THE FEYNMAN PROPAGATOR ON CURVED SPACETIMES

JAN DEREZIŃSKI

Dep. of Math. Meth. in Phys.

FACULTY OF PHYSICS

UNIVERSITY OF WARSAW

in collaboration with DANIEL SIEMSSEN

Quantum Field Theory on curved spacetimes is one of the most important formalisms of theoretical physics. It also has a curious mathematical structure. It leads to interesting problems in PDE's and operator theory.

Plan of my talk:

1. Propagators on the flat Minkowski space.
2. Propagators on a curved spacetime, including a construction of the distinguished Feynman propagator.
3. The question about the self-adjointness
   of the Klein-Gordon operator on a curved spacetime.

PART I. PROPAGATORS ON A FLAT SPACETIME.

Let me start with the Klein-Gordon equation the flat Minkowski space $\mathbb{R}^{1,3}$

$$(-\Box + m^2)\psi = 0.$$

The following propagators and 2-point functions should belong to the standard knowledge of every student of QFT:

- the forward/backward propagator

$$G^{\vee/\wedge}(x,y) \coloneqq \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2 \pm \mathrm{i}0\,\mathrm{sgn}\,p_0}\,\mathrm{d}p,$$

- the Feynman/anti-Feynman propagator

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) \coloneqq \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2 \mp \mathrm{i}0}\,\mathrm{d}p,$$

- the Pauli–Jordan propagator

$$G^{\mathrm{PJ}}(x,y) \coloneqq \frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{e}^{-\mathrm{i}(x-y)\cdot p}\,\mathrm{sgn}(p_0)\delta(p^2+m^2)\,\mathrm{d}p,$$

- the positive/negative frequency 2-point function

$$G^{(\pm)}(x,y) \coloneqq \frac{1}{(2\pi)^3}\int \mathrm{e}^{-\mathrm{i}(x-y)\cdot p}\theta(\pm p_0)\delta(p^2+m^2)\,\mathrm{d}p.$$

Define the Klein Gordon operator $K := -\Box + m^2$.
$G^{\vee/\wedge}, G^{\mathrm{F}/\overline{\mathrm{F}}}$ can be viewed as its inverses

$$KGf = GKf = f,$$

and $G^{(\pm)}, G^{\mathrm{PJ}}$ are its bisolutions

$$KGf = GKf = 0.$$

Note the identities satisfied by the propagators:

$$\begin{aligned} G^{\mathrm{PJ}} &= G^{\vee} - G^{\wedge} && (1)\\ &= \mathrm{i}G^{(+)} - \mathrm{i}G^{(-)}, && (2)\\ G^{\mathrm{F}} - G^{\overline{\mathrm{F}}} &= \mathrm{i}G^{(+)} + \mathrm{i}G^{(-)}, && (3)\\ G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} &= G^{\vee} + G^{\wedge}. && (4) \end{aligned}$$

The following facts are easy to see:

(1) The Klein-Gordon operator $K = -\Box + m^2$ is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^{1,3})$ in the sense of $L^2(\mathbb{R}^{1,3})$.

(2) For $s > \frac{1}{2}$, as an operator $\langle t\rangle^{-s}L^2(\mathbb{R}^{1,3}) \to \langle t\rangle^{s}L^2(\mathbb{R}^{1,3})$, the Feynman propagator is the boundary value of the resolvent of the Klein-Gordon operator:

$$\operatorname*{s-lim}_{\epsilon\searrow 0}(K \mp \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}/\overline{\mathrm{F}}}.$$

Here $\langle t\rangle$ denotes the so-called "Japanese bracket"

$$\langle t\rangle := \sqrt{1+t^2}.$$

After splitting the coordinates into time and space $\mathbb{R}^{1,3} = \mathbb{R} \times \mathbb{R}^3$, we can rewrite the Klein-Gordon equation as a 1st order equation for the Cauchy data. This is the evolution approach, which will be easy to generalize to curved spacetimes.

$$\left(\partial_t + \mathrm{i}B\right) \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = 0,$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} := \begin{bmatrix} u(t) \\ \mathrm{i}\partial_t u(t) \end{bmatrix}, \quad B := \begin{bmatrix} 0 & \mathbb{1} \\ -\Delta + m^2 & 0 \end{bmatrix}.$$

The evolution $\mathrm{e}^{-\mathrm{i}(t-s)B}$ preserves the charge form

$$(u|Qv) = (u_1|v_2) + (u_2|v_1), \qquad Q := \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}.$$

It is natural to introduce a whole scale of Hilbert spaces of the Cauchy data $\mathcal{W}_\lambda = \mathcal{K}_{\frac{1}{2}+\lambda} \oplus \mathcal{K}_{-\frac{1}{2}+\lambda}$,

where $\mathcal{K}_\beta = (-\Delta + m^2)^{-\frac{\beta}{2}} L^2(\mathbb{R}^d)$ is the Sobolev space. Their scalar product can be written as

$$(u|v)_\lambda := (|B|^{-\frac{1}{2}+\lambda} u \,|H| B|^{-\frac{1}{2}+\lambda} v).$$

B is self-adjoint on all of them. Among them the energy space has the scalar product given by the Hamiltonian

$$H := BQ = \begin{bmatrix} -\Delta + m^2 & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \qquad (u|v)_{\frac{1}{2}} := (u\,|Hv).$$

The dynamical space $\mathcal{W}_0$ becomes naturally the 1-particle space after quantization.

We define the propagators in the evolution approach:

Pauli-Jordan bisolution $E^{\mathrm{PJ}}(t,s) := \mathrm{e}^{-\mathrm{i}(t-s)B}$,

forward inverse $E^{\vee}(t,s) := \theta(t-s)\mathrm{e}^{-\mathrm{i}(t-s)B}$,

backward inverse $E^{\wedge}(t,s) := -\theta(s-t)\mathrm{e}^{-\mathrm{i}(t-s)B}$,

pos./neg. freq. bisolution $E^{(\pm)}(t,s) := \mathrm{e}^{-\mathrm{i}(t-s)B}\Pi^{(\pm)}$,

Feynman/anti-Feynman inverse

$$E^{\mathrm{F}/\overline{\mathrm{F}}}(t,s) := \theta(t-s)\mathrm{e}^{-\mathrm{i}(t-s)B}\Pi^{(\pm)} - \theta(s-t)\mathrm{e}^{-\mathrm{i}(t-s)B}\Pi^{(\mp)},$$

Here θ is the Heavyside function and $\Pi^{(\pm)} := \theta(\pm B)$.

They act on functions $t \mapsto w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ as follows:

$$(E^\bullet w)(t) := \int E^\bullet(t,s) w(s)\,\mathrm{d}s, \quad \bullet = \mathrm{PJ}, \vee, \wedge, (\pm), \mathrm{F}/\overline{\mathrm{F}}.$$

We obtain also the propagators in the spacetime approach:

$$G^\bullet := \mathrm{i}E^\bullet_{12}, \quad \bullet = \mathrm{PJ}, \vee, \wedge, \mathrm{F}/\overline{\mathrm{F}},$$
$$G^{(\pm)} := \pm E^{(\pm)}_{12}, \quad E^\bullet = \begin{bmatrix} E^\bullet_{11} & E^\bullet_{12} \\ E^\bullet_{21} & E^\bullet_{22} \end{bmatrix}.$$

Let $\mathcal{W}_{\mathrm{KG}}$ be the space of smooth complex space-compact solutions of the Klein-Gordon equation. The classical charged fields $\psi(x)$, $\psi^*(x)$ are the functionals on $\mathcal{W}_{\mathrm{KG}}$

$$\langle\psi(x)|\zeta\rangle = \zeta(x), \quad \langle\psi^*(x)|\zeta\rangle = \overline{\zeta(x)}.$$

$\mathcal{W}_{\mathrm{KG}}$ is naturally a symplectic space. The corresponding Poisson bracket is called the Peierls bracket and is given by

$$\{\psi(x), \psi^*(y)\} = -G^{\mathrm{PJ}}(x, y).$$

Quantization is performed in two steps.

Construction of the algebra. We put hats on ψ and ψ^*, replacing the Poisson bracket by i times the commutator, obtaining the CCR algebra over $\mathcal{W}_{\mathrm{KG}}$:

$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x, y).$$

Choice of a representation. There exists a natural Fock state whose expectation values are

$$(\Omega \,|\, \hat{\psi}(x)\hat{\psi}^*(y)\Omega) = G^{(+)}(x, y),$$
$$(\Omega \,|\, \hat{\psi}^*(x)\hat{\psi}(y)\Omega) = G^{(-)}(x, y).$$

This state by the GNS construction defines a representation acting on a bosonic Fock space.

The most important role in the evaluation of Feynman diagrams is played by the Feynman propagator, which expresses the vacuum expectation of time ordered products of fields. Similarly, the anti-Feynman propagator expresses the vacuum expectations of the reverse time-ordered products of fields:

$$\big(\Omega \,\big|\, \mathrm{T}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = -\mathrm{i}G^{\mathrm{F}}(x,y),$$
$$\big(\Omega \,\big|\, \overline{\mathrm{T}}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = \mathrm{i}G^{\overline{\mathrm{F}}}(x,y).$$

As a side remark, note that the Euclidean analog of the Klein-Gordon equation is the Helmholtz equation

$$(-\Delta_4 + m^2)\psi = 0.$$

Its theory is much simpler, we have only one propagator:

$$G^{\mathrm{E}}(x,y) := \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2}\,\mathrm{d}p$$

Performing the Wick rotation from the Euclidean propagator we obtain the Feynman and anti-Feynman propagator. Practitioners often prefer to use the Euclidean setting.

PART II. PROPAGATORS ON A CURVED SPACETIME.

Consider now a curved spacetime with a metric tensor $g_{\mu\nu}$ in the presence of an external electromagnetic potential A_μ and an external scalar potential Y. Introduce the Klein-Gordon operator

$$K := |g|^{-\frac{1}{4}}(\mathrm{i}\partial_\mu + A_\mu)|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}\partial_\nu + A_\nu)|g|^{-\frac{1}{4}} + Y.$$

The charged field satisfies the Klein-Gordon equation

$$K\psi = K\psi^* = 0.$$

Consider first the stationary stable case. More precisely, we assume that $M = \mathbb{R} \times \Sigma$ and g, A, Y do not depend on $t \in \mathbb{R}$. We can then define the generator of the evolution of the Cauchy data B and the corresponding Hamiltonian $H = BQ$. We assume that H is positive definite (which is called the stability condition).

The whole theory of propagators of the Klein-Gordon equation goes through from the Minkowski to the stationary stable case with obvious minor changes (except for the slide that used the Fourier transformation).

The Poincaré covariance is lost. However, the state $(\Omega| \cdot \Omega)$ is uniquely determined by the requirement that it is the ground state of the Hamiltonian implementing the evolution.

Suppose now that the Klein-Gordon equation is more generic. We assume only that M is globally hyperbolic. It is well known that under this assumption one can define the classical propagators, i.e. the forward, backward, and Pauli-Jordan propagators, all possessing a causal support:

$$G^{\vee}, \quad G^{\wedge}, \quad G^{\mathrm{PJ}} := G^{\vee} - G^{\wedge}.$$

We introduce the space of space-compact solutions $\mathcal{W}_{\mathrm{KG}}$ and the classical fields $\psi(x)$, $\psi^*(x)$. $\mathcal{W}_{\mathrm{KG}}$ has a natural symplectic structure, and the Peierls bracket is still expressed by G^{PJ}. Then we perform algebraic quantization, obtaining quantum fields $\hat{\psi}(x)$, $\hat{\psi}^*(x)$. Unfortunately, unlike in the stable stationary case, $G^{(\pm)}$ are not well defined, hence there is no natural state.

Let us make an additional assumption that the Klein-Gordon equation is asymptotically stationary and stable in the future and the past. More precisely, we assume that we can identify $M \simeq \mathbb{R} \times \Sigma$, the evolution of the Cauchy data is given by a time dependent generator $B(t)$ with the Hamiltonian $H(t) := B(t)Q$, such that

asymptotic stationarity: $\lim_{t \to \pm\infty} B(t) =: B_\pm$, $\lim_{t \to \pm\infty} H(t) =: H_\pm$ exist;

asymptotic stability: $H_\pm$ is positive definite.

Define the evolution generated by $B(t)$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}R(t,s) = B(t)R(t,s), \quad R(t,t) = \mathbb{1}.$$

Set $\Pi_{\pm}^{(+)} := \theta\big(B_{\pm}\big), \quad \Pi_{\pm}^{(-)} := \theta\big(-B_{\pm}\big).$

Lemma. *Under mild technical conditions, for any* s

$$\lim_{t\to-\infty} R(s,t)\,\mathrm{Ran}\,\Pi_{-}^{(+)}, \qquad \lim_{t\to+\infty} R(s,t)\,\mathrm{Ran}\,\Pi_{+}^{(-)},$$

$$\lim_{t\to-\infty} R(s,t)\,\mathrm{Ran}\,\Pi_{-}^{(-)}, \qquad \lim_{t\to+\infty} R(s,t)\,\mathrm{Ran}\,\Pi_{+}^{(+)}$$

are two pairs of complementary subspaces.

With help of the above lemma we define two pairs of projections onto these subspaces:

$$\Lambda^{\mathrm{F}(+)}(s), \quad \Lambda^{\mathrm{F}(-)}(s),$$

$$\Lambda^{\overline{\mathrm{F}}(+)}(s), \quad \Lambda^{\overline{\mathrm{F}}(-)}(s).$$

Now we can define all propagators in the Cauchy data setting:

$$
\begin{aligned}
E^{\mathrm{PJ}}(t,s) &\coloneqq R(t,s),\\
E^{\vee}(t,s) &\coloneqq \theta(t-s)R(t,s),\\
E^{\wedge}(t,s) &\coloneqq -\theta(s-t)R(t,s),\\
E_{\pm}^{(+)}(t,s) &\coloneqq \lim_{\tau\to\pm\infty} R(t,\tau)\Pi_{\pm}^{(+)}R(\tau,s),\\
E_{\pm}^{(-)}(t,s) &\coloneqq \lim_{\tau\to\pm\infty} R(t,\tau)\Pi_{\pm}^{(-)}R(\tau,s),\\
E^{\mathrm{F}}(t,s) &\coloneqq \theta(t-s)R(t,s)\Lambda^{\mathrm{F}(+)}(s)-\theta(s-t)R(t,s)\Lambda^{\mathrm{F}(-)}(s),\\
E^{\overline{\mathrm{F}}}(t,s) &\coloneqq \theta(t-s)R(t,s)\Lambda^{\overline{\mathrm{F}}(-)}(s)-\theta(s-t)R(t,s)\Lambda^{\overline{\mathrm{F}}(+)}(s).
\end{aligned}
$$

To obtain the propagators in the spacetime approach, set

$$G^{\bullet} := \mathrm{i}E^{\bullet}_{12}, \quad \bullet = \mathrm{PJ}, \vee, \wedge, \mathrm{F}/\overline{\mathrm{F}},$$

$$G^{(\pm)}_{\pm} := \pm E^{(\pm)}_{\pm 12}, \quad E^{\bullet} = \begin{bmatrix} E^{\bullet}_{11} & E^{\bullet}_{12} \\ E^{\bullet}_{21} & E^{\bullet}_{22} \end{bmatrix}.$$

The identities satisfied by the propagators in the generic case differ from the stationary case:

$$G^{\mathrm{PJ}} = G^{\vee} - G^{\wedge} \tag{1'}$$

$$= \mathrm{i}G^{(+)}_{\pm} - \mathrm{i}G^{(-)}_{\pm}, \tag{2'}$$

$$G^{\mathrm{F}} - G^{\overline{\mathrm{F}}} = \mathrm{i}G^{(+)}_{\pm} + \mathrm{i}G^{(-)}_{\pm} + \text{ smooth }, \tag{3'}$$

$$G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge} + \text{ smooth }. \tag{4'}$$

We can use the in/out positive/negative frequency bisolutions to define two Fock representations containing the in–vacuum and the out–vacuum:

$$(\Omega_\pm \,|\, \hat{\psi}(x)\hat{\psi}^*(y)\Omega_\pm) = G_\pm^{(+)}(x,y),$$
$$(\Omega_\pm \,|\, \hat{\psi}^*(x)\hat{\psi}(y)\Omega_\pm) = G_\pm^{(-)}(x,y).$$

The Feynman propagator yields the expectation value of the time-ordered product of fields between the in- and the out-vacuum:

$$-\mathrm{i}G^{\mathrm{F}}(x,y) = \frac{\left(\Omega_+|\mathrm{T}\left(\hat{\psi}(x)\hat{\psi}^*(y)\right)\Omega_-\right)}{\left(\Omega_+|\Omega_-\right)}.$$

Note that the RHS is defined only if Ω_+ and Ω_- are in the same representation (which is rare and mathematically equivalent to the so-called Shale condition). However the LHS is always well-defined! Therefore, physicists do not have to care too much about the "implementability of the dynamics on a Hilbert space". They can always compute Feynman diagrams using the Feynman propagator, even if mathematicians forbid them to do this!

In words, the Feynman propagator describes particles travelling forward in time and antiparticles travelling backward in time. Sometimes it is called the in-out Feynman propagator—in my opinion, the name the Feynman propagator is good enough: the definite article the is sufficient.

In a somewhat different setting, the construction of G^{F} was given by A.Vasy et al and by Gérard-Wrochna. But it seems that in its natural generality the above construction was realized only recently by me and D.Siemssen.

Thus on asymptotically stationary spacetimes we have two natural vacuum states and a single natural Feynman propagator. They depend globally on the whole spacetime. However, their singularities are given by the local data.

Perhaps some of you may be surprised that the so-called Hadamard condition has not been mentioned in my talk so far.

In words, a two-point function satisfies the Hadamard condition if it is a positive definite bisolution of the Klein-Gordon equation whose wave front set is the same as in the flat case. The state defined by such a two-point function is called a Hadamard state.

Note that there are many Hadamard states. In particular, the in- and out states, which we discussed, are automatically Hadamard, as proven by Gérard and Wrochna.

To my understanding, one can divide researchers interested in QFT on curved spacetimes into two categories.

1. The Feynmanists work with a global spacetime and use the distinguished in- and out states and the distinguished Feynman propagator. This is probably common among phenomenologically minded researchers.
2. The Hadamardists usually look at spacetimes locally and say that the refererence state can be arbitrary as long as it satisfies the Hadamard condition. Many researchers in the mathematical QFT community belong to this category.

There is no contradiction between the Feynmanist and Hadamardist philosophy. Nevertheless, the emphasis of both approaches is quite different. My presentation tries to be a mathematical exposition of the Feynmanist approach.

If one insists on locality, at least in the time variable, one can use the hybrid approach: Consider the spacetime $[t_-, t_+] \times \Sigma$, choose Hadamard states for the in-vacuum Ω_- and the out-vacuum Ω_+, and then use the corresponding Feynman propagator to compute scattering amplitudes between t_- and t_+.

To describe the (mild) technical assumptions needed to construct the propagators we discussed, let us first recall some concepts from functional analysis.

We say that a topological space $\mathcal{W}$ is Hilbertizable, if it is equipped with a topology of a Hilbert space. Suppose that a (complex) Hilbertizable space $\mathcal{W}$ is equipped with a non-degenerate Hermitian form Q, sometimes called a charge form

$$\mathcal{W} \times \mathcal{W} \ni (v, w) \mapsto (v|Qw) = \overline{(w|Qv)} \in \mathbb{C}.$$

Note that often one starts from a real space with a symplectic form ω. Then the charge form appears naturally as the complexification of $\mathrm{i}\omega$.

An operator $S_\bullet$ on $(\mathcal{W}, Q)$ will be called an admissible involution if $S_\bullet^2 = \mathbb{1}$ and there exists a scalar product $(\cdot|\cdot)_\bullet$ compatible with the structure of $\mathcal{W}$ such that

$$(v \,|\, Qw) = (v \,|\, S_\bullet w)_\bullet.$$

$(\mathcal{W}, Q)$ is called a Krein space if it possesses an admissible involution.

Note that if $S_\bullet$ is an admissible involution then its spectral subspaces are a pair of Q-orthogonal subspaces, one is maximal positive and the other maximal negative. The following lemma plays the crucial role in the construction of the Feynman propagator:

Lemma. *Let $\mathcal{Z}^{(+)}, \mathcal{Z}^{(-)}$ be subspaces of a Krein space $\mathcal{W}$. If $\mathcal{Z}^{(+)}$ is maximal positive and $\mathcal{Z}^{(-)}$ is maximal negative, then they are complementary.*

The technical assumptions that we need in our construction of propagators can be described as follows We assume the space of Cauchy data equipped with the charge form can be endowed with the Krein structure, such that $B(t)$ can be interpreted as a closed operator generating an evolution $R(t, s)$. Besides, we assume that $\mathrm{sgn}(B_+)$ and $\mathrm{sgn}(B_-)$ are admissible involutions.

PART III. THE KLEIN-GORDON OPERATOR.

Recall that the Klein-Gordon operator is

$$K := |g|^{-\frac{1}{4}}(\mathrm{i}\partial_\mu + A_\mu)|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}\partial_\nu + A_\nu)|g|^{-\frac{1}{4}} + Y.$$

It is clearly Hermitian (symmetric) in the sense of the Hilbert space $L^2(M)$:

$$\int \overline{f_1(x)}(Kf_2)(x)\,\mathrm{d}x = \int \overline{(Kf_1)(x)}f_2(x)\,\mathrm{d}x.$$

We use the so-called half-density formalism. Thus the scalar product of f_1 and f_2 is in coordinates $(f_1|f_2) = \int \overline{f_1(x)}f_2(x)\,\mathrm{d}x$.

Conjecture. *On a large class of spacetimes*

(1) the operator K is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$ in the sense of $L^2(M)$.

(2) in the sense $\langle t\rangle^{-s}L^2(M) \to \langle t\rangle^{s}L^2(M)$, where $s > \frac{1}{2}$,

$$\operatorname*{s-lim}_{\epsilon\searrow 0}(K - \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}}, \quad \operatorname*{s-lim}_{\epsilon\searrow 0}(K + \mathrm{i}\epsilon)^{-1} = G^{\overline{\mathrm{F}}}.$$

The above conjecture is easy to show in various special cases: In the static stable case, if the spatial dimension is zero (when the Klein-Gordon operator reduces to the 1-dimensional Schrödinger operator), on a large class of Friedmann type spacetimes, also for symmetric spacetimes.

Surprisingly, we have not found a trace of this question in the older mathematical literature. Many respected mathematicians and mathematical physicists react with disgust to this question, claiming that it is completely non-physical.

In recent papers of A. Vasy and also Nakamura-Taira this conjecture has been proven for asymptotically Minkowskian spaces by rather technical arguments.

However, in the physical literature there are many papers that take the self-adjointness of the Klein-Gordon operator for granted. The method of computing the Feynman propagator with external fields and possibly on curved spacetimes based on the identity

$$\lim_{\epsilon \searrow 0} \frac{1}{K - \mathrm{i}\epsilon} = \mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}\tau K} \, \mathrm{d}\tau \qquad (*)$$

has even a name:

the Fock–Schwinger or Schwinger–DeWitt method.

The variable τ is called the proper time or sometimes the fifth coordinate. Of course, without the self-adjointness of K, $(*)$ does not make sense.

THANK YOU FOR YOUR ATTENTION

(This is the end of the main part of my slides. Note that I have some additional slides with "remarks" and "appendices", which normally I do not have time to cover in a talk.)

REMARK ABOUT VACUUM ENERGY

Suppose we have two quantum field theories with the evolutions U_i and the in vacuum Ω_i^- and the out vacuum Ω_i^-, $i = 0, 1$. Then the difference of energies produced during the two evolutions is defined by

$$\mathrm{e}^{-\mathrm{i}\mathcal{E}} = \lim_{t\to\infty} \frac{\left(\Omega_1^+|U_1(t,-t)\Omega_1^-\right)}{\left(\Omega_0^+|U_0(t,-t)\Omega_0^-\right)},$$

$\mathcal{E}$ goes under various names, e.g. the (relative) vacuum energy $\mathcal{E}$ or the effective action. Note that its imaginary part describes the decay of the vacua.

Let K_i, $i = 0, 1$, be two Klein-Gordon operators corresponding to the same metric but two scalar potentials Y_i, $i = 0, 2$. Then the vacuum energy can be computed using the Feynman propagator G_0^{F}:

$$\begin{aligned}\mathcal{E} =& \mathrm{Tr}\Big(\log(K_1 - \mathrm{i}0) - \log(K_0 - \mathrm{i}0)\Big) \\ =& \mathrm{Tr} \log \Big(\mathbb{1} + (Y_1 - Y_0) G_0^{\mathrm{F}} \Big).\end{aligned}$$

This is usually infinite, but in some situations after renormalization it leads to useful finite expressions, e.g. it describes the Casimir effect.

REMARK ABOUT WICK ROTATION

One often prefers to replace Lorentzian manifold by Riemannian manifolds by doing the Wick rotation, which is described below.

For simplicity let us assume that the potentials are zero. By choosing appropriate coordinates the Klein-Gordon operator (or actually the d'Alembertian) can be written as

$$\begin{aligned}K = & -|g|^{-\frac{1}{4}}(\partial_0 - \beta^i\partial_i)|g|^{\frac{1}{2}}\alpha^{-2}(\partial_0 - \beta^i\partial_i)|g|^{-\frac{1}{4}} \\ & + |h|^{-\frac{1}{4}}\partial_i|h|^{\frac{1}{2}}h^{ij}\partial_j|h|^{-\frac{1}{4}}.\end{aligned}$$

Here, $\alpha^{-2} = -g^{00}$, β is the so-called lapse vector and $[h_{ij}]$ is the spatial part of the metric tensor $[g_{\mu\nu}]$.

The Wick rotation consists in replacing α with $\mathrm{i}\alpha$. The d'Alembertian becomes an elliptic operator:

$$K^{\mathrm{E}} = |g|^{-\frac{1}{4}}(\partial_0 - \beta^i\partial_i)|g|^{\frac{1}{2}}\alpha^{-2}(\partial_0 - \beta^i\partial_i)|g|^{-\frac{1}{4}} + |h|^{-\frac{1}{4}}\partial_i|h|^{\frac{1}{2}}h^{ij}\partial_j|h|^{-\frac{1}{4}}.$$

The Feynman propagator $(K - \mathrm{i}0)^{-1}$ can then be replaced by the Euclidean propagator K^{-1}. Life becomes much easier!

Note that to define the Wick rotation we needed to fix an identification $M \sim \mathbb{R} \times \Sigma$.

The Wick rotation is usually described in textbooks as replacing x^0 with $\mathrm{i}x^0$. This is not always correct, especially in the curved case.

Appendix I. Evolution in Hilbertizable spaces

Let $\mathcal{W}$ be a Banach space. We say that a two-parameter family of bounded operators

$$\mathbb{R} \times \mathbb{R} \ni (t, s) \mapsto R(t, s) \in B(\mathcal{W}) \qquad (*)$$

is a strongly continuous evolution family on $\mathcal{W}$ if for all r, s, t, we have the identities

$$R(t, t) = \mathbb{1}, \quad R(t, s)R(s, r) = R(t, r).$$

and the map $(*)$ is strongly continuous.

If $R(t, s) = R(t - s, 0)$ for all t, s, we say that the evolution is autonomous. Setting $R(t) := R(t, 0)$, we obtain a strongly continuous one-parameter group. As is well known, we can then write $R(t) = \mathrm{e}^{-\mathrm{i}tB}$, where $-\mathrm{i}B$ is a certain unique, densely defined, closed operator called the generator of $R(t)$.

If $\mathcal{W}$ is a Hilbert space, then B is self-adjoint if and only if R is unitary.

Let $\mathcal{W}$ be a topological vector space. We say that it is Hilbertizable if it has a topology of a Hilbert space for some scalar product $(\cdot\,|\,\cdot)_\bullet$ on $\mathcal{W}$.

Let $(\cdot\,|\,\cdot)_1$, $(\cdot\,|\,\cdot)_2$ be two scalar products compatible with a Hilbertizable space $\mathcal{W}$. Then there exist constants $0 < c \leq C$ such that

$$c(w\,|\,w)_1 \leq (w\,|\,w)_2 \leq C(w\,|\,w)_1.$$

Let $\{B(t)\}_{t\in\mathbb{R}}$ be a family of densely defined, closed operators on a Hilbertizable space $\mathcal{W}$. Let $\mathcal{V}$ be another Hilbertizable space densely and continuously embedded in $\mathcal{W}$. The following theorem, due essentially to Kato, gives sufficient conditions for the existence of a (non-autononomous) evolution generated by $\{B(t)\}_{t\in\mathbb{R}}$

Theorem. Suppose that the following conditions are satisfied:

(a) $\mathcal{V} \subset \operatorname{Dom} B(t)$ so that $B(t) \in B(\mathcal{V}, \mathcal{W})$ and $t \mapsto B(t) \in B(\mathcal{V}, \mathcal{W})$ is norm-continuous.

(b) For every t, scalar products $(\cdot \mid \cdot)_{\mathcal{W},t}$ and $(\cdot \mid \cdot)_{\mathcal{V},t}$ compatible with $\mathcal{W}$ resp. $\mathcal{V}$ have been chosen.

(c) $B(t)$ is self-adjoint in the sense of $\mathcal{W}_t$ and the part $\tilde{B}(t)$ of $B(t)$ in $\mathcal{V}_t$ is self-adjoint in the sense of $\mathcal{V}_t$.

(d) For $C \in L^1_{\mathrm{loc}}$ and all s, t

$$\|v\|_{\mathcal{W},s} \le \|v\|_{\mathcal{W},t} \exp\left|\int_s^t C(r)\,\mathrm{d}r\right|,$$
$$\|w\|_{\mathcal{V},s} \le \|w\|_{\mathcal{V},t} \exp\left|\int_s^t C(r)\,\mathrm{d}r\right|.$$

Then there exists a unique family of bounded operators $\{R(t,s)\}_{s,t}$ on $\mathcal{W}$, preserving $\mathcal{V}$, called the evolution generated by $B(t)$, such that:

(i) It is an evolution on $\mathcal{W}$ and $\mathcal{V}$,

(ii) For all $v \in \mathcal{V}$ and s, t,

$$\begin{aligned} \mathrm{i}\partial_t R(t,s)v &= B(t)R(t,s)v, \\ -\mathrm{i}\partial_s R(t,s)v &= R(t,s)B(s)v, \end{aligned}$$

where the derivatives are in the strong topology of $\mathcal{W}$.

Appendix II. Lemma about subspaces of a Krein spaces

Suppose that $(\mathcal{W}, Q)$ is a Krein space. A subspace $\mathcal{Z}$ of $\mathcal{W}$ is called positive/negative, iff $(\cdot|Q\cdot)$ restricted to $\mathcal{Z}$ is positive/negative. It is called maximal positive/negative if it cannot be extended to a larger positive/negative subspace.

Every admissible involution $S_\bullet$ defines a pair of projections

$$\text{the positive projection} \quad \Pi_\bullet^{(+)} := \frac{1}{2}(\mathbb{1} + S_\bullet),$$
$$\text{the negative projection} \quad \Pi_\bullet^{(-)} := \frac{1}{2}(\mathbb{1} - S_\bullet).$$

It is easy to see that $\mathrm{Ran}\,\Pi_\bullet^{(+)}$ is maximal positive and $\mathrm{Ran}\,\Pi_\bullet^{(-)}$ is maximal negative, and they are orthogonal wrt $(\cdot|Q\cdot)$.

Lemma. Let S_1, S_2 be a pair of admissible involutions on a Krein space $(\mathcal{W}, Q)$. Then we have two direct sum decompositions:

$$\begin{aligned}\mathcal{W} &= \operatorname{Ran}\Pi_1^{(+)} \oplus \operatorname{Ran}\Pi_2^{(-)} \\ &= \operatorname{Ran}\Pi_1^{(-)} \oplus \operatorname{Ran}\Pi_2^{(+)}.\end{aligned}$$

In other words, a pair of subspaces, one maximal positive and the other maximal negative is always complementary.

Let us sketch the proof. Set $K := S_2 S_1$. Then K is positive with respect to $(\cdot \mid \cdot)_1$ and $(\cdot \mid \cdot)_2$. Hence we can define $c := \Pi_1^{(+)} \frac{\mathbb{1}-K}{\mathbb{1}+K} \Pi_1^{(-)}$. Then the projections corresponding to the above direct sum decompositions are

$$\Lambda_{12}^{(+)} = \begin{bmatrix} \mathbb{1} & c \\ 0 & 0 \end{bmatrix}, \qquad \Lambda_{21}^{(-)} = \begin{bmatrix} 0 & -c \\ 0 & \mathbb{1} \end{bmatrix};$$

$$\Lambda_{12}^{(-)} = \begin{bmatrix} 0 & 0 \\ c^* & \mathbb{1} \end{bmatrix}, \qquad \Lambda_{21}^{(+)} = \begin{bmatrix} \mathbb{1} & 0 \\ -c^* & 0 \end{bmatrix}.$$

where we use the direct sum $\operatorname{Ran} \Pi_1^{(+)} \oplus \operatorname{Ran} \Pi_1^{(-)}$.