Fig. 24. $\phi = \frac{15}{8}\pi$
The singularity at $(\beta, m) = (0, -\frac{1}{2})$ is quite curious: it is invisible when we consider just the variable $m$. In fact,

$$m \mapsto H_m = H_{0,m}$$

is holomorphic around $m = -\frac{1}{2}$, and $H_{-\frac{1}{2}}$ has the Neumann boundary condition. It is also holomorphic around $m = \frac{1}{2}$, and $H_{\frac{1}{2}}$ has the Dirichlet boundary condition. Thus one has

$$H_{0,-\frac{1}{2}} \neq H_{0,\frac{1}{2}}.$$
If we introduce the Coulomb potential, then

whenever $\beta \neq 0$, \[ H_{\beta,-\frac{1}{2}} = H_{\beta,\frac{1}{2}}. \]

The function

\[ (\beta, m) \mapsto H_{\beta,m} \quad (*) \]

is holomorphic around $(0, \frac{1}{2})$, in particular,

\[ \lim_{\beta \to 0} (\mathbb{1} + H_{\beta,\frac{1}{2}})^{-1} = (\mathbb{1} + H_{0,\frac{1}{2}})^{-1}. \]

But

\[ \lim_{\beta \to 0} (\mathbb{1} + H_{\beta,-\frac{1}{2}})^{-1} = (\mathbb{1} + H_{0,\frac{1}{2}})^{-1} \neq (\mathbb{1} + H_{0,-\frac{1}{2}})^{-1} \]

Thus $(*)$ is not even continuous near $(0, -\frac{1}{2})$. 
ALMOST HOMOGENEOUS SCHRÖDINGER OPERATORS
(in collaboration with SERGE RICHARD)

Let us go back to

\[ L_\alpha = -\partial_x^2 + \left( -\frac{1}{4} + \alpha \right) \frac{1}{x^2}. \]

Recall that we defined \( L_{\max}^{m^2} \) and \( L_{\min}^{m^2} \).
For any $\kappa \in \mathbb{C} \cup \{\infty\}$ let $H_{m,\kappa}$ be the restriction of $L_{m^2}^{\max}$ to the domain

$$\text{Dom}(H_{m,\kappa}) = \left\{ f \in \text{Dom}(L_{m^2}^{\max}) \mid \text{for some } c \in \mathbb{C}, \quad f(x) - c(x^{1/2-m} + \kappa x^{1/2+m}) \in \text{Dom}(L_{m^2}^{\min}) \text{ around } 0 \right\}, \quad \kappa \neq \infty;$$

$$\text{Dom}(H_{m,\infty}) = \left\{ f \in \text{Dom}(L_{m^2}^{\max}) \mid \text{for some } c \in \mathbb{C}, \quad f(x) - cx^{1/2+m} \in \text{Dom}(L_{m^2}^{\min}) \text{ around } 0 \right\}.$$
For \( \nu \in \mathbb{C} \cup \{\infty\} \), let \( H_0^\nu \) be the restriction of \( L_0^{\max} \) to

\[
\text{Dom}(H_0^\nu) = \{ f \in \text{Dom}(L_0^{\max}) \mid \text{for some } c \in \mathbb{C}, \quad f(x) - c\left(x^{1/2}\ln x + \nu x^{1/2}\right) \in \text{Dom}(L_0^{\min}) \text{ around 0} \}, \quad \nu \neq \infty;
\]

\[
\text{Dom}(H_0^{\infty}) = \{ f \in \text{Dom}(L_0^{\max}) \mid \text{for some } c \in \mathbb{C}, \quad f(x) - cx^{1/2} \in \text{Dom}(L_0^{\min}) \text{ around 0} \}.
\]
Proposition 11.

1. For any $|\text{Re}(m)| < 1$, $\kappa \in \mathbb{C} \cup \{\infty\}$

$$H_{m,\kappa} = H_{-m,\kappa^{-1}}.$$ 

2. $H_{0,\kappa}$ does not depend on $\kappa$, and these operators coincide with $H_0^\infty$. 
Proposition 12.

\[ H^*_{m,\kappa} = H_{m,\kappa} \text{ and } H^{\nu*}_0 = H^\nu_0. \]

In particular,

(i) \( H_{m,\kappa} \) is self-adjoint for \( m \in ]-1,1[ \) and \( \kappa \in \mathbb{R} \cup \{\infty\} \), and for \( m \in i\mathbb{R} \) and \( |\kappa| = 1 \).

(ii) \( H^{\nu}_0 \) is self-adjoint for \( \nu \in \mathbb{R} \cup \{\infty\} \).
Proposition 13. For any $m$ with $|\Re (m)| < 1$ and any $\kappa, \nu \in \mathbb{C} \cup \{\infty\}$, we have

$$U_{\tau}H_{m,\kappa}U_{-\tau} = e^{-2\tau}H_{m,e^{-2\tau}m,\kappa},$$

$$U_{\tau}H_{0,\nu}U_{-\tau} = e^{-2\tau}H_{0,\nu+\tau}.$$ 

In particular, only

$$H_{m,0} = H_m,$$

$$H_{m,\infty} = H_{-m},$$

$$H_{0,\infty} = H_0$$

are homogeneous.

The renormalization group action: $R_{\tau}(A) := e^{2\tau}U_{\tau}AU_{-\tau}$. The homogeneous extensions are its only fixed points.
Self-adjoint extensions of the Hermitian operator

\[ L_\alpha = -\partial_x^2 + \left( -\frac{1}{4} + \alpha \right) \frac{1}{x^2}. \]

K—Krein, F—Friedrichs, dashed line—single bound state, dotted line—infinite sequence of bound states.
The essential spectrum of $H_{m, \kappa}$ and $H_0^{\nu}$ is $[0, \infty]$.

**Theorem 14.**

1. $z \in \mathbb{C} \setminus [0, \infty]$ belongs to the point spectrum of $H_{m, \kappa}$ iff
   \[
   (-z)^{-m} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}.
   \]

2. $H_0^{\nu}$ possesses an eigenvalue iff $-\pi < \text{Im } 2\nu < \pi$, and then it is $z = -e^{-2\nu}$. 


For a given $m, \kappa$ all eigenvalues form a geometric sequence that lies on a logarithmic spiral, which should be viewed as a curve on the Riemann surface of the logarithm. Only its “physical sheet” gives rise to eigenvalues. For $m$ which are not purely imaginary, only a finite piece of the spiral is on the “physical sheet” and therefore the number of eigenvalues is finite.

If $m$ is purely imaginary, this spiral degenerates to a half-line starting at the origin.

If $m$ is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.
Theorem 15. Let $m = m_r + i m_i \in \mathbb{C}^\times$ with $|m_r| < 1$.

(i) Let $m_r = 0$.

(a) If $\frac{\ln |\kappa \frac{\Gamma(m)}{\Gamma(-m)}|}{m_i} \in ] - \pi, \pi[$, then $\# \sigma_p(H_m, \lambda) = \infty$,

(a) if $\frac{\ln |\kappa \frac{\Gamma(m)}{\Gamma(-m)}|}{m_i} \notin ] - \pi, \pi[$, then $\# \sigma_p(H_m, \lambda) = 0$.

(ii) If $m_r \neq 0$ and if $N \in \mathbb{N}$ satisfies $N < \frac{m_r^2 + m_i^2}{|m_r|} \leq N + 1$, then

$$\# \sigma_p(H_m, \lambda) \in \{N, N + 1\}.$$
HOMOGENEOUS RANK ONE PERTURBATIONS

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$ and operator $X$

$$Xf(x) := xf(x).$$

Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. We consider a family of operators formally given by

$$H_{m,\lambda} := X + \lambda |x^{\frac{m}{2}}\rangle \langle x^{\frac{m}{2}}|.$$
Note that $X$ is homogeneous of degree $1$.

$|x^m \rangle \langle x^m |$ is homogeneous of degree $1 + m$. However strictly speaking, it is not a well defined operator, because $x^m$ is never square integrable.
If $-1 < \text{Re} m < 0$, the perturbation $|x^2_m\rangle\langle x^2_m|$ is form bounded relatively to $X$ and then $H_{m,\lambda}$ can be defined. The perturbation is formally rank one. Therefore,

$$(z - H_{m,\lambda})^{-1} = (z - X)^{-1}$$

$$+ \sum_{n=0}^{\infty} (z - X)^{-1} |x^2_m\rangle (-\lambda)^{n+1} \langle x^2_m| (z - X)^{-1} |x^2_m\rangle n \langle x^2_m| (z - X)^{-1}$$

$$= (z - X)^{-1}$$

$$+ \left( \lambda^{-1} - \langle x^2_m| (z - X)^{-1} |x^2_m\rangle \right)^{-1} (z - X)^{-1} |x^2_m\rangle \langle x^2_m| (z - X)^{-1}. $$
By straightforward complex analysis methods we obtain

\[
\langle x^m | (z - X)^{-1} | x^m \rangle = \int_0^\infty x^m (z - x)^{-1} \, dx = (-z)^m \frac{\pi}{\sin \pi m}.
\]

Therefore, the resolvent of \( H_{m,\lambda} \) can be given in a closed form:

\[
(z - H_{m,\lambda})^{-1} = (z - X)^{-1} + \left( \lambda^{-1} - (-z)^m \frac{\pi}{\sin \pi m} \right)^{-1} (z - X)^{-1} \langle x^m | x^m \rangle (z - X)^{-1}.
\]
The above formula defines a resolvent of a closed operator for all $-1 < \text{Re} \, m < 1$ and $\lambda \in \mathbb{C} \cup \{\infty\}$. This allows us to define a holomorphic family of closed operators $H_{m,\lambda}$.

Note that $H_{m,0} = X$.

$m = 0$ is special: $H_{0,\lambda} = X$ for all $\lambda$. 
We introduce $H^\rho_0$ for any $\rho \in \mathbb{C} \cup \{\infty\}$ by

$$(z - H^\rho_0)^{-1} = (z - X)^{-1} - (\rho + \ln(-z))^{-1}(z - X)^{-1}|x^0\rangle\langle x^0|(z - X)^{-1}.$$ 

In particular, $H^\infty_0 = X$. 
The group of dilations ("the renormalization group") acts on our operators in a simple way:

\[ U_\tau H_{m,\lambda} U_{\tau}^{-1} = e^{\tau} H_{m,e^{\tau}m\lambda}, \]
\[ U_\tau H_0^\rho U_{\tau}^{-1} = e^{\tau} H_0^{\rho+\tau}. \]
Define the unitary operator

$$(I f)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}).$$

Its inverse is

$$(I^{-1} f)(x) := \left(\frac{y}{2}\right)^{\frac{1}{2}} f\left(\frac{y^2}{4}\right).$$

Note that

$$I^{-1} X I = \frac{X^2}{4},$$

$$I^{-1} A I = \frac{A}{2}. $$
We change slightly notation: the almost homogeneous Schrödinger operators $H_m$, $H_{m,\kappa}$ and $H_0^\nu$ will be denoted $\tilde{H}_m$, $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^\nu$.

Recall that we introduced the Hankel transformation $\mathcal{F}_m$, which is a bounded invertible involution satisfying

$$\mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} = X^2,$$

$$\mathcal{F}_m A \mathcal{F}_m^{-1} = -A.$$
Theorem 16.

1. \[ \mathcal{F}_m^{-1} I^{-1} H_{m, \lambda} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_{m, \kappa}, \]

where \[ \lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}, \]

2. \[ \mathcal{F}_m^{-1} I^{-1} H_{0}^\rho I \mathcal{F}_m = \frac{1}{4} \tilde{H}_0^\nu, \]

where \( \rho = -2\nu. \)